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FOURIER ANALYSIS OF GRIDFUNCTIONS, PROLONGATIONS AND RESTRICTIONS

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Fourier analysis of gridfunctions, prolongations and restrictions

by

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ABSTRACT

In this report we present in some detail the elementary Fourier analysis of gridfunctions. These gridfunctions are functions defined on an n-dimensional, rectangular, regularly spaced and infinite grid. We consider the effect of operators of a general convolution type; in particular we study prolongations and restrictions. These are operators which transform functions on coarse to functions on fine grids, vice versa. Special attention is payed to the combination of particular restrictions and prolongations.

KEY WORDS & PHRASES: *Fourier Transformation, gridfunctions, prolongation, restriction*

'ACE

This report is meant as a general introduction to the Fourier analysis ridfunctions. Fourier transforms (FT) are a part of mathematics of which may suppose that every mathematician is familiar with. Many texts are lable and the approach to FT may range from very applied to really ract. Hence, much material on the FT is available and almost all prines that are used in this report are well known in one form or another. ver, no text is known to the author, in which the theory of FT of gridtions is analysed in detail. Moreover, the implications of the theory not always immediately clear and, in particular for numerical analysts, od understanding of this theory and its implications may be really ful in different areas of their interest.

The motive for us to consider Fourier transforms of gridfunctions is d in our need to avail of the elementary material to give solid aments to the local mode analysis of the behaviour of multigrid algors. This kind of analysis of multigrid algorithms is already found in DT [1977] where it is one of the essential justifications for the multi- approach.

CONTENTS

0. FOURIER TRANSFORMS OF CONTINUOUS FUNCTIONS

In this section we collect well-known results with respect to Fourier transforms of functions that are defined (almost everywhere) on domains in the real n-dimensional space.

All results mentioned in this section can be found in general texts as e.g. KATZNELSON [1968], LIONS and MAGENES [1968], PAPOULIS [1962], RUDIN [1973].

Let u be a real or complex valued function defined (almost everywhere) on the real n-dimensional space $\mathbb{R}^n$ and let u be square integrable: $u \in L^2(\mathbb{R}^n)$, then its Fourier transform $\hat{u}$ is defined by

$$\hat{u}(y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ixy}\, u(x)\,dx.$$

Furthermore, a back-transformation formula is available

$$\tilde{u}(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{+ixy}\, \hat{u}(y)\,dy,$$

such that $\tilde{u}(x) = u(x)$ almost everywhere on $\mathbb{R}^n$. Moreover, we know

$$\hat{u} \in L^2(\mathbb{R}^n) \quad \text{and} \quad \|u\|_{L^2(\mathbb{R}^n)} = \|\hat{u}\|_{L^2(\mathbb{R}^n)}.$$

In words we can express this by saying that the Fourier transformation is a norm-invariant bijection $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. The above definition of a Fourier transform can be generalized to more general functions than just $L^2(\mathbb{R}^n)$ - functions. The same definition applies to the set of "tempered distributions" (see e.g. RUDIN [1973]), in this case -again- the back-transformation is available.

A number of useful relations is known when operations are performed on functions or their Fourier transforms. In table I we mention a number of these relations for the case n = 1. Most relations are easily generalized to the n-dimensional case.

REMARK. From the definition of a FT it is clear that the FT of a symmetric real function is again symmetric and real. For these functions the Fourier transformation is identical with the Fourier back-transformation. A few examples of such functions and their FT are given in table II.

	u	û	Remark
1.	$u(\lambda x)$	$\frac{1}{\lvert\lambda\rvert}\,\hat{u}\,(\frac{y}{\lambda})$	$\lambda \in \mathbb{R}$ $\lambda \neq 0$
2.	$u(x+\lambda)$	$e^{i\lambda y}\,\hat{u}\,(y)$	$\lambda \in \mathbb{R}$
3.	$e^{i\lambda x}\,u(x)$	$\hat{u}(y-\lambda)$	$\lambda \in \mathbb{R}$
4.	$Du(x)$	$iy\,\hat{u}\,(y)$	$D = (\frac{d}{dx})$
5.	$xu(x)$	$iD\,\hat{u}\,(y)$	$D = (\frac{d}{dy})$
6.	$p(D)\;u(x)$	$p(iy)\,\hat{u}\,(y)$	$p(x) = \sum_{0}^{n} c_k\,x^k$ (rows 6 and 7)
7.	$p(x)\;u(x)$	$p(iD)\,\hat{u}\,(y)$	
8.	$u_1 u_2$	$\hat{u}_1 * \hat{u}_2$	$(u_1 * u_2)\,(x) =$
9.	$u_1 * u_2$	$\hat{u}_1 \hat{u}_2$	$(2\pi)^{-n/2} \int_{\mathbb{R}} u_1\,(x-y)\,u_2\,(y)\,dy$

Table I. The effect of operations on the Fourier transform of a function.

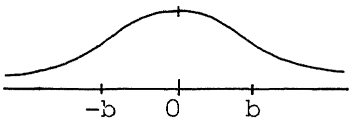
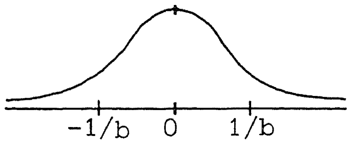
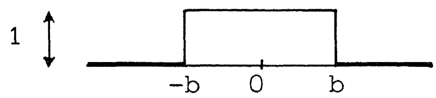
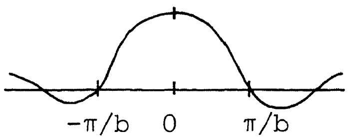
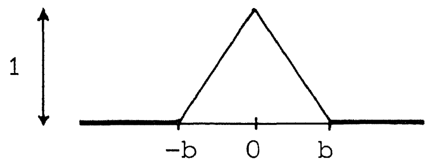
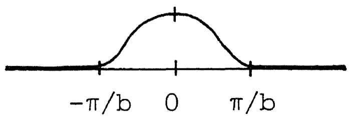
1.	$e^{-(x/b)^2/2}$ 	$be^{-(by)^2/2}$ 	2.
3.	piecewise constant block 	$\sqrt{\frac{2}{\pi}}\ \frac{\sin by}{y}$ 	4.
5.	piecewise linear hat-function 	$b\left(\frac{\sin(by/2)}{by/2}\right)^2 = \frac{2}{b}\ \frac{1-\cos(by)}{y^2}$ on $[-\pi/b,\pi/b]$ 	6.
7.	unit function 1	Dirac delta-function δ	8.

Table II. Some symmetric functions and their mutual Fourier transforms

REMARK. Above we saw that the FT of a function defined on $\mathbb{C}^n$ itself is a function defined on $\mathbb{C}^n$. It is also well-known that a Fourier transformation is defined for a finite set of equally spaced data (Finite Fourier Transform). In this case the FT of a set of N data is again a set of N coefficients (see e.g. HAMMING [1977]).

The FT of a periodic function (or, what is the same, the FT of a function defined on a torus) is a countable infinite set of coefficients. Analogously, in the following sections we shall introduce the Fourier transformation on an infinite set of equally spaced data. In this case the FT of such a "gridfunction" will be a periodic function (which is the same as a function defined on a torus).

The periodization of a function.

DEFINITION. Let $h \in \mathbb{R}^n$ be given, then the h-*periodization of a function* $u:\mathbb{R}^n \to \mathbb{C}$ is defined by

$$\tilde{u}(x) = \sum_{k\in\mathbb{Z}^n} u(x-kh).$$

We notice that $\tilde{u}(x)$ is a periodic function on $\mathbb{R}^n$ with period h; it is completely defined by a mapping $[0,h) \to \mathbb{C}$, where $[0,h)$ is defined by

$$[0,h) = [0,h_1) \times [0,h_2) \times \ldots \times [0,h_n).$$

The FT of a function $\tilde{u}(x)$ defined on a torus $[0,h)$ is (cf. KATZNELSON [1968]) a sequence $\{c_k\}_{k\in\mathbb{Z}^n}$ defined by

$$c_k = \frac{1}{h^n(2\pi)^{n/2}} \int_0^h e^{-2\pi ikx/h}\, \tilde{u}(x)\,dx \;,$$

from which it is clear that $c_k = \hat{u}(2\pi k)/h^n$. Also the Fourier transformation on the torus $[0,h)$ has its back-transformation. From this we see that the knowledge of $\hat{u}(y)$ only at certain equally spaced points is enough to restore a periodization of the original function u, whereas the complete definition of $\hat{u}(y)$ (almost everywhere on $\mathbb{R}^n$) is necessary to find the function u(x) itself.

