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APPLICATION OF THE OSHER-ENGQUIST DIFFERENCE SCHEME AND THE FULL MULTI-GRID METHOD TO A TWO DIMENSIONAL NONLINEAR ELLIPTIC MODEL EQUATION

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Application of the Osher-Engquist difference scheme and the full multi-grid method to a two dimensional nonlinear elliptic model equation*)

by

E.J. van Asselt

ABSTRACT

We consider a nonlinear boundary value problem in two dimensions. For the discretization of this boundary value problem the Osher-Engquist difference scheme is used and the discrete equations are solved by a full multi-grid method (FMGM). In the FMGM a coarse to fine sequence of grids with uniform meshes is applied.

The result obtained on a coarser grid serves as initial approximation to the solution on the finer grid.

On each grid the discrete equations are solved by Newton iteration. The Newton equations are approximately solved by the iterative use of a linear multi-level algorithm.

Numerical results are given and comparisons with the method of time steps show that the multi-grid approach is far more efficient than the method with explicit time steps.

KEY WORDS & PHRASES: *Osher-Engquist difference scheme, multi-grid method, incomplete LU-relaxation*

*) This report will be submitted for publication elsewhere.

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1. THE DISCRETIZATION

Let Ω be the unit square in $\mathbb{R}^2$: $\Omega = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1\}$.

Consider the boundary value problem:

$$\varepsilon_1 \frac{\partial^2}{\partial x^2} u + \varepsilon_2 \frac{\partial^2}{\partial y^2} u - a_1(u) \frac{\partial u}{\partial x} - a_2(u) \frac{\partial u}{\partial y} - g(u) = 0 \tag{1.1}$$

$$u(x,y) = h(x,y) \text{ on } \delta\Omega.$$

For the discretization of this problem we use the one sided Osher-Engquist difference scheme (cf. OSHER [5]).

Let G_k be a uniform $N_1 \times N_2$ grid ($(N_1-1)(N_2-1)$ inner points) with mesh size $h_k = (h_{k_1}, h_{k_2})$.

On G_k the discrete equation reads:

$$\begin{aligned} &\frac{\varepsilon_1}{h_1^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + \frac{\varepsilon_2}{h_2^2}(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}) \\ &- \frac{1}{h_1}[{}_x\Delta_+ f_{1-}(u_{i,j}) + {}_x\Delta_- f_{1+}(u_{i,j})] \\ &- \frac{1}{h_2}[{}_y\Delta_+ f_{2-}(u_{i,j}) + {}_y\Delta_- f_{2+}(u_{i,j})] \\ &- g(u_{i,j}) \\ &= 0 \qquad ; \quad i = 1,\ldots,N_1-1; \; j = 1,\ldots,N_2-1, \end{aligned} \tag{1.2}$$

with

$$f_{m+}(u) = \int^u a_m^+(s)ds, \qquad a_m^+(s) = \max(0, a_m(s)),$$

$$f_{m-}(u) = \int^u a_m^-(s)ds, \qquad a_m^-(s) = \min(0, a_m(s)), \quad m = 1,2,$$

$$x^{\Delta}_{+}\, u_{i,j} = u_{i+1,j} - u_{i,j},$$

$$x^{\Delta}_{-}\, u_{i,j} = u_{i,j} - u_{i-1,j},$$

$$y^{\Delta}_{+}\, u_{i,j} = u_{i,j+1} - u_{i,j},$$

$$y^{\Delta}_{-}\, u_{i,j} = u_{i,j} - u_{i,j-1}.$$

2. THE FULL MULTI-GRID METHOD

Equation (1.2) can be efficiently solved by the full multi-grid method (FMGM) in the following way.

In the FMGM a coarse to fine sequence of grids is used: $G_0, G_1 \ldots, G_N$. On each grid G_k we have the discrete nonlinear problem (1.2) which is solved by Newton iteration.

On the coarsest grid G_0, the Newton-equations are solved directly by Gaussian-elimination with partial pivoting. If the Newton process has sufficiently converged the approximating solution on G_0 is interpolated to G_1 by linear interpolation and corrected on the boundary. The result serves as initial approximation to the solution process on G_1.

On G_k, $k = 1,2,\ldots,N$, we apply also Newton iteration, but now the linear equations are approximately solved by the iterative use of the CS-multi-grid-algorithm with a fixed strategy (cf. BRANDT [1], HACKBUSCH [2]).

