stichting
mathematisch
centrum

AFDELING NUMERIEKE WISKUNDE NW 108/81 JUNI
(DEPARTMENT OF NUMERICAL MATHEMATICS)

P.H.M. WOLKENFELT

# MODIFIED MULTILAG METHODS FOR VOLTERRA FUNCTIONAL EQUATIONS

Preprint

kruislaan 413 1098 SJ amsterdam

*Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.*

*The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).*

1980 Mathematics subject classification: 65R20

Modified multilag methods for Volterra functional equations*)

by

P.H.M. Wolkenfelt

ABSTRACT

Linear multistep methods for ordinary differential equations in conjunction with a family of computationally efficient quadrature rules are employed to define a class of so-called multilag methods for the solution of Volterra integral and integro-differential equations. In addition, modified multilag methods are proposed which have the property that the stability behaviour is independent of the choice of the quadrature rules. High-order convergence of the methods is established. In particular, a special class of high-order convergent methods is presented for the efficient solution of first kind Volterra equations. Numerical experiments are reported.

KEY WORDS & PHRASES: *Numerical analysis, Volterra integral and integro-differential equations, multilag methods, convergence and stability*

---

*) This report will be submitted for publication elsewhere.

## 1. INTRODUCTION

Consider the second kind Volterra integral equation

$$(1.1) \qquad f(x) = g(x) + \int_0^x K(x,y,f(y))dy, \quad 0 \le x \le X,$$

whose kernel K and forcing function g are assumed to be sufficiently smooth.

In order to discretize (1.1) at $x = x_n$ we need an approximation of the Volterra integral operator at $x = x_n$. A conventional approach is to consider a family of quadrature rules $\mathcal{W}$ with weights $w_{nj}$ which yields the direct quadrature methods

$$(1.2) \qquad f_n = g(x_n) + h \sum_{j=0}^{n} w_{nj} K(x_n, x_j, f_j).$$

Here, h denotes the stepsize, $x_j = jh$ are equidistant gridpoints and $f_j$ denotes a numerical approximation to $f(x_j)$. A wide variety of specific methods (1.2) is discussed e.g. in [2].

The stability behaviour of a numerical method for (1.1) is analyzed by applying that method with a fixed positive stepsize h to the test equation (cf. [3])

$$(1.3) \qquad f(x) = 1 + \lambda \int_0^x f(y)dy, \qquad \lambda \in \mathbb{C}.$$

Thus applying (1.2) to (1.3) yields the equations

$$(1.4) \qquad f_n = 1 + h\lambda \sum_{j=0}^{n} w_{nj} f_j.$$

It is well-known that the weights $w_{nj}$ frequently display a certain structure which makes it possible to reduce the discrete Volterra equation (1.4) to a finite term recurrence relation. Particular attention has been paid (cf. [16,20]) to the class of $(\rho,\sigma)$-reducible quadrature methods which have the property that the equations (1.4) reduce to the relations

$$(1.5) \qquad \sum_{i=0}^{k} a_i f_{n-i} = h\lambda \sum_{i=0}^{k} b_i f_{n-i}.$$

In (1.5) $a_i$ and $b_i$ represent the coefficients of a linear multistep (LM) method for ordinary differential equations (see e.g. [14]) which we shall denote by $(\rho,\sigma)$. Here, $\rho$ and $\sigma$ are polynomials defined as

$$(1.6) \qquad \rho(\zeta) := \sum_{i=0}^{k} a_i \zeta^{k-i}, \qquad \sigma(\zeta) := \sum_{i=0}^{k} b_i \zeta^{k-i}.$$

The main advantage of constructing methods for (1.1) which reduce to (1.5), lies in the fact that the stability behaviour, determined by the stability polynomial $\rho(\zeta) - h\lambda\sigma(\zeta)$, can be prescribed by choosing a suitable LM method. For example, the backward differentiation methods generate highly stable quadrature rules (cf. [20]). A disadvantage of $(\rho,\sigma)$-reducible quadrature methods however concerns their implementation. For instance in the case of the backward differentiation methods just mentioned, either the weights must be generated numerically (cf. [20]) in each integration step which results in a rather awkward implementation and extra overhead costs, or the methods must be implemented following the imbedding approach described in [18] (see also §2) at the cost of a rather large number of additional arithmetic operations.

In this paper, we propose two new classes of methods which are more efficient than the $(\rho,\sigma)$-reducible quadrature methods since they can be constructed and implemented in a simple and straightforward fashion. The methods, which we have called *multilag methods* and *modified multilag methods*, are composed of an LM method $(\rho,\sigma)$ and a family of efficient quadrature rules $\mathcal{W}$.

It turns out, however, that the stability behaviour of the multilag methods is not identical to that of the $(\rho,\sigma)$-reducible quadrature methods. In fact, stability is determined by $(\rho,\sigma)$ as well as by the quadrature rules $\mathcal{W}$*). Adopting the idea of "modification" proposed by VAN DER HOUWEN [11,12] in connection with mixed Runge-Kutta methods for (1.1), we change the multilag methods by adding suitable perturbation terms (residuals) to obtain the modified multilag methods the stability behaviour of which is determined only by $(\rho,\sigma)$ *irrespective of the choice of the quadrature rules* $\mathcal{W}$. As a result the modified multilag methods combine the advantages of the multilag methods

*) We intend to report on the stability behaviour of the multilag methods for various choices of $\mathcal{W}$ in future work.

and the $(\rho,\sigma)$-reducible quadrature methods. To be specific, the methods are easy to construct, simple to implement and computationally efficient. Moreover, they reduce to (1.5) when applied to (1.3).

The derivation of the multilag methods for (1.1) is essentially based on an appropriate approximation of the Volterra integral operator (see §2) and therefore it is not surprising that the same approximations can also be employed in connection with the numerical solution of other types of Volterra equations. To demonstrate this, we shall apply our techniques also to derive numerical methods for Volterra integro-differential equations

$$(1.7) \qquad f'(x) = F(x,f(x), \int_0^x K(x,y,f(y))dy), \qquad f(0) = f_0,$$

and for first kind Volterra integral equations

$$(1.8) \qquad \int_0^x K(x,y,f(y))dy = g(x), \qquad g(0) = 0.$$

We shall establish, in §3 and 4, the order of convergence of the multilag methods as well as their modification for the solution of (1.1) and (1.7).