1. BASIC DEFINITIONS

For a fixed "mesh" $h = (h_1,h_2,\dots,h_n)$ with $h_j > 0$, $j = 1,2,\dots,n$, the *regular infinite n-dimensional grid* $\mathbb{Z}_h^n$ is defined by

$$\mathbb{Z}_h^n = \{jh \mid j \in \mathbb{Z}^n\}.$$

For $h \in \mathbb{R}_+^n$ and $j \in \mathbb{Z}^n$ the expressions $jh \in \mathbb{R}^n$, $h/j \in \mathbb{R}^n$ and $h^j \in \mathbb{R}$ are defined by

$$jh = (j_1h_1, j_2h_2, \dots, j_nh_n),$$
$$h/j = (h_1/j_1, h_2/j_2, \dots, h_n/j_n),$$
$$h^j = h_1^{j_1} . h_2^{j_2} . \;\dots\; . h_n^{j_n} .$$

Further we define $h^n = h_1.h_2. \;\dots\; .h_n$.

We define the *n-dimensional torus* $[2\pi/h]^n$ by

$$[2\pi/h]^n = (-\pi/h_1,\pi/h_1]\times \dots \times(-\pi/h_n,\pi/h_n] \subset \mathbb{R}^n .$$

2. SPACES OF GRIDFUNCTIONS

A complex or a real *gridfunction* is defined as a mapping

$$\mathbb{Z}_h^n \to \mathbb{C}^d,$$

respectively

$$\mathbb{Z}_h^n \to \mathbb{R}^d,$$

where d is the dimension of the image space.

In this report we mostly restrict ourselves to the scalar real grid-function

$$\mathbb{Z}_h^n \to \mathbb{R}$$

and, unless stated otherwise, we shall use the word gridfunction for this kind of gridfunction exclusively.

It is immediate that, with the usual addition and scalar multiplication, the set of all gridfunctions is a vector space. This vector space we denote by

$$\ell_h(\mathbb{Z}_h^n)$$

or, shortly, by ℓ_h.

For any $p \geq 1$ or $p = \infty$ the space $\ell_h(\mathbb{Z}_h^n)$ can be provided with a norm $\|.\|_p$, which is defined by

$$\|u_h\|_p^p = h^n \sum_{j\in\mathbb{Z}^n} |u_h(jh)|^p, \qquad 1 < p < \infty,$$

or

$$\|u_h\|_p = \sup_{j\in\mathbb{Z}^n} |u_h(j^h)|, \qquad p = \infty .$$

For a fixed p, $1 \leq p \leq \infty$, all gridfunctions with a finite norm $\|.\|_p$ form a subspace of $\ell_h(\mathbb{Z}_h^n)$, which is denoted by

$$\ell_h^p(\mathbb{Z}_h^n) .$$

It is obvious that for any p, $1 \leq p \leq \infty$, $\ell_h^p(\mathbb{Z}_h^n)$ is a Banach space (cf. YOSIDA p.35). Moreover, for $p = 2$, $\ell_h^2(\mathbb{Z}_h^n)$ is a Hilbert space with the inner product

$$\langle u_h, v_h \rangle_h = h^n \sum_{j\in\mathbb{Z}^n} u_h(jh)\, v_h(jh).$$

3. THE FOURIER TRANSFORM OF A GRIDFUNCTION

We define $\hat{u}_h : [2\pi/h]^n \to \mathbb{C}$, the Fourier transform of $u_h : \mathbb{Z}_h^n \to \mathbb{C}$ by

$$\hat{u}_h(\omega) = \left(\frac{h}{\sqrt{2\pi}}\right)^n \sum_{j\in\mathbb{Z}^n} e^{-ijh\omega}\, u_h(jh).$$

The backtransformation formula reads

$$u_h(jh) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \int_{\omega\in[2\pi/h]^n} e^{+ijh\omega}\, \hat{u}_h(\omega)d\omega.$$

<u>REMARK 1</u>. $\hat{u}_h$ can also be considered as a $[2\pi/h]^n$- periodic function $\hat{u}_h:\mathbb{R}^n \to \mathbb{C}$.

<u>REMARK 2</u>. The back transformation formula is easily derived from the usual Fourier-transformation theory for periodic functions.

<u>REMARK 3</u>. By the Parseval equality we have

$$\|u_h\|_{\ell^2_u(\mathbb{Z}^n_h)} = \|\hat{u}_h\|_{L^2([2\pi/h]^n)} .$$

<u>NOTE</u>. For this equality we need the special balancing of the transformation and its back transformation by the scalar factor $(2\pi)^{-n/2}$!

In the backtransformation formula we see that any gridfunction u_h, for which $\hat{u}_h$ exists, can be considered as a linear combination of gridfunctions $u_{h,\omega}$ of the form

$$u_{h,\omega}(jh) = e^{ijh\omega}, \qquad \omega \in [-\pi/h,\pi/h]^n,$$

i.e. a periodic gridfunction with period $\frac{2\pi}{h\omega}$. The parameter ω is called the *frequency* of the gridfunction $u_{h,\omega}$.

We see that for a given "mesh width" h the range of ω is limited to the halfopen interval $[-\pi/h,\pi/h)$ or to the interval $[0,2\pi/h)$. The equivalence of both intervals as the range of definition of ω is caused by the fact that

$$u_{h,\omega} \equiv u_{h,\omega} + 2\pi k/h$$

for all $k \in \mathbb{Z}^n$. This phenomenon, that gridfunction with a frequency ω can be identified with a gridfunction with a gridfunction with frequency ω mod $2\pi/h$ is known as *aliassing*.

REMARK 4. In the above formulation of the Fourier transform, where we carry on the meshwidth h as a parameter, we see that the range of frequencies that can be represented on a fine grid (small h) is larger than the range of those which can be represented on a coarser grid (large h).

4. THE RELATION BETWEEN THE FT OF A CONTINUOUS FUNCTION AND THE FT OF ITS RESTRICTION TO A GRID

In this section we describe the relation between the FT of a continuous function defined on $\mathbb{R}^n$ and the FT of its restriction to a regular infinite n-dimensional grid $\mathbb{Z}_h^n$.

THEOREM.

Let $u:\mathbb{R}^n \to \mathbb{C}$ *be a continuous function with FT* $\hat{u}$. *Let* u_h *be defined by*

$$u_h(jh) = u(jh) \qquad \forall j \in \mathbb{Z}^n,$$

then

$$\hat{u}_h(\omega) = \sum_{k\in\mathbb{Z}^n} \hat{u}(\omega+2\pi k/h).$$

PROOF.

$$\begin{aligned}
\hat{u}_h(\omega) &= \left(\frac{h}{\sqrt{2\pi}}\right)^n \sum_j e^{-ijh\omega}\, u_h(jh) \\
&= \left(\frac{h}{\sqrt{2\pi}}\right)^n \sum_j e^{-ijh\omega}\, u(jh) \\
&= \left(\frac{h}{\sqrt{2\pi}}\right)^n \sum_j e^{-ijh\omega} \left(\frac{1}{\sqrt{2\pi}}\right)^n \int_{\mathbb{R}^n} e^{ijhy}\, \hat{u}(y)\, dy \\
&= \left(\frac{h}{2h}\right)^n \sum_j e^{-ijh\omega} \sum_{k\in\mathbb{Z}^n} \int_{-\pi/h}^{\pi/h} e^{ijh(y-2\pi k/h)}\, \hat{u}(y)\,dy
\end{aligned}$$

$$= \left(\frac{h}{2\pi}\right)^n \sum_j e^{-ijh\omega} \sum_k \int_{-\pi/h}^{\pi/h} e^{ijhz} \hat{u}(z+2\pi k/h)dz$$

$$= \left(\frac{h}{2\pi}\right)^n \sum_j e^{-ijh\omega} \int_{-\pi/h}^{\pi/h} e^{ijhz} \sum_k \hat{u}(z+2\pi k/h)dz$$

$$= \sum_k \hat{u}(z+2\pi k/h). \qquad \square$$

REMARK. We see that $\hat{u}_h$ is the $[2\pi/h]$ - periodization of $\hat{u}$.