The smoothing step consists of Incomplete-LU relaxation (ILU) (cf. HEMKER [3]). In fact we applied a more up to date version of the ILU-relaxation which uses 7 diagonals in the decomposed matrix (HEMKER, to be published).

The restriction $\bar{R}$ in the multi-grid algorithm is the adjoint of the linear interpolation matrix, and linear interpolation is used as the prolongation.

On the lowest level the correction equations are always solved directly by Gaussian-elimination.

In coarse-grid corrections each coarse grid operator is the last Jacobian of the Newton process before a next finer grid was introduced in

the full multi-grid method.

3. NUMERICAL RESULTS OF THE APPLICATION OF FMGM FOR $\varepsilon/h = 0(1)$

Consider the following problem:

(3.1) $\varepsilon\Delta u - (\frac{u^2}{2})_x - (\frac{u^2}{2})_y - u = 0$, on the unit square

$\{(x,y) \mid 0\le x\le 1,\ 0\le y\le 1\}$ with boundary conditions

$$u(0,y) = \begin{cases} -\frac{1}{2}, & 0\le y<\frac{1}{2} \\ \frac{1}{2}, & \frac{1}{2}\le y\le 1 \end{cases}$$

$$u(1,y) = \begin{cases} \frac{1}{2}, & 0\le y<\frac{1}{2} \\ -\frac{1}{2}, & \frac{1}{2}\le y\le 1 \end{cases}$$

$$u(x,0) = \begin{cases} -\frac{1}{2}, & 0\le x<\frac{1}{2} \\ \frac{1}{2}, & \frac{1}{2}\le x\le 1 \end{cases}$$

$$u(x,1) = \begin{cases} \frac{1}{2}, & 0\le x<\frac{1}{2} \\ -\frac{1}{2}, & \frac{1}{2}\le x\le 1 \end{cases}$$

G_i $(i=0,1,\ldots,4)$ are uniform grids with mesh-sizes

$h_i = (2^{-i-1}, 2^{-i-1})$, $\varepsilon = 1.0$.

(With $\varepsilon = 10^{-6}$, $h = 0.1$, this problem was suggested to us by S. Osher).

On each grid the discretized equation reads:

$$
\begin{aligned}
(3.2)\qquad & \varepsilon(u_{i+1,j} + u_{i-1,j} - 4u_{i,j} + u_{i,j+1} + u_{i,j-1}) \\
& - h(f_-(u_{i+1,j}) - 2f_-(u_{i,j}) + 2f_+(u_{i,j}) \\
& \quad -f_+(u_{i-1,j}) + f_-(u_{i,j+1}) - f_+(u_{i,j-1})) \\
& - h^2 u_{ij} \qquad = \qquad 0,
\end{aligned}
$$

with

$$
(3.3)\qquad f_-(u) = \begin{cases} \frac{u^2}{2}, & u < 0 \\ 0, & u \geq 0 \end{cases},
$$

$$
f_+(u) = \begin{cases} \frac{u^2}{2}, & u > 0 \\ 0, & u \leq 0 \end{cases}.
$$

In each Newton iteration step s cycles of the multi-grid iteration were applied and on each grid the Newton-process is terminated when

$$
\max_{i,j} \left| u_{i,j}^{n+1} - u_{i,j}^{n} \right| < \tfrac{1}{2}.10^{-3}.
$$

For the initial approximation on G_0 we took the grid function u_0 which satisfies the boundary conditions, and is zero on the inner points of G_0.

In our numerical experiments with the multi-grid method we have used different values for the following parameters:

p: number of relaxation steps before the coarse grid correction.

σ: number of coarse-grid correction steps.

q: number of relaxation steps after the coarse grid correction.
(cf. HEMKER [4]).

Table (3.3) shows the number of Newton iterations on the grids $G_0,\ldots,G_4$

for different values of s,p,σ and q.

The last column shows the effect of parameter-continuation with decreasing values of the parameter ε (cf. POLAK, WACHTERS, BEELEN, HEMKER [6], WACHER [7]).

G_i	h	ε	s=1,p=1 σ=2,q=0	ε	s=2,p=1 σ=2,q=0	ε	s=2,p=1 σ=0,q=0	ε	s=2,p=1 σ=0,q=0
G_0	0.5	1.0	2	1.0	2	1.0	2	10.0	2
G_1	0.25	1.0	3	1.0	3	1.0	3	8.0	3
G_2	0.125	1.0	4	1.0	3	1.0	3	6.0	4
G_3	0.0625	1.0	4	1.0	3	1.0	5	3.0	4
G_4	0.03125	1.0	4	1.0	3	1.0	8	1.0	5

Table (3.3). *The number of Newton iterations to obtain an approximate solution with an accuracy of* $0.5.10^{-3}$.