It is well-known that for the solution of first kind equations (1.8) by means of direct quadrature methods special stabilized quadrature rules must be constructed (see e.g. [1,6]). In §5, we shall present a class of high-order convergent modified multilag methods which combine *conventional* quadrature rules with a highly stable LM method.

To illustrate the theoretical results we have included in §6 some numerical experiments with modified multilag methods in which we chose for $(\rho,\sigma)$ the highly stable backward differentiation methods and for $\mathcal{W}$ the Gregory quadrature rules.

## 2. PRELIMINARIES AND NOTATIONS

In this section we shall derive approximations of the Volterra integral operator $\int_0^x K(x,y,f(y))dy$, which occurs in the functional equations (1.1), (1.7) and (1.8). For this derivation it is convenient to introduce the function $\Psi(t,x)$ defined as

$$(2.1) \qquad \Psi(t,x) = \int_0^t K(x,y,f(y))dy,$$

where (for the moment) f is a given function. Following Pouzet (see e.g. [2]), we regard $\Psi(t,x)$ as the solution of the ordinary differential equation (with parameter x)

$$(2.2) \qquad \frac{d}{dt} \Psi(t,x) = K(x,t,f(t))$$

with initial condition $\Psi(0,x) = 0$. This observation suggests the use of methods for ordinary differential equations (cf. [9,18]). Using an LM method $(\rho,\sigma)$ (with normalization $a_0 = 1$), we may define an approximation $\hat{\psi}_n(x)$ of $\psi_n(x)$ $(\psi_n(x) := \Psi(hh,x))$ by the recurrence relation

$$(2.3) \qquad \hat{\psi}_\nu(x) = - \sum_{i=1}^{k} a_i \hat{\psi}_{\nu-i}(x) + h \sum_{i=0}^{k} b_i K(x,x_{\nu-i},f(x_{\nu-i})),$$

$$\nu = k(1)n,$$

provided that the starting values $\hat{\psi}_0(x),\ldots,\hat{\psi}_{k-1}(x)$ are given. In the treatment of second kind Volterra equations WOLKENFELT et al. [18] discuss methods employing such approximations and indicate the equivalence with $(\rho,\sigma)$-reducible quadrature methods. A disadvantage of this approach is that for the computation of $\hat{\psi}_n(x)$ the recurrence relation (2.3) must be evaluated for $\nu = k(1)n$, which may give a considerable amount of overhead, especially when dealing with systems of Volterra integral equation. This drawback can be avoided by the following approach: instead of defining starting values $\hat{\psi}_0(x),\ldots,\hat{\psi}_{k-1}(x)$ followed by a recursive evaluation of (2.3), we compute approximations $\tilde{\psi}_{n-k}(x),\ldots,\tilde{\psi}_{n-1}(x)$ by means of *computationally efficient* quadrature rules followed by *one single application* of $(\rho,\sigma)$. To be specific, we define

$$(2.4) \qquad \hat{\psi}_n(x) := - \sum_{i=1}^{k} a_i \tilde{\psi}_{n-i}(x) + h \sum_{i=0}^{k} b_i K(x,x_{n-i},f(x_{n-i}))$$

where

$$(2.5) \qquad \tilde{\psi}_n(x) := h \sum_{j=0}^{n} w_{nj} K(x,x_j,f(x_j)).$$

Here, $W = \{w_{nj} | n \geq n_0, 0 \leq j \leq n\}$ denotes a family of quadrature rules. The value of $n_0$ depends on the accuracy of these rules. Obviously, (2.4) can only be applied for $n \geq n_k = n_0 + k$.

REMARK. Examples of computationally efficient quadrature rules are the rules with a finite repetition factor (see e.g. [3]). In the case of a repetition factor of one the weights satisfy

$$w_{nj} - w_{n-1,j} = \begin{cases} 0 & \text{if } 0 \leq j < n-\kappa, \\ \nabla w_{nj} & \text{if } n-\kappa \leq j \leq n, \end{cases}$$

so that $\tilde{\psi}_{n-k+1}(x),\ldots,\tilde{\psi}_{n-1}(x)$ defined in (2.5) can be computed recursively as follows

$$\text{(2.6)} \qquad \tilde{\psi}_m(x) = \tilde{\psi}_{m-1}(x) + h \sum_{j=m-\kappa}^{m} \nabla w_{mj} K(x,x_j,f(x_j)), \quad m = n-k+1(1)n-1.$$

Specific examples are the Gregory quadrature rules ([2]). It is easily verified that for the evaluation of $\hat{\psi}_n(x)$ by means of (2.4), (2.5) and (2.6) roughly 2nk multiplications and additions are saved in comparison with (2.3). □

So far we assumed that the function f is known. Now assume that only approximations $f_j$ to $f(x_j)$ are available. In this case we replace in (2.4) and (2.5), $f(x_j)$ by $f_j$, $\hat{\psi}_n(x)$ by $\hat{I}_n(x)$ and $\tilde{\psi}_n(x)$ by $\tilde{I}_n(x)$ to obtain the approximations

$$\text{(2.7)} \qquad \hat{I}_n(x) := -\sum_{i=1}^{k} a_i \tilde{I}_{n-i}(x) + h \sum_{i=0}^{k} b_i K(x,x_{n-i},f_{n-i}), \quad n \geq n_k$$

where

$$\text{(2.8)} \qquad \tilde{I}_n(x) := h \sum_{j=0}^{n} w_{nj} K(x,x_j,f_j), \qquad n \geq n_0.$$

Since the function $\tilde{I}_n(x)$ which depends on all previously computed $f_j$-values, is usually called a *lag term* (or history term), we shall call the function $\hat{I}_n(x)$ a multilag approximation to $\psi_n(x)$.