5. OPERATORS DEFINED ON GRIDFUNCTIONS AND q-CONVOLUTIONS

In this section we introduce the notion of a q-convolution. This is a generalization of the usual convolution.

DEFINITION. Let $q \in \mathbb{Z}^n$ ($q_j > 0$, $j = 1,2,\dots,n$) then $[0,q)$ is defined by

$$[0,q) = \{m \in \mathbb{Z}^n \mid 0 \le m_j < q_j, \qquad j = 1,2,\dots,n\}.$$

DEFINITION. Let a_h, $u_h \in \ell_h$ be two gridfunctions, then the a_h- *q-convolution* of u_h is denoted by $a_h \overset{q}{*} u_h \in \mathbb{Z}_h^n$ and defined by

$$(a_h \overset{q}{*} u_h)((mq+p)h) \quad \sum_{j\in\mathbb{Z}^n} a_h((mq-jq+p)h)\, u_h((jq+p)h)$$

for all $m \in \mathbb{Z}^n$ and $p \in [0,q)$.

DEFINITION. The a_h-*convolution* of u_h is defined by

$$a_h * u_h = a_h \overset{1}{*} u_h.$$

REMARK 1. We see that the a_h-convolution of u_h is simply given by

$$(a_h * u_h)(mh) = \sum_{j\in\mathbb{Z}^n} a_u(m-j)h).\, u_h(jh).$$

NOTATION. Given a gridfunction a_h, by A_h^q we denote the linear convolution mapping $\ell_h(\mathbb{Z}_h^n) \to \ell_h(\mathbb{Z}_h^n)$ defined by

$$A_h^q u_h = a_h \overset{q}{*} u_h \text{ for all } u_h \in \ell_h(\mathbb{Z}_h^n);$$

moreover, we denote $A_h = A_h^1$.

REMARK 2. The mapping $A_h^q : \ell_h(\mathbb{Z}_h^n) \to \ell_h(\mathbb{Z}_h^n)$ is a linear operator.

REMARK 3. A large number of well known difference operators on a regular rectangular mesh and with constant coefficients, yield mappings of the form A_h^q. If the same difference equation is applied at each gridpoint then the mappings are of the form A_h^1. If, in a periodic way, different difference equations are used at different points of the mesh (such as e.g. in regular finite element discretizations) the the discretized operator is of the form A_h^q, $q > 1$.

DEFINITION. A gridfunction *with a finite support* is a function u_h on $(\mathbb{Z}_h^n)$ for which a $M > 0$ exists that

$$u_h(jh) = 0 \qquad \text{for all } j \text{ with } |j| > M.$$

DEFINITION. A *smoothing operator* is a convolution operator a_h* where a_h has a finite support and satisfies

$$\sum_{j \in \mathbb{Z}^n} a_h(jh) = 1.$$

DEFINITION. A special case of an operator A_h^1 is the *translation operator* T_{qh}, $q \in \mathbb{Z}^n$, which is defined by

$$(T_{qh} u_h)(mh) = (a_h * u_h)(mh) = u_h(mh-qh).$$

For this translation operator the generating gridfunction a_h is given by $a_h(jh) = \delta_{jq}$ (with Kronecker symbol δ_{jq}).

REMARK 4. Clearly the inverse operator of T_{qh} is

$$E_{qh} = T_{qh}^{-1} = T_{-qh}.$$

DEFINITION. A special case of an operator A_h^q, $q \in \mathbb{Z}^n$, is the *flat q-restriction operator* R_q^0, which is defined by

$$(R_q^0 u_h)(jh) = \begin{cases} u_h(jh) & \text{if } j \ \underline{\text{mod}}\ q = 0, \\ 0 & \text{if } 0 \ \underline{\text{mod}}\ q \neq 0. \end{cases}$$

For this restriction operator the generating gridfunction a_h is given by

$$a_h(jh) = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{if } j \neq 0. \end{cases}$$

REMARK 5. [*Translation decomposition*].
With the translation operator and the flat q-restriction we can construct a partition of the identity operator

$$u_h = \sum_{p \in [0,q)} T_{ph} R_q^0 T_{-ph} u_h = \sum_{p \in [0,q)} T_{-ph} R_q^0 T_{ph} u_h.$$

6. CONVOLUTION OR TOEPLITZ OPERATORS

Let $A: \ell_h(\mathbb{Z}_h^n) \to \ell_h(\mathbb{Z}_h^n)$ be a linear operator. What eigenvalues λ_ω correspond with eigenfunctions v_ω of the form $v_\omega(jh) = e^{i\omega jh}$, if any?

In other words:

can we find $\lambda_\omega \in \mathbb{C}$, $\omega \in [2\pi/h]^n$ such that

$$Av_\omega = \lambda_\omega v_\omega \qquad ?$$

If it would be the case, then, with $A = (a_{mj})$,

$$\sum_j a_{mj}\, v_\omega(jh) = \lambda_\omega\, v_\omega(mh),$$

$$\sum_j a_{m,j}\, e^{i\omega jh} = \lambda_\omega\, e^{i\omega mh},$$

$$\lambda_\omega = \sum_j a_{m,j}\, e^{i\omega(j-m)h} = \sum_j a_{m,m+k}\, e^{i\omega kh}$$

should be independent of m.

That is the case if $\{a_{m,j}\}$ is such that

$$a_{m,m+k} = a_{-k} \text{ for all } m \in \mathbb{Z}^n .$$

CONCLUSION. If the linear operator $A:\ell_h(\mathbb{Z}_h^n) \to \ell_h(\mathbb{Z}_h^n)$ is such that its matrix elements $\{a_{m,j}\}$ satisfy

$$a_{m,m+k} = a_{-k} \text{ for all } m \in \mathbb{Z}^n$$

then, for any $\omega \in [2\pi/h]^n$, $\lambda_\omega = \sum_k a_{-k}\, e^{i\omega kh}$ is an eigenvalue corresponding to the eigenfunction v_ω of the form

$$v_\omega(jh) = e^{i\omega jh}.$$

DEFINITION. A linear operator $A:\ell_h(\mathbb{Z}_h^n) \to \ell_n(\mathbb{Z}_h^n)$ of which the matrix $\{a_{m,j}\}$ satisfies $a_{m,m+k} = a_{-k}$ $\forall m \in \mathbb{Z}^n$ is called a *Toeplitz* or *convolution operator* or *matrix*.

DEFINITION. The function $\lambda:[2\pi/h]^n \to \mathbb{C}^n$ is called *the spectrum* of the Toeplitz operator.

REMARK. A Toeplitz or convolution operator A can be defined by means of convolution with a grid function a_h if the element $a_{m,m+k} = a_{-k}$ is identified with $a_h(-kh)$, $\forall k \in \mathbb{Z}^n$.

$$\begin{pmatrix} \ddots & & & & \\ \cdots\; a_h(2h) & a_h(h) & a_h(0) & a_h(-h) & a_h(-2h)\;\cdots \\ & & & & \ddots \end{pmatrix} = (a_{m,j})$$

or

$$Au_h = \sum_j a_{m,j}\, u_h(jh) = a_h * u_h .$$

The spectrum of this operator is given by

$$\lambda(\omega) = \sum_{k\in\mathbb{Z}} a_h(kh)\, e^{-i\omega hk} = \frac{\sqrt{2\pi}}{h}\, \hat{a}_h(\omega) .$$

EXAMPLE. An infinite tridiagonal matrix with constant coefficients on the diagonals is a Toeplitz matrix

$$\begin{pmatrix} \ddots & \ddots & & \phi \\ \ddots & \beta \quad \alpha \quad \gamma & \ddots & \\ \phi & & \ddots & \ddots \end{pmatrix} = A \qquad \begin{array}{l} a_1 = \beta,\ a_0 = \alpha,\ \ a_{-1} = \gamma; \\ a_j = 0, \quad \text{if } |j| > 1. \end{array}$$

$$\lambda_\omega = a_{-1}\, e^{i\omega h} + a_0 + a_1\, e^{-i\omega h} = \alpha = (\gamma+\beta)\cos(\omega h) + i(\gamma-\beta)\sin\omega h,$$

$$|\lambda_\omega|^2 = \alpha^2 + \beta^2 + \gamma^2 + 2\alpha(\beta+\gamma)\cos(\omega h) + 2\beta\gamma\cos(2\omega h).$$