We note that for a real multi-grid algorithm (σ>0) the number of iteration steps is independent of the meshwidth, whereas it increases with $h \to 0$ for $\sigma = 0$.

If we use Gauss-Seidel relaxation instead of ILU relaxation for the inner points we get for ε = 1.0, s = 2, p = 1, σ = 2, q = 0 the results in table (3.4).

Grids	G_0	G_1	G_2	G_3	G_4
Meshwidth h	0.5	0.25	0.125	0.0625	0.03125
Newtoniterations	2	≥ 10	≥ 10	≥ 10	≥ 10

Table (3.4) *The number of Newton iterations by Gauss-Seidel relaxation.*

4. NUMERICAL RESULTS OF THE APPLICATION OF FMGM FOR SMALL ε/h

For ε/h small we obtain the following results for problem 3.1. We take $\varepsilon = 10^{-6}$.

G_i	h	ε	s=1,p=1 σ=0,q=0	s=1,p=1 σ=2,q=0
G_0	0.5	10^{-6}	2	2
G_1	0.25	10^{-6}	5	5
G_2	0.125	10^{-6}	6	6
G_3	0.0625	10^{-6}	7	7
G_4	0.03125	10^{-6}	8	8

Table (4.1) *The number of Newton-iterations to obtain an approximate solution with an accuracy of* $0.5.10^{-3}$.

On another mesh-sequence we obtain similar results.

H_i	h	ε	s=1,p=1 σ=0,q=0	s=1,p=1 σ=2,q=0
H_0	0.2	10^{-6}	6	6
H_1	0.1	10^{-6}	6	6
H_2	0.05	10^{-6}	7	7
H_3	0.025	10^{-6}	8	8

Table (4.2) *The number of Newton-iterations to obtain an approximate solution with an accuracy of* $0.5.10^{-3}$.

We notice that, with this value of the parameter ε, the problem is of singular perturbation type.

In its solution boundary-layers and corner-layers appear. This makes the problem hard for numerical solution, because these layers cannot be represented on coarse grids.

This may explain the - probably inevitable - increase of the number of Newton iterations on the finer grids.

In order to give an impression of the solution of our singular perturbation problem we give the solution on H_1 in table (4.3).

0.500	0.500	0.500	0.500	0.500	-0.500	-0.500	-0.500	-0.500	-0.500	-0.500
0.500	0.406	0.311	0.127	-0.231	-0.412	-0.414	-0.419	-0.429	-0.452	-0.500
0.500	0.402	0.300	0.122	-0.226	-0.330	-0.338	-0.354	-0.382	-0.429	-0.500
0.500	0.392	0.279	0.126	-0.182	-0.259	-0.277	-0.308	-0.354	-0.419	-0.500
0.500	0.368	0.242	0.108	-0.133	-0.201	-0.232	-0.277	-0.338	-0.414	-0.500
0.500	0.307	0.173	0.063	-0.097	-0.157	-0.201	-0.259	-0.330	-0.412	-0.500
-0.500	-0.000	-0.001	-0.015	-0.059	-0.097	-0.133	-0.182	-0.226	-0.231	0.500
-0.500	-0.000	-0.000	-0.002	-0.015	0.063	0.108	0.126	0.122	0.127	0.500
-0.500	-0.000	-0.000	-0.000	-0.001	0.173	0.242	0.279	0.300	0.311	0.500
-0.500	-0.000	-0.000	-0.000	-0.000	0.307	0.368	0.392	0.402	0.406	0.500
-0.500	-0.500	-0.500	-0.500	-0.500	0.500	0.500	0.500	0.500	0.500	0.500

Table (4.3) *The solution of problem (3.1) on* H_1; $\varepsilon = 10^{-6}$.