For the convergence analysis of our methods we need the local truncation error $T_n(h;x)$ of (2.4) at $t = nh$ defined as

$$(2.9) \qquad \psi_n(x) = -\sum_{i=1}^{k} a_i\psi_{n-i}(x) + h\sum_{i=0}^{k} b_iK(x,x_{n-i},f(x_{n-i})) + T_n(h;x).$$

Note that for an LM method of order p

$$(2.10) \qquad T_n(h;x) = C_{p+1}h^{p+1}\frac{d^p}{dt^p}K(x,t,f(t))\Big|_{t=nh} + O(h^{p+2}) \text{ as } h \to 0$$

where $C_{p+1} \neq 0$ denotes the error constant of $(\rho,\sigma)$ (cf. [8]). For the rules (2.5) we define the quadrature error

$$(2.11) \qquad Q_n(h;x) := \psi_n(x) - \tilde{\psi}_n(x).$$

Furthermore we assume that the quadrature weights $w_{nj}$ are uniformly bounded, i.e. $|w_{nj}| \le \bar{w}$ for all n and j.

In our theorems we shall establish a bound on the global discretization error in terms of quadrature errors, local truncation errors and errors in the starting values using the following notation:

$$(2.12) \qquad \delta_1(h) = \max\{|f(x_j)-f_j)|: 0 \le j \le n_0-1\};$$

$$(2.13) \qquad \delta_2(h) = \max\{|f(x_j)-f_j|: n_0 \le j \le n_k-1\};$$

$$(2.14) \qquad T_N(h) = \max\{|T_n(h;x_n)|: n_k \le n \le N\};$$

$$(2.15) \qquad Q_N(h) = \max\{|Q_{n-i}(h;x_n)|: n_k \le n \le N,\ 1 \le i \le k\};$$

$$(2.16) \qquad \Delta Q_N(h) = \max\{|Q_{n-i}(h;x_n) - Q_{n-i}(h;x_{n-i})|: n_k \le n \le N,\ 1 \le i \le k\}$$

In order not to distract the reader's attention from the main results, all theorems are stated without proof. However, for those interested, the technical details can be found in the Appendix of [21].

## 3. METHODS FOR SECOND KIND VOLTERRA INTEGRAL EQUATIONS

The second kind Volterra equation (1.1) can be written as

$$(3.1)\qquad f(x) = g(x) + \Psi(x,x), \qquad 0 \le x \le X,$$

where we have used the notation (2.1).

### 3.1. Multilag methods

In order to discretize (3.1) at $x = x_n$, we replace $f(x_n)$ by $f_n$ and $\Psi(x_n,x_n) = \psi_n(x_n)$ by $\hat{I}_n(x_n)$ defined in (2.7) to obtain the *multilag method*

$$(3.2)\qquad f_n = g(x_n) - \sum_{i=1}^{k} a_i \tilde{I}_{n-i}(x_n) + h \sum_{i=0}^{k} b_i K(x_n, x_{n-i}, f_{n-i}), \quad n \ge n_k$$

where $\tilde{I}_n(x)$ is defined in (2.8). The required starting values are $f_j$, $j = 0(1)n_k-1$.

For the global error $f(x_n)-f_n$ the following result can be derived.

THEOREM 3.1. *Assume that* K *satisfies the Lipschitz condition*

$$(3.3)\qquad |K(x,y,\phi_1) - K(x,y,\phi_2)| \le L_1|\phi_1-\phi_2|.$$

*Let* $f(x)$ *be the solution of* (3.1) *and let* $f_n$ *be defined by* (3.2). *Then for* h *sufficiently small* $(X = Nh)$

$$(3.4)\qquad \max_{n_k \le n \le N} |f(x_n)-f_n| \le C \max\{h\delta_1(h), h\delta_2(h), Q_N(h), T_N(h)\}$$

*where* C *is a constant independent of* N *and* h, *and where* $\delta_1(h), \delta_2(h), Q_N(h)$ *and* $T_N(h)$ *are defined in* (2.12) *to* (2.15). □

Using this theorem high-order convergence of the methods (3.2) is now readily established.

THEOREM 3.2. *Let the condition* (3.3) *be satisfied and assume that* g *and* K *are sufficiently smooth. In addition, let*

(i) *the* LM *method* $(\rho,\sigma)$ *be convergent of order* p;

(ii) *the quadrature rules* W *be of order* q;

(iii) *the errors in the starting values be of order* s.

*Then the multilag method* (3.2) *is convergent of order* r, *where*

$r = \min\{s+1, q, p+1\}$. *To be specific*

$$\max_{n_k \le n \le N} |f(x_n) - f_n| \le Ch^r \text{ as } h \to 0,\ N \to \infty,\ Nh = X$$

*where* C *is a constant independent of* N *and* h. □

With respect to the stability analysis we remark that the application of (3.2) to the basic test equation (1.3) yields the relations

$$\text{(3.5)} \qquad \begin{aligned} f_n &= 1 - \sum_{i=1}^{k} a_i \tilde{I}_{n-i} + h\lambda \sum_{i=0}^{k} b_i f_{n-i}, \\ \tilde{I}_n &= h\lambda \sum_{j=0}^{n} w_{nj} f_j. \end{aligned}$$

which clearly indicates that the stability behaviour of (3.2) depends on $(\rho,\sigma)$ as well as on the quadrature rules $\mathcal{W}$. Under suitable assumptions on the quadrature weights (e.g. reducibility [20] or finite repetition factor [3]) the relations (3.5) can be reduced to a recurrence relation in terms of $f_n$-values only and the stability behaviour is then determined by a root condition on the associated stability polynomial. A systematic study along these lines for various choices of quadrature rules $\mathcal{W}$ will be the subject of future research.

In this paper we concentrate on a modification of (3.2) which has been constructed in such a way that the stability behaviour with respect to (1.3) is independent of the choice of the quadrature rules $\mathcal{W}$ used for the lag terms $\tilde{I}_n(x)$.