EXAMPLE. The translation operator T_{qh} is a convolution operator a_h* with a_h such that $a_h(jh) = \delta_{jq}$. Its spectrum is

$$\lambda(\omega) = e^{-i\omega qh}.$$

EXAMPLE. The forward difference operator:

$$a_0 = -1/h, \quad a_{-1} = +1/h,$$
$$\lambda(\omega) = (e^{i\omega h}-1)/h.$$

EXAMPLE. The backward difference operator:

$$a_{+1} = -1/h, \quad a_0 = 1/h,$$
$$\lambda(\omega) = (1-e^{-i\omega h})/h.$$

7. THE RELATION BETWEEN $\hat{u}_h$ AND $\widehat{a_h * u_h}$

In table I we saw that - for functions defined on $\mathbb{R}$ - a simple relation exists between the FT of a convolution product and the function product of two FTs. In this section we show that a similar relation exists for convolutions of gridfunctions.

THEOREM

$$\widehat{a_h * u_h} = \left(\frac{\sqrt{2\pi}}{h}\right)^n \hat{a}_h \, \hat{u}_h .$$

PROOF.

$$\begin{aligned}
\widehat{a_h * u_h}(\omega) &= \left(\frac{h}{\sqrt{2\pi}}\right)^n \sum_m e^{-imh} \sum_j a_h((m-j)h) . \, u_h(jh) = \\
&= \left(\frac{h}{2\pi}\right)^n \sum_{jm} \int_y a_h(mh-jh) \, e^{-imh+ijhy} \, \hat{u}_h(y)\,dy = \\
&= \left(\frac{h}{2\pi}\right)^n \sum_{jk} \int_y a_h(kh) \, e^{-ikh\omega + ijh(y-\omega)} u_h(y)\,dy = \\
&= \sum_k a_h(kh) \, e^{-ikh\omega} \, \hat{u}_h(\omega) \\
&= \left(\frac{\sqrt{2\pi}}{h}\right)^n \hat{a}_h(\omega) \, \hat{u}_h(\omega)
\end{aligned}$$

□

REMARK 1. We see that the result is similar to the result in table 1 except for the factor $(2\pi)^{n/2}/h^n$. This factor is due to the fact that in table 1 convolution is defined with a similar factor $(2\pi)^{-\frac{1}{2}}$.

REMARK 2. Clearly $a_h * u_h = u_h$

$$\text{iff} \quad \widehat{a_h * u_h} = \hat{u}_h$$

$$\text{iff} \quad \left(\frac{h}{\sqrt{2\pi}}\right)^n = \hat{a}_h$$

$$\text{iff} \quad a_h(jh) = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{if } j \neq 0. \end{cases}$$

8. THE RELATION BETWEEN THE FOURIER TRANSFORMS OF A GRIDFUNCTION AND ITS (CANONICAL) q-RESTRICTION

DEFINITION. Let $u_h \in \ell_h(\mathbb{Z}_h^n)$ be a gridfunction defined on $\mathbb{Z}_h^n$, then its *canonical q-restriction* $R_q u_h$, $(q \in \mathbb{Z}^n)$, is the gridfunction u_H defined on $\mathbb{Z}_H^n = \mathbb{Z}_{qh}^n$ defined by

$$(R_q u_h)(jH) = u_H(jH) = u_H(jqh) = u_h(qjh).$$

LEMMA.

If $u_h \in \ell_h^p(\mathbb{Z}_h^n)$ *then* $R_q u_h \in \ell_H^p(\mathbb{Z}_H^n)$ *with* $H = qh$.

PROOF.

$$\begin{aligned} \|u_H\|^p_{\ell_H^p(\mathbb{Z}_H^n)} &= H^n \sum_{j\in\mathbb{Z}^n} |u_H(jH)|^p \\ &= q^n h^n \sum_{j\in\mathbb{Z}^n} |(u_n(jqh)|^p \\ &\leq q^n h^n \sum_{j\in\mathbb{Z}^n} |u_n(jh)|^p \\ &= q^n \|u_n\|_{\ell_h^p(\mathbb{Z}_H^n)} < \infty \end{aligned}$$

With $q^n = q_1, q_2, \ldots, q_n$. □

<u>COROLLARY</u>.

Let $H = qh$ *and* $R_q : \ell_h^P(\mathbb{Z}_h^n) \to \ell_H^P(\mathbb{Z}_H^n)$ *then*

$$\|R_q u_h^P\|_{\ell_h^P}^P \le q^n \|u_h\|_{\ell_h^P}^P \qquad \textit{for all } u_h^p;$$

i.e.

$$\|R_q\| \le q^{n/P}. \qquad \Box$$

<u>THEOREM</u>.

$$\widehat{(R_q u_h)(z)} = \sum_{p\in[0,q)} \hat{u}_h(z + 2\pi p/H) \qquad \textit{for all } z \in [2\pi/H],$$

$$H = qh,\ q > 0,\ \ q \in \mathbb{Z}^n.$$

<u>PROOF</u>. We denote $u_H = R_q u_h$, the gridfunction defined on $\mathbb{Z}_H^n$, $H = qh$).

$$\hat{u}_H(\omega) = (2\pi)^{-n/2} H^n \sum_{j\in\mathbb{Z}^n} e^{-ijH\omega} u_H(jH)$$

$$= (2\pi)^{-n/2} H^n \sum_{j\in\mathbb{Z}^n} e^{-ijH\omega} (2\pi)^{-h/2} \int_{z\in[2\pi/h]^n} e^{ijqhz} \hat{u}_n(z)\,dz$$

$$= (\tfrac{H}{2\pi})^n q^n \sum_{j\in\mathbb{Z}^n} \int_{z\in[2\pi/h]^n} e^{ijH(z-\omega)} \hat{u}_h(z)\,dz$$

$$= (\tfrac{H}{2\pi})^n \sum_{j\in\mathbb{Z}^n} \sum_{p\in[0,q)} \int_{z\in 2\pi p/H + [2\pi/H]^n} e^{ijH(z-\omega)} \hat{u}_h(z)\,dz$$

$$= (\tfrac{H}{2\pi})^n \sum_{j\in\mathbb{Z}^n} \sum_{p\in[0,q)} \int_{z\in[2\pi/H]^n} e^{ijH(z-\omega)} \hat{u}_h(z+2\pi P/H)\,dz$$

$$= (\tfrac{H}{2\pi})^n \sum_{j\in\mathbb{Z}^n} \int_{z\in[2\pi/H]^n} e^{ijH(z-\omega)} \sum_{p\in[0,q)} \hat{u}_h(z+2p\pi/H)\,dz$$

Hence, by Fourier's integral identity,

$$\hat{u}_H(z) = \sum_{p\in[0,q)} \hat{u}_h(z+2\pi p/H). \qquad \Box$$

REMARK. The Fourier transforms of gridfunctions and coarse grid restrictions behave similarly to those of continuous functions and their restriction to a grid (cf. section 4.)

DEFINITION. A *(weighted) q-restriction* of a gridfunction u_h defined on $\mathbb{Z}_h^n$ to a gridfunction u_H, $H = qh$, defined on $\mathbb{Z}_H^n$, denoted by $R_q(a_h)u_h$, is defined by

$$(R_q(a_h)u_h)(jH) = u_H(jH) = \sum_{k \in \mathbb{Z}^n} a_h(kh)\; u_h((qj-k)h).$$

It is clear that any weighted q-restriction can be written as

$$u_H = R_q(a_h * u_h) = R_q A_h u_h$$

and therefore the relation between $\hat{u}_H$ and $\hat{u}_h$ is given by

$$\hat{u}_H(z) = \left(\frac{\sqrt{2\pi}}{h}\right)^n \sum_{p \in [0,q)} \hat{a}(z+2\pi p/H)\; \hat{u}_h(z+2\pi p/H).$$

REMARK. As was the case with the canonical q-restriction, the range of definition (i.e. the range of periodicity) of the Fourier transform of a gridfunction is decreased by a factor q.