5. THE METHOD OF TIME STEPS

This method is based on convergence to the steady state solution of

$$\varepsilon_1 \frac{\partial^2}{\partial x^2} u + \varepsilon_2 \frac{\partial^2}{\partial y^2} u - a_1(u) \frac{\partial u}{\partial x} - a_2(u) \frac{\partial u}{\partial y} - g(u) = \frac{\partial u}{\partial t} \tag{5.1}$$

with the time independent boundary condition:

$$u(x,y,t) = h(x,y) \text{ on } \partial\Omega$$

and initial condition

$$u(x,y,0) = u_0(x,y).$$

Discretization of (5.1) yields

$$(5.2)\qquad u_{i,j}^{n+1} = u_{i,j}^{n} + \Delta t^n\Big[\frac{\varepsilon_1}{h_1^2}(u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}) + \frac{\varepsilon_2}{h_2^2}(u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n})$$

$$- \frac{1}{h_1}\big[{}_x\Delta_+ f_{1-}(u_{i,j}^{n}) + {}_x\Delta_- f_{1+}(u_{i,j}^{n})\big]$$

$$- \frac{1}{h_2}\big[{}_y\Delta_+ f_{2-}(u_{i,j}^{n}) + {}_y\Delta_- f_{2+}(u_{i,j}^{n})\big]$$

$$-g(u_{i,j}^{n})\Big];$$

The time steps Δt^n are limited by the Courant-Friedrich-Levy (CFL) condition. Since the scheme satisfies a maximum principle it is sufficient to check this condition at $t = 0$ and take $\Delta t^n = \Delta t^0$ for all n. Application of this method to problem (3.1) on the grid H_1 (section 4) with $\varepsilon = 10^{-6}$ and as initial approximation v_1 which satisfies the boundary conditions and is zero on the inner points, gives for $\Delta t = 0.25$ the same results as in table (4.3) after 300 time steps (Osher, private communication).

6. COMPARISON OF THE EFFICIENCY OF FMGM AND THE METHOD OF TIME STEPS

6.1 The method of time steps

For problem (3.1) equation (5.2) becomes

$$(6.1.1)\qquad u_{i,j}^{n+1} = u_{i,j}^{n} - \Delta t u_{i,j}^{n} - \frac{\Delta t}{h}\big[f_-(u_{i+1,j}^{n}) - 2\{f_-(u_{i,j}^{n}) - f_+(u_{i,j}^{n})\}$$

$$-f_+(u_{i-1,j}^{n}) + f_-(u_{i,j+1}^{n}) - f_+(u_{i,j-1}^{n})\big]$$

$$+ \frac{\varepsilon}{h^2}\Delta t(u_{i+1,j}^{n} + u_{i-1,j}^{n} - 4u_{i,j}^{n} + u_{i,j+1}^{n} + u_{i,j-1}^{n}),$$

with f_- and f_+ as in (3.3).

The number of operations and function evaluations for T time steps on a grid with N inner points is:

additions	:	12TN,
multiplications	:	5TN,
f_+ evaluations	:	TN,
f_- evaluations	:	TN.

6.2 The full multi-grid method

6.2.1 The matrix of the linearized problem and the right-hand side

First we determine the amount of work to compute the matrix and right hand side of the linearized problem on a uniform grid with N inner points.

The matrix $A = (a_{ij})$ has a 5-diagonal form with

$$a_{i,j} = -4\varepsilon + 2h(a^-(u_{i,j}) - a^+(u_{i,j}))-h^2 \text{ (main diagonal elements)},$$

$$a_{i-1,j} = \varepsilon+h\, a^+(u_{i-1,j}),$$

$$a_{i+1,j} = \varepsilon-h\, a^-(u_{i+1,j}),$$

$$a_{i,j-1} = \varepsilon+h\, a^+(u_{i,j-1}),$$

$$a_{i,j+1} = \varepsilon-h\, a^-(u_{i,j+1}),$$

$$a^+(u) = \max(0,u),$$

$$a^-(u) = \min(0,u).$$

The right-hand side reads:

$$h\,a^{+}(u_{i,j-1})u_{i,j-1} + h\,a^{+}(u_{i-1,j})u_{i-1,j} + 2\,h[a^{-}(u_{i,j}) - a^{+}(u_{i,j})]u_{i,j}$$
$$- h\,a^{-}(u_{i+1,j})u_{i+1,j}$$
$$- h\,a^{-}(u_{i,j+1})u_{i,j+1}$$
$$+ h[f_{-}(u_{i+1,j}) - 2(f_{-}(u_{i,j}) - f_{+}(u_{i,j}))$$
$$- f_{+}(u_{i-1,j}) + f_{-}(u_{i,j+1}) - f_{+}(u_{i,j-1})].$$

Thus, for one Newton linearization we need:

additions	:	17N,
multiplications	:	12N,
a^{+} evaluations	:	N,
a^{-} evaluations	:	N,
f_{+} evaluations	:	N,
f_{-} evaluations	:	N.

6.2.2. ILU-relaxation

The decomposition.