### 3.2. Modified multilag methods

In [12] a modification of mixed Runge-Kutta methods was proposed (see also [11]) with the aim of improving the stability behaviour. This modified method was derived by modifying the lag term by a suitable perturbation term which can be regarded as a residual (see [13]). Motivated by this approach, we present the following modification of (3.2)

$$\text{(3.6a)} \qquad f_n = g(x_n) - \sum_{i=1}^{k} a_i\{\tilde{I}_{n-i}(x_n) + r_{n-i}\} + h \sum_{i=0}^{k} b_i K(x_n, x_{n-i}, f_{n-i}), \qquad n \ge n_k$$

$$(3.6b) \qquad r_n = f_n - g(x_n) - \tilde{I}_n(x_n), \qquad n \geq n_0,$$

where $\tilde{I}_n(x)$ is defined in (2.8). The *modified multilag method* (3.6) requires the starting values $f_j$, $j = 0(1)n_k-1$. Note that $r_n$ defined in (3.6b) can be regarded as a residual.

We remark that the class (3.6) includes as a special case the methods proposed by VAN DER HOUWEN [11] (who chose, in the notation (3.6), $a_1 = -1$, $a_2 = \ldots = a_k = 0$).

It is easily verified that application of (3.6) to the test equation (1.3) yields, due to cancellation of the $\tilde{I}_n$ terms, the recurrence relation (1.5). Thus the stability behaviour of (3.6) is determined only by $(\rho,\sigma)$, and therefore identical to that of the $(\rho,\sigma)$-reducible quadrature methods.

Before establishing the high-order convergence of the modified methods (3.6) we first state the following result.

THEOREM 3.3. *Let* K *satisfy the Lipschitz condition*

$$(3.7) \qquad \begin{aligned} &|K(x,y,\phi_1) - K(x,y,\phi_2) - K(x_n,y,\phi_1) + K(x_n,y,\phi_2)| \leq \\ &\leq L_1^* \, |x-x_n| \, |\phi_1-\phi_2|, \end{aligned}$$

*and let the* LM *method* $(\rho,\sigma)$ *be convergent. Furthermore let* $f(x)$ *be the solution of* (3.1) *and let* $f_n$ *be defined by* (3.6). *Then for* h *sufficiently small*

$$(3.8) \qquad \max_{n_k \leq n \leq N} |f(x_n)-f_n| \leq C \max\{h\delta_1(h), \delta_2(h), h^{-1}\Delta Q_N(h), h^{-1}T_N(h)\}$$

*where* C *is a constant independent of* N *and* h *and where* $\delta_1(h), \delta_2(h)$, $\Delta Q_N(h)$ and $T_N(h)$ *are defined in* (2.12) *to* (2.16). □

The Lipschitz condition (3.7) required in the above theorem is satisfied if, for example, $K_x$ satisfies a Lipschitz condition with respect to f. We then may write the left-hand side of (3.7) as

$$\left| \int_{x_n}^{x} \{K_x(t,y,\phi_1) - K_x(t,y,\phi_2)\}dt \right|$$

from which the right-hand side of (3.7) is immediate. It can also be shown that $h^{-1}\Delta Q_N(h)$ has the same order of accuracy as $Q_N(h)$ provided that $K$ and $K_x$ are sufficiently smooth. This fact together with Theorem 3.3 yields

THEOREM 3.4. *Let the assumptions of Theorem* 3.3 *and* 3.2 *be valid. Then the modified multilag method* (3.6) *is convergent of order* $r^*$, *where* $r^* = \min\{s,q,p\}$. □

Comparison of the results of Theorem 3.2 and 3.4 clearly shows the effect of the modification on the order of convergence: if $s \geq p+1$ and $q \geq p+1$, the order of the modified methods is lowered by one.

## 4. METHODS FOR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

Using (2.1), equation (1.7) can be written as

$$f'(x) = F(x,f(x),\Psi(x,x)), \qquad 0 \leq x \leq X \tag{4.1}$$

with initial condition $f(0) = f_0$. Application of an LM method for ordinary differential equations to (4.1) in which $\Psi(x_n,x_n)$ is replaced by a numerical approximation yields a wide class of numerical methods (cf. [5,15,16,20]).

### 4.1. Multilag methods

We shall employ a linear $k^*$-step method $(\rho^*,\sigma^*)$ with coefficients $a_i^*$ and $b_i^*$, and numerical approximations $\hat{I}_n = \hat{I}_n(x_n)$ as defined in (2.7) to obtain the methods

$$\sum_{i=0}^{k} a_i^* f_{n-i} = h \sum_{i=0}^{k} b_i^* F(x_{n-i},f_{n-i},\hat{I}_{n-i}), \qquad n \geq n_k = n_0 + k \tag{4.2a}$$

$$\hat{I}_n = -\sum_{i=1}^{k} a_i \tilde{I}_{n-i}(x_n) + h \sum_{i=0}^{k} b_i K(x_n,x_{n-i},f_{n-i}), \quad n \geq n_k \tag{4.2b}$$

$$\hat{I}_n = \tilde{I}_n(x_n) \text{ if } n_0 \leq n \leq n_k-1, \tag{4.2c}$$

where $\tilde{I}_n(x)$ is defined in (2.8). Note that we have assumed, without loss of generality, that $k^* = k$. The required starting values for (4.2) are

$f_j$, $j = 0(1)n_k-1$.

A bound for the global discretization error is established in the following theorem.