If a significant part of the frequencies available is a fine mesh gridfunction u_h lie in $[-\pi/h,-\pi/H]$ or $[\pi/H,\pi/h]$ (i.e. are high frequencies), then by the canonical restriction also the representation of the low frequencies is disturbed.

To get a better representation of the low frequencies on the coarse grid, it seems wise to apply a low-pass high-cut filter in the form of an operator A_h; i.e. a good choice of a_h in $R_q(a_h)$ may cause a closer representation of the low frequencies on the coarse grid.

EXAMPLE. [*Transposed linear interpolation*]
In one dimension (n=1) we consider the following restriction on a twice coarser grid (q=2).

$$\begin{aligned} u_H(jH) &= R_2(a_h)u_h(jH) = \\ &= \tfrac{1}{4}u_h((2j-1)h) + \tfrac{1}{2}u_h(2jh) + \tfrac{1}{4}u_h((2j+1)h). \end{aligned}$$

(By reasons to be explained later this restriction is called transposed linear interpolation.) Thus we have

$$u_H = R_2(a_h * u_h)$$

with

$$a_h(jh) = \begin{cases} \frac{1}{2} & j = 0, \\ \frac{1}{4} & |j| = 1, \\ 0 & |j| > 1. \end{cases}$$

Hence

$$\hat{a}_h(y) = \frac{h}{\sqrt{2\pi}} \tfrac{1}{2}(1+\cos(hy))$$

and

$$\hat{u}_H(y) = \tfrac{1}{2}(1+\cos(hy))\hat{u}_h(y) + \tfrac{1}{2}(1-\cos(hy))\ \hat{u}_h(y+2\pi/H).$$

This clearly gives a better representation of the lower frequencies than the canonical q-restriction where we find

$$\hat{u}_H(y) = \hat{u}_h(y) + \hat{u}_h(y+2\pi/H).$$

9. A GRIDFUNCTION AND ITS (FLAT) q-PROLONGATION

DEFINITION. Let $u_H \in \ell_H(\mathbb{Z}^n_H)$ be a gridfunction defined on $\mathbb{Z}^n_H$, then its *a_h-q-prolongation* $P_q(a_h)u_H$, ($a_h \in \ell_h(\mathbb{Z}^n_h)$, $q \in \mathbb{Z}^n$) is the gridfunction u_h defined on $\mathbb{Z}^n_h = \mathbb{Z}^n_{H/q}$, defined by

$$(P_q(a_h)u_H(mqh+ph) = u_h((mq+p)h) = \sum_{j\in\mathbb{Z}^n} a_h((mq-jq+p)h).u_H(jH)$$

$$\forall m \in \mathbb{Z}^n, \quad p \in [0,q).$$

REMARK. As a special case we introduce the *flat q-prolongation operator* P^0_q, which is defined by $u_h = P^0_q u_H = P_q(a_h)u_H$ with

$$\begin{cases} u_h(mgh) = u_H(mH), & \\ u_h(mgh+ph) = 0 & \text{if } p \neq 0,\ p \in [0,q). \end{cases}$$

This P_q^0 can be written as $P_q(a_h)$ with

$$a_h(jh) = \begin{cases} 1 \text{ if } j = 0 \\ 0 \text{ if } j \neq 0 \end{cases} .$$

<u>REMARK</u>. From the definition of P_q^0 it is immediately clear that $R_q^0 = P_q^0 R_q$.

<u>LEMMA</u>. *If* $u_H \in \ell_H^p(\mathbb{Z}_H^n)$ then $P_q^0\, u_H \in \ell_h^p(\mathbb{Z}_h^n)$ *with* $h = H/q$, *and*

$$\|P_q^0\| \leq q^{-n/p}.$$

<u>PROOF</u>.

$$\begin{aligned} \|P_q^0\, u_H\|^p &= h^n \sum_{j\in\mathbb{Z}^n} |\ (P_q^0\, u_H)(jh)\ |^p \\ &\leq h^n \sum_{m\in\mathbb{Z}^n} |\ (P_q^0\, u_H)(mqh)\ |^p \\ &= h^n \sum_{m\in\mathbb{Z}^n} |\ u_H(mH)\ |^p \\ &= q^{-n}\, \|u_H\|^p < \infty\ . \end{aligned}$$ □

<u>COROLLARY</u>. *With the corresponding lemma in section 8 we find*

$$\|P_q^0 R_q\| \leq q^{+n/p} q^{-n/p} = 1,$$

$$\|R_q P_q^0\| = \|I_H\| = 1.$$ □

Between the operators P_q^0, $P_q(ah)$, T_{ph} and the q-convolution we easily verify the following relations

(9.1) $$R_q T_{-ph} P_q(ah) u_H = R_q T_{-ph}(a_h \overset{q}{*} T_{ph} P_q^0 u_H),$$

(9.2) $$R_q T_{-ph}(a_h \overset{q}{*} u_h) = R_q T_{-ph} A_h^q u_h = (R_q T_{-ph} a_h) * (R_q T_{-ph} u_h),$$

(9.3) $a_h \overset{q}{*} u_h = \sum_{p\in[0,q)} T_{ph} P_q^0((R_q T_{-ph} a_h) * (R_q T_{-ph} u_h))$,

(9.4) $R_q T_{-sh} T_{ph} P_q^0 = \delta_{p-s} I, \qquad p,s \in [0,q)$,

(9.5) $P_q(ah) u_H = \sum_p T_{ph} P_q^0((R_q T_{-ph} a_h) * u_H)$.

10. THE RELATION BETWEEN THE FOURIER TRANSFORMS OF A GRIDFUNCTION AND ITS (FLAT) q-PROLONGATION

We first consider the flat q-prolongation. Let u_H be a gridfunction defined on $\mathbb{Z}_H^n$. Its FT is denoted by $\hat{u}_H$. Let $h = H/q$ and let $u_h = P_q^0 u_H$, its FT is denoted by $\hat{u}_h$; $\hat{u}_H$ is defined on $[2\pi/H]^n$ and $\hat{u}_h$ is defined on $[2\pi/h]^n$.

The relation between $\hat{u}_H$ and $\hat{u}_h$ is given in the following

<u>THEOREM</u>. The FT of $P_q^0 u_H$ is a scalar multiple of the periodic continuation of the FT of u_H to $[2\pi q/H]$: i.e. $\widehat{P_q^0 u_H}(\omega) = \hat{u}_h(\omega) = q^{-n}\, \hat{u}_H(\omega)$ on $[2\pi q/H]$.

<u>PROOF</u>. $\hat{u}_h(\omega) =$

$$= \left(\frac{h}{\sqrt{2\pi}}\right)^n \sum_{j\in\mathbb{Z}^n} \sum_{p\in[0,q)} e^{-i(jq+p)h\omega}\, u_h((jq+p)h)$$

$$= \left(\frac{h}{\sqrt{2\pi}}\right)^n \sum_{j\in\mathbb{Z}^n} \sum_{p=0} e^{-i(jq+p)h\omega}\, u_H(jH)$$

$$= q^{-n} \left(\frac{H}{\sqrt{2\pi}}\right)^n \sum_{j\in\mathbb{Z}^n} e^{-ijqh\omega}\, u_H(jH).$$

Notice that $\hat{u}_h(\omega)$ appears to be a periodic function with period $2\pi/(qh)$ defined on $[2\pi/h]^n$!

$$\hat{u}_h(\omega) = q^{-n}\left(\frac{H}{\sqrt{2\pi}}\right)^n \sum_{j\in\mathbb{Z}^n} e^{-ijH\omega}\, u_H(jH) = q^{-n}\, \hat{u}_H(\omega) \qquad \square$$

For a general a_h-q-prolongation we find the following relation between Fourier transforms.