In our version of the ILU-relaxation the matrix A is decomposed as follows:

$$A = LU + R.$$

The lower triangular matrix L has 4 nonzero diagonals, the upper triangular matrix U has 4 nonzero diagonals with main diagonal-elements 1, and the matrix R has 2 nonzero diagonals.
On a uniform grid with N inner points the number of operations of this decomposition including the determination of R (which need 2N multiplications) is:

additions	:	7N,
multiplications	:	9N,
divisions	:	3N.

The relaxation sweep.

The relaxation procedure for Au = b reads

6.2.2.1 $$u_{i+1} = (I - BA)u_i + Bb, \quad \text{with } B = (LU)^{-1}.$$

This defect correction process can be written as

6.2.2.2 $$LUu_{i+1} = -Ru_i + b.$$

Since L and U are triangular the solutions of these systems can be easily obtained.

The number of operations for one relaxation sweep on a uniform grid with N inner points is:

additions	:	8N,
multiplications	:	8N,
divisions	:	N.

6.2.3 Coarse grid corrections

Let the uniform fine grid have $N = O(h^2)$ inner points, and the uniform course grid have $M = O(h^2/4)$ inner points.
The right hand side for the coarse grid correction reads:

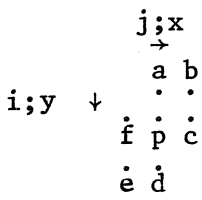
$$\bar{R}(b - Au_n),$$

where u_n is the last iterate of the defect correction process 6.2.2.1.

The 7-points restriction $\bar{R}$.

The 7-points restriction $\bar{R}$ is the adjoint of the linear interpolation matrix:

The value of the coarse grid function u_H in the coarse grid point p is determined by the values of the fine grid function u_h in 7 points (See figure 6.2.3.1).



$$u_H(p) = 0.5(u_h(a)+u_h(b)+u_h(f) + u_h(c) + u_h(e) + u_h(d)) + u_h(p).$$

Figure 6.2.3.1. *The 7-points restriction* $\bar{R}$.

The number of operations for $\bar{R}$ is:

additions : 6M,

multiplications : M.

The Residual.

The residual is determined by means of the matrix R:

$$b - Au_n = R(u_{n-1} - u_n).$$

The number of operations is:

additions : 2N,

multiplications: 2N.

6.2.4. Linear interpolation

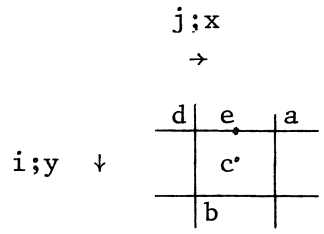
Each value of the find grid function u_h in a point that does not belong to the coarse grid, is determined by 2 neighbouring points, whereas the function value is copied at a point that does belong to the coarse grid (see figure 6.2.4.1).

The number of new points in the fine grid is 3M.

Thus the number of operations is:

additions : 3M,

multiplications: 3M.



$u_h(d) = u_H(d)$.

$u_h(c) = 0.5(u_H(a) + u_H(b))$.

$u_h(e) = 0.5(u_H(d) + u_H(a))$.

figure 6.2.4.1. *Linear interpolation.*

6.2.5. Direct solution by Gaussian-elimination with partial pivoting

Let the number of gridpoints by M. Then the number of operations is:

additions : $M^2 + 2M^{3/2}$,

multiplications: $M^2 + 2M^{3/2}$,

divisions : $M^{3/2}$.

Here we use the fact that the matrix A has a band structure.

6.3. The total number of operations for the method of time steps and FMGM

In table (6.3.1) we compare the total number of operations for both methods, applied on problem (3.1) with only two grids H_0 and H_1 and parameters s = 1, p = 1, σ = 1, q = 0, $\varepsilon = 10^{-6}$.

	Method of time steps on H_1	FMGM
additions	291600	23676
multiplications	121500	20868
divisions		2712
a^+ evaluations		582
a^- evaluations		582
f_+ evaluations	24300	582
f_- evaluations	24300	582

Table (6.3.1) *The number of operations for the method of time steps and FMGM.*

7. CONCLUSIONS

The total number of operations for FMGM is much less than the total number of operations for the method of time steps (Table 6.3.1).

ILU-relaxation is faster than Gauss-Seidel relaxation for these problems (Table 3.4).

For finer grids the advantage of the multi-grid method is even more pronounced than for the relatively coarse 10 × 10 grid for which the explicit operations count was made in section 6.3.

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