THEOREM 4.1. *Let* K *satisfy the condition* (3.3) *and let* F *satisfy the Lipschitz conditions*

(4.3a) $$|F(x,\phi_1,z) - F(x,\phi_2,z)| \le L_2|\phi_1-\phi_2|,$$

(4.3b) $$|F(x,\phi,z_1) - F(x,\phi,z_2)| \le L_3|z_1-z_2|,$$

*and assume that the* LM *method* $(\rho^*,\sigma^*)$ *is convergent. Let* $f(x)$ *be the solution of* (4.1) *and let* $f_n$ *be defined by* (4.2). *Then for* h *sufficiently small*

(4.4) $$\max_{n_k\le n\le N} |f(x_n)-f_n| \le C \max\{h\delta_1(h),\delta_2(h),h\delta_3(h),Q_N(h),T_N(h),h^{-1}T_N^*(h)\}$$

*where* C *is a constant independent of* N *and* h *and where* $\delta_1(h),\delta_2(h),Q_N(h)$ *and* $T_N(h)$ *are defined in* (2.12) *to* (2.15). *Furthermore*

(4.5) $$\delta_3(h) = \max\{|Q_n(h;x_n)|: n_0 \le n \le n_k-1\},$$

(4.6) $$T_N^*(h) = \max\{|T_n^*(h;x_n)|: n_k \le n \le N\},$$

*where* $T_n^*(h;x_n)$ *denotes the local truncation error at* $x = x_n$ *of the* LM *method* $(\rho^*,\sigma^*)$ *when applied to* (4.1). □

An immediate consequence of the above theorem is

THEOREM 4.2. *Let the conditions* (3.3) *and* (4.3) *be satisfied and assume that* F *and* K *are sufficiently smooth. In addition, let*

(i) *the* LM *method* $(\rho^*,\sigma^*)$ *be convergent of order* $p^*$;

(ii) *the* LM *method* $(\rho,\sigma)$ *be convergent of order* p;

(iii) *the quadrature rules* $W$ *be of order* q;

(iv) *the errors in the starting values be of order* s.

*Then the multilag method* (4.2) *is convergent of order* r, *where* $r = \min\{s,q,p+1,p^*\}$. □

Concerning the stability behaviour we note that the application of (4.2) to the basic test equation (cf. [16])

$$(4.7)\qquad f'(x) = \xi f(x) + \eta \int_0^x f(y)dy, \qquad \xi,\eta \in \mathbb{C}$$

yields relations which depend also on the quadrature rules $\mathcal{W}$. In order to eliminate the effect of these quadrature rules on the stability behaviour we construct a modification of (4.2).

## 4.2. Modified multilag methods

Along the same lines as in §3.2 we define the modified multilag methods by

$$(4.8a)\qquad \sum_{i=0}^{k} a_i^* f_{n-i} = h \sum_{i=0}^{k} b_i^* F(x_{n-i}, f_{n-i}, \hat{I}_{n-i}), \qquad n \geq n_k,$$

$$(4.8b)\qquad \hat{I}_n = - \sum_{i=1}^{k} a_i \{\tilde{I}_{n-i}(x_n) + r_{n-i}\} + h \sum_{i=0}^{k} b_i K(x_n, x_{n-i}, f_{n-i}), \quad n \geq n_k$$

$$(4.8c)\qquad r_n = \hat{I}_n - \tilde{I}_n(x_n), \qquad n \geq n_k.$$

As in (4.2c) we define $\hat{I}_n = \tilde{I}_n(x_n)$ if $n_0 \leq n \leq n_k - 1$, which implies that $r_n = 0$ if $n_0 \leq n \leq n_k - 1$.

Due to this modification the method (4.8) applied to (4.7) yields the recurrence relations

$$(4.9)\qquad \begin{aligned} \sum_{i=0}^{k} a_i^* f_{n-i} &= h \sum_{i=0}^{k} b_i^* (\xi f_{n-i} + \eta \hat{I}_{n-i}), \\ \sum_{i=0}^{k} a_i \hat{I}_{n-i} &= h \sum_{i=0}^{k} b_i f_{n-i}. \end{aligned}$$

Elimination of $\hat{I}_n$ yields a recurrence relation in $f_n$-values only whose characteristic (or stability) polynomial is given by

$$(4.10)\qquad \rho(\zeta)[\rho^*(\zeta) - h\xi\sigma^*(\zeta)] - h^2\eta\sigma(\zeta)\sigma^*(\zeta),$$

which is independent of $W$. Note that the same stability polynomials were found by MATTHYS [16] who considered $(\rho,\sigma)$-reducible quadrature rules.

We shall now deal with the convergence of (4.8). First we give the following bound for the global error.

THEOREM 4.3. *Let* K *satisfy the condition* (3.7) *and let* F *satisfy* (4.3) *and let the* LM *methods* $(\rho^*,\sigma^*)$ *and* $(\rho,\sigma)$ *be convergent. Further, let* $f(x)$ *be the solution of* (4.1) *and let* $f_n$ *be defined by* (4.8). *Then for* h *sufficiently small*

$$\text{(4.11)} \qquad \max_{n_k\le n\le N} |f(x_n)-f_n| \le C\max\{h\delta_1(h),\delta_2(h),\delta_3(h),h^{-1}\Delta Q_N(h), h^{-1}T_N(h),h^{-1}T_N^*(h)\}$$

*where* C *is a constant independent of* N *and* h, *where* $\delta_1(h),\delta_2(h)$, $T_N(h)$ *and* $\Delta Q_N(h)$ *are defined in* (2.12) *to* (2.16) *and where* $\delta_3(h)$ *and* $T_N^*(h)$ *are defined in* (4.5),(4.6). □

As a consequence we have

THEOREM 4.4. *Let the assumptions of Theorem* 4.3 *and* 4.2 *be valid. Then the modified multilag method* (4.8) *is convergent of order* $r^*$, *where* $r^* = \min\{s,q,p,p^*\}$. □

From the results of Theorem 4.2 and 4.4 it is evident that the modified methods may lose one order of accuracy (cf. §3.2).

## 5. MODIFIED MULTILAG METHODS FOR FIRST KIND VOLTERRA INTEGRAL EQUATIONS

In section 3 and 4 we considered general LM methods in conjunction with general quadrature rules. It turned out that convergent LM methods together with convergent quadrature rules generate convergent methods for second kind Volterra equations and integro-differential equations.

It is well known, however, that for the solution of first kind equations convergence of the quadrature rules does not generally imply convergence of the associated direct quadrature method and additional assumptions are necessary (see e.g. [1,6,7,10,17,19]).