THEOREM.

$$\widehat{P_q(a_h)u_H} = q^{-n}\left(\frac{\sqrt{2\pi}}{H}\right)^n \left(\sum_{p\in[0,q)} e^{-iph}\cdot \widehat{R_qT_{-ph}a_h}\right)\hat{u}_H.$$

PROOF. We use relation (9.5):

$$\hat{u}_h(\omega) = \widehat{P_q(a_h)u_H}(\omega) = \sum_{p\in[0,q)} \widehat{T_{ph}P_q^0((R_qT_{-ph}a_h) * u_H)}(\omega)$$

$$= \sum_p e^{-iph\omega}\, q^{-n}\, \widehat{(R_qT_{-ph}a_h) * u_H}$$

$$= \sum_p e^{-iph\omega}\, q^{-n} \left(\frac{\sqrt{2\pi}}{H}\right)^n \widehat{R_qT_{-ph}a_h}\,\hat{u}_H$$

$$= q^{-n}\left(\frac{\sqrt{2\pi}}{H}\right)^n \hat{u}_H(\omega) \sum_p e^{-iph\omega}\, \widehat{R_qT_{-ph}a_h}(\omega) \qquad \square$$

EXAMPLE. In 1 dimension (n=1) we consider the mean-interpolation M_{hH}, which is defined by

$$\begin{cases} M_{hH}u_H(2jh) = u_h(2jh) = 0\ , \\ M_{hH}u_H(2jh+h) = u_h(2jh+h) = \tfrac{1}{2}(u_H(jH)+u_H((j+1)H)). \end{cases}$$

For the FT $\hat{u}_h$ we easily derive

$$\widehat{M_{hH}u_H}(\omega) = \hat{u}_h(\omega) = \tfrac{1}{2}\cos(h\omega)\hat{u}_H(\omega).$$

EXAMPLE. [*Linear interpolation*]
We define the 1-dimensional linear interpolation by

$$u_h = M_{hH}u_H + P^0_{hH}u_H$$

i.e.

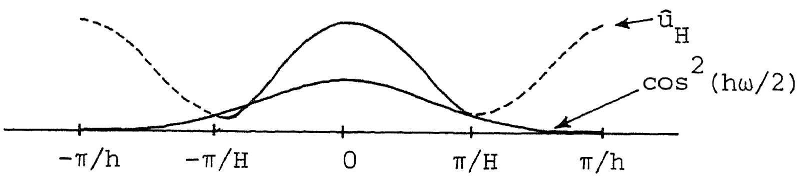
$$\begin{cases} u_h(2jh) = u_H(jH) \\ u_h(2jh+h) = \tfrac{1}{2}(u_H(jH) + u_H(jH+H)) \end{cases}$$

then

$$\hat{u}_h = \widehat{M_{hH}u_H + \overset{0}{P}_{hH}u_H} = \widehat{M_{hH}u_H} + \widehat{\overset{0}{P}_{hH}u_H}$$

$$= \tfrac{1}{2}\cos(h\omega)\hat{u}_H + \tfrac{1}{2}\hat{u}_H$$

$$= \tfrac{1}{2}(1+\cos(h\omega))\hat{u}_H = \cos^2(h\omega/2)\hat{u}_H(\omega)$$



Notice that the $2\pi/H$ periodization of $\hat{u}_h$ returns the old function $\hat{u}_H$!

11. RESTRICTIONS AND PROLONGATIONS AND TRANSPOSED GRIDFUNCTIONS

The general form of the q-Restriction $R_q(a_h)$, formally given by $(H=qh)$

$$R_q(a_h)u_h = u_H = R_q(a_h * u_h) = R_q A_h u_h,$$

is explicitly given by $(j \in \mathbb{Z}^n)$

$$(R_q(a_h)u_h)(jH) = \sum_{m\in\mathbb{Z}^n} a_h(jqh-mh).u_h(mh).$$

Representing the linear mapping $R_q(a_h):\ell_h(\mathbb{Z}_h^n) \to \ell_H(\mathbb{Z}_H^n)$ by a matrix (R_{jm}), the elements of (R_{jm}) are given by

$$R_{jm} = a_h(jqh-mh).$$

The general form of the q-Prolongation $P_q(b_h)$ formally given by $(h=H/q)$

$$P_q(b_h)u_H = u_h = \sum_p T_{ph} P_q^0((R_q T_{-ph} b_h) * u_H),$$

is explicitly given by ($m \in \mathbb{Z}^h$, $p \in [0,q)$)

$$(P_q(b_h)u_H)(mqh+ph) = u_h(mqh+ph) =$$

$$= \sum_{j \in \mathbb{Z}^n} b_h(mqh-jqh+ph)uh(jH).$$

Representing the linear mapping $P_q(b_h): \ell_H(\mathbb{Z}_H^n) \to \ell_h(\mathbb{Z}_h^n)$ by a matrix (P_{mj}): $\ell_H(\mathbb{Z}_H^n) \to \ell_h(\mathbb{Z}_h^n)$, the elements $P_{mq+p,j}$, $m \in \mathbb{Z}^n$, $p \in [0,q)$ are given by

$$P_{mq+p,j} = b_h(mqh-jqh+ph),$$

i.e. the elements P_{ij} are given by

$$P_{ij} = b_h(ij-jqh).$$

<u>DEFINITION</u>. The *restriction* $R_q(a_h)$ and the *prolongation* $P_q(b_h)$ *are* called *each others transpose* if their corresponding matrices satisfy the relation

$$q^n(R_{ij})^T = (P_{ij}),$$

i.e.

$$q^n R_{ji} = P_{ij} \qquad \forall\, ij \in \mathbb{Z}^n.$$

From this we conclude that $R_q(a_h)$ and $P_q(b_h)$ are each others transpose iff

$$q^n a_h(mh) = b_k(-mh) \qquad \forall\, m \in \mathbb{Z}^h.$$

<u>DEFINITION</u>. The gridfunction b_h is called the *transpose of the gridfunction* a_h if $a_h(jh) = b_h(-jh)$ $\forall j \in \mathbb{Z}^n$.

DEFINITION. The transpose of the gridfunction a_h we denote by a_h^T. The *symmetric part of* a_h, denoted by a_h^S, is defined by

$$a_h^S = (a_h + a_h^T)/2 .$$

Similarly we define the *anti-symmetric part*

$$a_h^A = (a_h - a_h^T)/2. \qquad \square$$

Clearly $a_h^{ST} = a_h^S$, $a_h^{AT} = -a_h^A$, $a_h = a_h^S + a_h^A$.

COROLLARY. The *restriction* $R_q(a_h)$ and the *prolongation* $P_q(b_h)$ are called *each others transpose* if

$$b_h^T = q^h a_h .$$

ILLUSTRATION. Here we illustrate for n = 1, q = 2 the matrices corresponding to $R_h(a_h)$ and $P_h(a_h)$.

R_h(ah):

$$\begin{pmatrix} a_0 & a_{-1} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & a_1 & a_0 & a_{-1} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & a_1 & a_0 & a_{-1} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & a_1 & a_0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix} = \begin{pmatrix} U_0 \\ U_1 \\ U_2 \\ U_3 \end{pmatrix} ;$$

$P_h(ah)$:

$$\begin{pmatrix} a_0 & \cdot & \cdot & \cdot \\ a_1 & a_{-1} & \cdot & \cdot \\ \cdot & a_0 & \cdot & \cdot \\ \cdot & a_1 & a_{-1} & \cdot \\ \cdot & \cdot & a_0 & \cdot \\ \cdot & \cdot & a_1 & a_{-1} \\ \cdot & \cdot & \cdot & a_0 \end{pmatrix} \begin{pmatrix} U_0 \\ U_1 \\ U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix}.$$

As we mentioned before: for the canonical restriction $R_q = R_q(a_h)$ we have

$$\begin{cases} a_h(jh) = 1 & j = 0, \\ a_h(jh) = 0 & j \neq 0; \end{cases}$$

and for the linear interpolation $P_q(a_h)$ we have

$$\begin{aligned} a_h(jh) &= 1 & j &= 0 \\ a_h(jh) &= \tfrac{1}{2} & |j| &= 1, \\ a_h(jh) &= 0 & |j| &> 1. \end{aligned}$$

We already met its transpose in Section 8.

12. COMBINATION OF A PROLONGATION AND A SUBSEQUENT RESTRICTION: $R_q(a_h)\ P_q(b_h)$.

The following theorem is easily verified.

THEOREM.

$$R_q(a_h)P_q(b_h)u_H = (R_q(a_h*b_h))*u_H.$$

COROLLARY. *As a direct consequence of the above theorem we find for the* FT

$$\widehat{R_q(a_h)P_q(b_h)u_H}(\omega) =$$

$$= (\frac{\sqrt{2\pi}}{H})^n \, \widehat{(R_q(a_h*b_h)}(\omega)\hat{u}_H(\omega)$$

$$= (\frac{\sqrt{2\pi}}{H})^n \, \hat{u}_H(\omega) \sum_{p\in[0,q)} \widehat{a_h*b_h}(\omega+2\pi p/H)$$

$$= (\frac{2\pi}{hH})^n \, \hat{u}_H(\omega) \sum_{p\in[0,q)} \hat{a}_h.\hat{b}_h(\omega+2\pi p/H).$$

EXAMPLE 1. We consider in the one-dimensional case the flat restriction $R(a_h)$ and the linear interpolation $P(b_h)$ $(q=2)$. Then

$$\hat{a}_h(\omega) = \frac{h}{\sqrt{2\pi}} \quad \text{and} \quad \hat{b}_h(\omega) = \frac{h}{\sqrt{2\pi}} \quad (1+\cos(h\omega)).$$

We find

$$\widehat{RPu_H}(\omega) = \frac{2\pi}{hH} \hat{u}_H(\omega) \sum_{p=0,1} \hat{a}_h\hat{b}_h(\omega+2\pi p/H)$$

$$= \hat{u}_H(\omega) \frac{1}{2} \sum_{p=0,1} (1+\cos(h\omega+p\pi))$$

$$= \hat{u}_H(\omega).$$

Which is correct, because RP is the identity on $\ell_H(\mathbb{Z}_H)$!

EXAMPLE 2. We now take the transpose of the linear interpolation as the restriction operator: $R(\frac{1}{2}b_h)$. (Notice that b_h is a symmetric gridfunction!) We find

$$\widehat{RPu_H}(\omega) = \frac{2\pi}{hH} \hat{u}_H(\omega) \sum_{p=0,1} \frac{1}{2} \hat{b}_h^{\,2}(\omega+2\pi p/H)$$

$$= \hat{u}_H(\omega)\ \frac{1}{2} \sum_{p=0,1} \frac{1}{2}\{(1+\cos(h\omega+\pi p))\}^2$$

$$= \hat{u}_H(\omega) \sum_{p=0,1} \{\cos^2(\frac{h\omega+\pi p}{2})\}^2$$

$$= \hat{u}_H(\omega)\ \{\cos^4(h\omega/2) + \sin^4(h\omega/2)\}$$

$$= \hat{u}_H(\omega)\ \{\frac{3}{4} + \frac{1}{4}\cos(\omega H)\}, \qquad \omega \in [-\pi/H, \pi/H].$$

We see that here the operator RP damps the higher frequencies to some extent.

The function $\{\frac{3}{4} + \frac{1}{4}\cos(\omega H)\}$ is called the *transfer-function* of the operator RP.

13. COMBINATION OF A RESTRICTION AND A SUBSEQUENT PROLONGATION: $R_q(b_h)R_q(a_h)$.

Using the equalities (9.1) - (9.5), we easily derive

<u>THEOREM</u>.

$$P_q(b_h)R_q(a_h)u_h = \sum_{p\in[0,q)} T_{ph}R_q^0 T_{-ph}(b_h \overset{q}{*} T_{ph}(a_h * u_h)). \qquad \square$$

This theorem implies for all $m \in \mathbb{Z}^n$ and $p \in [0,q)$

$$[P_q(b_h)R_q(a_h)u_h](mqh+ph) =$$

$$[b_h \overset{q}{*} T_{ph}(a_h * u_h)](mqh+ph) =$$

$$\sum_{j,k\in\mathbb{Z}^n} b_h(jqh+ph)\,a_h(mqh-jqh-kh)\,u_h(kh).$$

<u>COROLLARY</u>. *With the aid of the above theorem we immediately derive*

$$\widehat{P_q(b_h)R_q(a_h)u_h}(\omega) =$$

$$= \sum_{p\in[0,q)} \widehat{T_{ph}P_q^0((R_q T_{-ph} b_h) * (R_q(a_h * u_h)))}(\omega)$$

$$= \left(\frac{\sqrt{2\pi}}{H}\right)^n \sum_p e^{-i\omega ph} q^{-n} \widehat{R_q T_{-ph} b_h} \widehat{R_q(a_h * u_h)} (\omega)$$

$$= \left(\frac{2\pi}{hH}\right)^n b_h(\omega) \sum_{s\in[0,q)} \hat{a}_h \hat{u}_h(\omega - 2\pi s/H).$$

<u>EXAMPLE</u>. [*General three-term* P *and* R, n = 1, q = 2]

We consider $R_2(a_h)$ and $P_2(b_h)$ with n = 1 and a_h and b_h defined by

$$a_h(jh) = \begin{cases} 1 & j = 0 \\ a & |j| = 1 \\ 0 & |j| > 1 \end{cases} \quad ; \quad b_h(jh) = \begin{cases} 1 & j = 0 \\ b & |j| = 1 \\ 0 & |j| > 1 \end{cases} .$$

To analyze the properties of $R_2(a_h)P_2(b_h)$ we compute $c_H = R_q(a_h * b_h)$:

$$c_H(jH) = \begin{cases} 1 + 2ab & j = 0 \\ ab & |j| = 1 \\ 0 & |j| > 1 \end{cases} .$$

Hence the spectrum of $R_2(a_h)P_2(b_h)$ is

$$\lambda(\omega) = \frac{\sqrt{2\pi}}{H} R_q(a_h * b_h) = 1 + 2ab(1+\cos(\omega H)).$$

The matrix corresponding with RP has the form

$$\begin{pmatrix} \cdot & \cdot & & & & & \\ \cdot & \cdot & & & & & \\ \cdot & 1+2ab & ab & & & & \\ & ab & 1+2ab & ab & & & \\ & & ab & 1+2ab & ab & & \\ & & & ab & \cdot & \cdot & \\ & & & & \cdot & \cdot & \end{pmatrix} .$$

In order to find the matrix corresponding with PR we have to compute

$$b_h \overset{2}{*} (T_{ph} A_h) \qquad p = 0,1.$$

This yields

$$\begin{pmatrix} \ddots & & & & & & & \\ & & b & ab & & & & \\ & a & 1 & a & 0 & & & \\ & ab & b & 2ab & b & ab & & \\ & & 0 & a & 1 & a & 0 & \\ & & & ab & b & 2ab & b & ab \\ & & & & 0 & a & 1 & a \\ & & & & & ab & b & \ddots \end{pmatrix} \begin{matrix} \vdots \\ \\ \\ (p=1) \\ (p=0) \\ (p=1) \\ (p=0) \\ \vdots \end{matrix}$$

EXAMPLE. [*The linear-flat filter*]
Now we want to compute the transfer function of I-PR or PR in the (one-dimensional) case where P is linear interpolation and R is the canonical 2-restriction (cf. example 12.1 for the transfer-function of RP). Application of the corollary yields

$$\widehat{PRu_h}(\omega) = \frac{2\pi}{hH} \hat{b}_h(\omega) \sum_{s=0,1} \hat{a}_h \hat{u}_h(\omega-\pi s/h)$$

$$= \frac{1}{2} (1+\cos(\omega h))[\hat{u}_h(\omega)+\hat{u}_h(\omega+\pi/h)] .$$

This clearly is a low-pass filter which suffers from perturbations in the low frequencies due to the high frequencies originally available. The corresponding high-pass filter has

$$\widehat{(I-PR)u_h}(\omega) = \frac{1}{2}(1-\cos(\omega h))\hat{u}_h(\omega) - \frac{1}{2}(1+\cos(\omega h))u_h(\omega+{}^{\pi}/h) .$$

NOTE. By a low(high)-pass *filter* we denote an operator $\ell_h(\mathbb{Z}_h^n) \to \ell_h(\mathbb{Z}_h^n)$ for which in the image-function and in the original function approximately the same amount of low (high) frequencies is present.