In this section we do not pursue complete generality and present the convergence results of a particular class of modified multilag methods. To be specific, we consider the methods

(5.1a) $$-\sum_{i=1}^{k} a_i\{\tilde{I}_{n-i}(x_n) + r_{n-i}\} + hb_0K(x_n,x_n,f_n) = g(x_n), \qquad n \ge n_k,$$

(5.1b) $$r_n = g(x_n) - \tilde{I}_n(x_n), \qquad n \ge n_0,$$

where $\tilde{I}_n(x)$ is defined in (2.8). The required starting values are $f_j$, $j = 0(1)n_k-1$.

The methods (5.1) can be derived as follows. Using (2.1) the first kind Volterra equation (1.8) can be written as

(5.2) $$\Psi(x,x) = g(x), \qquad 0 \le x \le X.$$

Discretization of (5.2) at $x = x_n$ using the approximation (2.7) in which we take $b_2=\ldots=b_k = 0$, and modification by the "residual approach" then yields (5.1). Note that we have chosen a particular class of LM methods (i.e. $\sigma(\zeta) = b_0\zeta^k$) which includes the well-known backward differentiation methods. We emphasize that the quadrature rules $W$ are still free to choose.

For the global error the following bound can be derived.

<u>THEOREM 5.1</u>. *In addition to the condition* (3.7) *assume that*

(5.3) $$|K(x,x,\phi_1) - K(x,x,\phi_2)| \ge L_4|\phi_1-\phi_2|, \qquad (L_4 > 0).$$

*Let the* LM *method* $(\rho,\sigma)$ *with* $\sigma(\zeta) = b_0\zeta^k$ *be convergent. Furthermore let* $f(x)$ *be the solution of* (1.8) *and let* $f_n$ *be defined by* (5.1). *Then for* $h$ *sufficiently small*

(5.4) $$\max_{n_k\le n\le N} |f(x_n)-f_n| \le C \max\{h\delta_1(h), h\delta_2(h), h^{-1}\Delta Q_N(h), h^{-1}T_N(h)\}$$

*where* $C$ *is a constant independent of* $N$ *and* $h$, *and where* $\delta_1(h), \delta_2(h), \Delta Q_N(h)$ *and* $T_N(h)$ *are defined in* (2.12) *to* (2.16). □

We remark that the Lipschitz condition (5.3) is implied by the conditions for the existence of a unique continuous solution to (1.8) given in [7]. To be specific, one of the conditions is that $|\frac{\partial K}{\partial f}(x,x,f)|$ should be bounded away from zero.

As an immediate consequence of Theorem 5.1 we have.

THEOREM 5.2. *Let the assumptions of Theorem* 5.1 *be valid and let* K *and* g *be sufficiently smooth. In addition, let*

(i) *the* LM *method* $(\rho,\sigma)$ *with* $\sigma(\zeta) = b_0\zeta^k$ *be convergent of order* p;

(ii) *the quadrature rules* $W$ *be of order* q;

(iii) *the errors in the starting values be of order* s.

*Then the method* (5.1) *is convergent of order* $r^*$, *where* $r^* = \min\{s+1,q,p\}$. □

It is easily verified that the methods (5.1) applied to the test equation

$$(5.5) \qquad \int_0^x f(y)dy = g(x)$$

reduce to $f_n = (hb_0)^{-1}\Sigma_{i=0}^k a_i g(x_{n-i})$, *irrespective of the choice of the quadrature rules* $W$. As a result, the methods (5.1) correspond to "local differentiation formulae" which is a desirable property with respect to stability (see e.g. [17,p.417]).

## 6. NUMERICAL EXPERIMENTS

In this section we report on numerical experiments with modified multilag methods (3.6), (4.8) and (5.1). For the LM method $(\rho,\sigma)$ and the quadrature rules $W$ we chose, for p = 2(1)6, the pth order backward differentiation (BD) methods ([14]) and the pth order Gregory quadrature rules, respectively. In the methods (4.8) we took $(\rho^*,\sigma^*)$ identical to $(\rho,\sigma)$. The methods are denoted by BDGp (p=2(1)6).

The methods were applied to test problems (taken from [5],[6] and [20]) with known exact solution. Integration was performed with a constant stepsize, and the necessary starting values were computed from the exact solution. In consequence of the Theorems 3.4, 4.4 and 5.2 the methods BDGp are of order p, asymptotically.

In the tables of results we have tabulated for different orders and a sequence of stepsizes, the number of correct decimal digits cd (defined by $-{}^{10}\log$ (absolute error)) at the endpoint of integration. Moreover we have listed in the convergence experiments the computed order $p^*$ (defined by $\{cd(h) - cd(2h)\}/{}^{10}\log 2$).

All calculations have been performed on a CDC CYBER 750 installation using 14 significant digits.

## 6.1. Second kind Volterra integral equations

In order to test their high-order convergence we have applied the BDG methods to the following problem

$$(6.1.1) \qquad f(x) = \tfrac{1}{2}x^2 \exp(-x) + \tfrac{1}{2}\int_0^x (x-y)^2 \exp(y-x) f(y)dy, \qquad 0 \le x \le 6,$$

with exact solution $f(x) = \frac{1}{3} - \frac{1}{3}\exp(-3x/2)\{\cos(\frac{1}{2}x\sqrt{3}) + \sqrt{3}\sin(\frac{1}{2}x\sqrt{3})\}$. In Table 6.1.1 the results are tabulated for various choices of h.

| $h^{-1}$ | p=2 | p=3 | p=4 | p=5 | p=6 |
|---|---|---|---|---|---|
| 4 | 1.89 | 1.86 | 2.34 | 2.97 | 3.51 |
| 8 | $2.22^{1.1}$ | $2.57^{2.4}$ | $3.25^{3.0}$ | $4.21^{4.1}$ | $4.92^{4.7}$ |
| 16 | $2.70^{1.6}$ | $3.37^{2.7}$ | $4.31^{3.5}$ | $5.60^{4.6}$ | $6.55^{5.4}$ |
| 32 | $3.25^{1.8}$ | $4.23^{2.9}$ | $5.44^{3.8}$ | $7.05^{4.8}$ | $8.28^{5.8}$ |
| 64 | $3.83^{1.9}$ | $5.11^{2.9}$ | $6.61^{3.9}$ | $8.53^{4.9}$ | $10.10^{6.1}$ |

Table 6.1.1. Number of correct digits at x=6 and the computed order $p^*$ for the BDG methods applied to (6.1.1).