EXAMPLE. [*The linear-transposed filter*]
We compute the transfer-function of I-PR and PR in the (one-dimensional) case where P is again linear interpolation and R is its transpose. (cf. example 12.2 for the transfer function of RP).

Application of the corollary yields

$$\widehat{PRu_h}(\omega) = \frac{1}{2}(1+\cos(\omega h))\,\frac{1}{2}\,[(1+\cos(\omega h))\hat{u}_h(\omega) + + (1+\cos(\omega h+\pi))\hat{u}_h(\omega+\pi/h)]$$

$$= \frac{1}{4}(1+\cos(\omega h)\,[(1+\cos(\omega h))\hat{u}_h(\omega) + + (1-\cos(\omega h))\hat{u}_h(\omega+\pi/h)]$$

$$= \frac{1}{4}(1+\cos(\omega h))^2\, u_h(\omega) + \frac{1}{4}\sin^2(\omega h) u_h(\omega+\pi/h).$$

Clearly this is a low-pass filter again, but we see that in this case the low frequency perturbation caused by the high frequencies is considerably less, than in the preceding example.

Here, for the corresponding high-pass filter, we can write

$$\widehat{(I-PR)u_h}(\omega) = \frac{1}{2}(1-\cos(\omega h))\hat{u}_h(\omega) + \frac{1}{4}\sin^2(\omega h)[\hat{u}_h(\omega)-\hat{u}_h(\omega+\pi/h)].$$

14. MORE ON NORMS AND HILBERT-SPACES OF GRIDFUNCTIONS

As we saw in section 2, $\ell_h^2(\mathbb{Z}_h^n)$ is a Hilbert space with the inner product

$$(u_h,v_h)_h = h^n \sum_{j\in\mathbb{Z}^n} u_h(jh)\,\bar{v}_h(jh).$$

From remark 3.3 it is easily seen that

$$(u_h,v_h)_h = (\hat{u}_h,\hat{v}_h)_{L^2[2\pi/h]^n}\,,$$

and we have also

$$y^T A x = (Ax,y)_h = h^n \sum_{j\in \mathbb{Z}_h^n} (Ax)(jh)\bar{y}(jh) =$$

$$= (A(\omega)\hat{x}(\omega),\hat{y}(\omega))_{L^2[2\pi/h]^n}$$

$$= (\frac{\sqrt{2\pi}}{h})^n \; (\hat{a}(\omega)\hat{x}(\omega),\hat{y}(\omega))_{L^2[2\pi/h]^n}.$$

<u>The transpose of a mapping</u>:

Let P_{hH} be a mapping $P_{hH}: \ell_H(\mathbb{Z}_H^n) \to \ell_h(\mathbb{Z}_h^n)$ then we can define the transpose of P_{hH} by

$$P_{hH}^T = R_{Hh}, \quad R_{Hh}:\ell_h(\mathbb{Z}_h^n) \to \ell_H(\mathbb{Z}_H^n)$$

such that

$$(P_{hH}v_H,\bar{u}_h)_h = (v_H,R_{Hh}\bar{u}_h)_H$$

for all $v_H \in \ell_H(\mathbb{Z}_H^n)$ and all $u_h \in \ell_h(\mathbb{Z}_h^n)$. Taking for v_H and u_h the i-th and j-th unit-vector respectively, we immediately see that

$$P_{ji} = (H/h)^n \; R_{ij} \qquad \forall_{i,j} \in \mathbb{Z}^n.$$

Taking $H = qh$ we see that the present definition of P_{hH} and R_{Hh} being each others transpose is equivalent with the definition given in section 11.

<u>Difference operators</u>.

In remark 5.4 we have already defined the *translation operator* $E_{ph}:\ell_h(\mathbb{Z}_h^n) \to \ell_h(\mathbb{Z}_h^n)$ such that

$$E_{ph}u_h(jh) = u_h(jh+ph).$$

We now define the *forward difference operator* $\Delta_{ph}:\ell_h(\mathbb{Z}_h^n) \to \ell_h(\mathbb{Z}_h^n)$ by

$$\Delta_{ph} = (E_{ph}-I)/h,$$

i.e.

$$\Delta_{ph}u_h(jh) = (u_h(jh+ph) - u_h(jh))/h .$$

In particular, for $1 \le k \le n$ we define $\Delta_k = \Delta_{e_k h}$, e_k the k-th unitvector; thus

$$\Delta_k\, u_h(jh) = (u_h(jh+h_k) - u_h(jh)/h_k.$$

REMARK. Clearly the spectrum of Δ_k is

$$\lambda(\omega) = (e^{ih_k\omega}-1)e_k/h_k$$

and

$$|\lambda(\omega)| = \frac{2}{h_k}\sin(h_k\omega/2).$$

Similarly we can define the *backward difference operator* $\nabla_{ph} = (I-E_{-ph})/h = (I-T_{ph})/h$ for which the spectrum is

$$(1-e^{-iph\omega})/h.$$

We notice that $\|\Delta_{ph}\| = \|\nabla_{ph}\|$.

Hilbert-space norms.

On the space $\ell_h(\mathbb{Z}_h^n)$ we have introduced a norm $\|\cdot\|_2$, which - in the context of Hilbert spaces - we shall denote by $\|\cdot\|_0$. This norm was defined by

$$\|u_h\|_0^2 = h^n \sum_{j\in\mathbb{Z}^n} |u_h(jh)|^2.$$

Now we introduce also norms $\|\cdot\|_k$, that are defined by

$$\|u_k\|_k^2 = \sum_{|\alpha|\le k} \|\Delta^\alpha u_h\|_0^2.$$

Here α is a multi-integer $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_i \geq 0$ and $|\alpha| = \Sigma\alpha_i$. With Δ^α we denote

$$\Delta^\alpha = \Delta_1^{\alpha_1} \Delta_2^{\alpha_2} \ldots \Delta_n^{\alpha_n} ,$$

where Δ_j is the forward divided difference operator in the j-th direction:

$$\Delta_j u_h(mh) = (u_h((m+e_j)h) - u_n(mh))/h;$$

here e_j is the j-th unit-vector.

By the Parseval identity we know

$$\|u_h\|_k^2 = \sum_{|\alpha|\leq k} \|\widehat{\Delta^\alpha u_n}\|^2_{L^2(-\pi/h,\pi/h)} .$$

Moreover, we know

$$\widehat{\Delta_j u_h}(\omega) = (\widehat{E_{e_j h} u_h}(\omega) - \widehat{u_h}(\omega))e_j/h_j$$

$$= (e^{ie_j h\omega} - 1)\hat{u}_h(\omega)e_j/h_j$$

$$= \hat{u}_h(\omega)e^{ie_j h_j \omega_j/2} \frac{2i}{h_j} \sin(\frac{h_j\omega_j}{2})e_j .$$

Hence

$$|\Delta_j u_h(\omega)| = |\hat{u}_h(\omega)| \; |\frac{2}{h_j} \sin(\frac{h_j\omega_j}{2})|$$

and therefore

$$|\Delta^\alpha u_h(\omega)| = |\hat{u}_h(\omega)| \prod_{j=1}^{n} |\frac{2}{h_j} \sin(\frac{h_j\omega_j}{2})|^{\alpha_j} .$$

Thus we find

$$\|u_h\|_k^2 = \sum_{|\alpha|\leq k} \|\hat{u}_h \prod_{j=1}^{n} |\frac{2}{h_j} \sin(\frac{h_j\omega_j}{2})|^{\alpha_j}\|^2 .$$

By the usual techniques it can be shown that this is equivalent with

$$\|u_h\|_k^2 \approx \|\{1+\sum_{j=1}^{n}\left[\frac{\sin(h_j\omega_j/2)}{h_j/2}\right]^2\}^{k/2}\hat{u}_h\|$$

This gives us the possibility to introduce discrete Sobolev norms of real order for gridfunctions by

$$\|u_h\|_s = \|\{1+\sum_{j=1}^{n}(\frac{\sin(h_j\omega_j/2)}{h_j/2})^2\}^{s/2}\hat{u}_h\|_{L^2(-\pi/h,\pi/h)}$$

for any real $s \geq 0$.

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