From this table it is obvious that the computed order tends to the theoretical order of convergence.

The favourable stability behaviour of the BDG methods is demonstrated in the following example:

$$(6.1.2) \qquad f(x) = g(x) - \lambda \int_0^x \frac{1+x}{1+y} f^2(y)dy, \qquad 0 \le x \le xe,$$

with exact solution $f(x) = [1+(1+x)\exp(-x)]^{\frac{1}{2}}$ if we choose $g(x) = f(x) + \lambda(1+x)[\ln(1+x) + 1-\exp(-x)]$. We considered the values $\lambda = 1,10,100,1000$ and $10000$ which makes (6.1.2) increasingly stiff. The endpoint of integration was 192h. The results are given in Table 6.1.2.

| $h^{-1}$ | $\lambda$ | p=2 | p=3 | p=4 | p=5 | p=6 |
|---|---|---|---|---|---|---|
| | 1 | 3.23 | 3.83 | 4.97 | 5.10 | 5.84 |
| | 10 | 3.23 | 3.84 | 4.98 | 5.11 | 5.85 |
| 4 | 100 | 3.24 | 3.84 | 4.98 | 5.11 | 5.85 |
| | 1000 | 3.24 | 3.84 | 4.98 | 5.11 | 5.85 |
| | 10000 | 3.24 | 3.84 | 4.98 | 5.11 | 5.85 |
| | 1 | 3.84 | 4.93 | 6.19 | 7.01 | 8.21 |
| | 10 | 3.87 | 4.96 | 6.22 | 7.04 | 8.24 |
| 16 | 100 | 3.87 | 4.97 | 6.23 | 7.05 | 8.24 |
| | 1000 | 3.87 | 4.97 | 6.23 | 7.05 | 8.24 |
| | 10000 | 3.87 | 4.97 | 6.23 | 7.05 | 8.24 |
| | 1 | 5.18 | 6.41 | 8.06 | 9.29 | 10.46 |
| | 10 | 4.99 | 6.42 | 8.08 | 9.35 | 10.65 |
| 64 | 100 | 4.99 | 6.42 | 8.09 | 9.37 | 10.36 |
| | 1000 | 4.99 | 6.42 | 8.09 | 9.43 | 10.63 |
| | 10000 | 4.99 | 6.42 | 8.09 | 9.40 | 10.44 |

Table 6.1.2. The number of correct digits at x = 192h for problem (6.1.2)

The results show that for fixed h the accuracy is hardly affected by increasing stiffness and justify the conclusion that the BDG methods are highly-stable.

## 6.2. Volterra integro-differential equations

To test high-order convergence we applied the BDG methods to

$$(6.2.1)\qquad f'(x) = -x - (1+x)^{-2} + \frac{1}{f(x)} \ln \frac{2+2x}{2+2x} + \int_0^x \frac{dy}{1+(1+x)f(y)}, \quad 0 \le x \le 10.$$

Taking $f(0) = 1$ yields the exact solution $f(x) = (1+x)^{-1}$. The results summarized in Table 6.2.1 clearly show that the computed order tends to the theoretical order of convergence, except for the sixth order method.

| $h^{-1}$ | p=2 | | p=3 | | p=4 | | p=5 | | p=6 | |
|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 5.85 | | 5.76 | | 6.32 | | 7.00 | | 7.60 | |
| | | 0.8 | | 1.4 | | 1.8 | | 2.0 | | 2.3 |
| 8 | 6.10 | | 6.19 | | 6.86 | | 7.59 | | 8.30 | |
| | | 1.0 | | 2.2 | | 2.7 | | 3.2 | | 3.4 |
| 16 | 6.40 | | 6.84 | | 7.67 | | 8.51 | | 9.33 | |
| | | 1.6 | | 2.6 | | 3.3 | | 3.9 | | 4.4 |
| 32 | 6.89 | | 7.61 | | 8.65 | | 9.67 | | 10.65 | |
| | | 1.9 | | 2.8 | | 3.6 | | 4.3 | | 7.1 |
| 64 | 7.45 | | 8.45 | | 9.73 | | 10.97 | | 12.79 | |

Table 6.2.1. Number of correct digits at x=10 and computed order $p^*$ for the BDG methods applied to (6.2.1).

For the stability test we applied the methods to

$$(6.2.2)\qquad \begin{cases} f'(x) = [d(x) - \alpha f(x) - \beta z(x)]^3 - 1, & f(0) = 1 \\ z(x) = \int_0^x (x+\gamma y)^\delta f^3(y)\,dy. & \end{cases}$$

Choosing $d(x) = 1 + \alpha + \gamma^{-1}(1+\delta)^{-1}\beta x^{\delta+1}\{(1+\gamma)^{\delta+1}-1\}$ yields the exact solution $f(x) \equiv 1$. As in [20] we considered the values $\alpha = 40$, $\beta = 15$, $\gamma = 2$ and $\delta = 3/2$, and integration was performed with $h = 1/8$. On the basis of the stability regions of the BDG methods (which are identical to those of the [BD;BD] methods given in [20]), we expect the methods to yield stable results. In Table 6.2.2 the results are given at some gridpoints.

| x | p=2 | p=3 | p=4 | p=5 | p=6 |
|---|---|---|---|---|---|
| 1.0 | 3.23 | 4.37 | 5.44 | 6.17 | * |
| 3.0 | 4.25 | 6.07 | 6.72 | 8.54 | 7.72 |
| 5.0 | 4.45 | 6.93 | 7.06 | 8.47 | 8.25 |
| 7.0 | 4.60 | 7.49 | 7.28 | 8.68 | 8.15 |
| 16.0 | 5.00 | 8.15 | 7.79 | 9.23 | 9.82 |

Table 6.2.2. Number of correct digits for problem (6.2.2) obtained with the BDG methods with h = 1/8.

The asterisk in this table indicates that x = 1 is a point where an exact starting value was given. The numerical results clearly display the stable behaviour of the BDG methods.

## 6.3. First kind Volterra integral equations

We applied the BDG methods to the following problems taken from [6]

$$(6.3.1)\qquad 2\int_0^x \cos(x-y)f(y)dy = \exp(x) + \sin(x) - \cos(x),$$

$$(6.3.2)\qquad \int_0^x \exp(y-x)f(y)dy = \sinh(x).$$

Both problems have the exact solution f(x) = exp(x). The endpoint of integration was x = 4. The correct order of convergence of the BDG methods up to order five is shown by the Tables 6.3.1 and 6.3.2.

| $h^{-1}$ | p=2 | | p=3 | | p=4 | | p=5 | | p=6 | |
|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 0.87 | 2.3 | 1.50 | 2.8 | 2.20 | 3.9 | 3.20 | 4.9 | 4.55 | 6.6 |
| 20 | 1.55 | 2.2 | 2.33 | 2.9 | 3.36 | 3.9 | 4.68 | 5.0 | 6.54 | 6.5 |
| 40 | 2.20 | 2.1 | 3.20 | 3.0 | 4.54 | 4.0 | 6.18 | 5.0 | 8.50 | 3.5 |
| 80 | 2.83 | | 4.09 | | 5.73 | | 7.68 | | 9.54 | |

Table 6.3.1. Number of correct digits at x=4 and the computed order $p^*$ of the BDG methods applied to (6.3.1)

| $h^{-1}$ | p=2 | p=3 | p=4 | p=5 | p=6 |
|---|---|---|---|---|---|
| 10 | -0.02 | 0.81 | 1.64 | 2.45 | 1.81 |
| | 1.9 | 2.8 | 3.8 | 4.7 | 1.4 |
| 20 | 0.54 | 1.66 | 2.77 | 3.87 | 2.23 |
| | 1.9 | 2.9 | 3.9 | 4.8 | 13.9 |
| 40 | 1.12 | 2.54 | 3.94 | 5.32 | 6.40 |
| | 2.0 | 3.0 | 3.9 | 4.9 | 6.8 |
| 80 | 1.71 | 3.43 | 5.12 | 6.80 | 8.43 |

Table 6.3.2. Number of correct digits at x=4 and computed order $p^*$ of the BDG methods applied to (6.3.2)

Although not displayed in the tables of results, the global error turns out to be a smooth function except for the sixth order method when h is small (h = 1/40, 1/80). This may explain the uncertain behaviour of BDG6.

## 7. CONCLUDING REMARKS

The results of section 6 justify the conclusion that the construction presented in this paper yields high order convergent methods which can be made highly stable by choosing a highly stable LM method.

To emphasize we repeat that the modified multilag methods applied to the basic test equations of (1.1), (1.7) and (1.8) yield exactly the same stability polynomials as those obtained with $(\rho,\sigma)$-reducible quadrature methods. As a consequence, all stability results previously derived for $(\rho,\sigma)$-reducible quadrature methods (e.g. A-stability results [16], stability regions [4,5,20]) also hold for the modified multilag methods.

Finally we remark that the class of methods presented here can easily be extended by considering cyclic LM methods for ordinary differential equations. In this case the method (3.6) for example takes the form

$$f_n = g(x_n) - \sum_{i=1}^{k} a_i^{(n)}\{\tilde{I}_{n-i}(x_n)+r_{n-i}\} + h\sum_{i=0}^{k} b_i^{(n)} K(x_n,x_{n-i},f_{n-i}), \tag{7.1}$$

with $r_n$ defined as in (3.6b) and where $a_i^{(n)}$ and $b_i^{(n)}$ are periodic functions of n. The proof of high-order convergence of (7.1) will probably be more complicated than for the methods presented in this paper. On the other hand, the stability properties of cyclic LM methods are well-known for ODE-theory and thus can be exploited to construct in a straightforward fashion highly accurate, highly stable modified multilag methods for the efficient solution of Volterra equations.

ACKNOWLEDGEMENTS

The author would like to thank Professor Dr. P.J. van der Houwen for his constructive remarks and valuable advice. He is also grateful to Mrs. Joke Blom who carried out the computations summarized in section 6 of this paper.

REFERENCES

[1] ANDRADE, C. & S. McKEE, *On optimal high accuracy linear multistep methods for first kind Volterra integral equations*, BIT 19 (1979), 1-11.

[2] BAKER, C.T.H., *The numerical treatment of integral equations*, Oxford, Clarendon Press, 1977.

[3] BAKER, C.T.H. & M.S. KEECH, *Stability regions in the numerical treatment of Volterra integral equations*, SIAM J. Numer. Anal. 15 (1978), 394-417.

[4] BAKER, C.T.H., A. MAKROGLOU & E. SHORT, *Regions of stability in the numerical treatment of Volterra integro-differential equations*, SIAM J. Numer. Anal. 16 (1979), 890-910.

[5] BRUNNER, H. & J.D. LAMBERT, *Stability of numerical methods for Volterra integro-differential equations*, Computing 12 (1974), 75-89.

[6] GLADWIN, C.J., *Quadrature rule methods for Volterra integral equations of the first kind*, Math. Comp. 33 (1979), 705-716.

[7] GLADWIN, C.J. & R. JELTSCH, *Stability of quadrature rule methods for first kind Volterra integral equations*, BIT 14 (1974), 144-151.

[8] HENRICI, P., *Discrete variable methods in ordinary differential equations*, New York, Wiley, 1962.

[9] HOCK, W.H., *Asymptotic expansions for multistep methods applied to non-linear Volterra integral equations of the second kind*, Numer. Math. 33 (1979), 77-100.

[10] HOLYHEAD, P.A.W., S. McKEE & P.J. TAYLOR, *Multistep methods for solving linear Volterra integral equations of the first kind*, SIAM J. Numer. Anal. 12 (1975), 698-711.

[11] HOUWEN, P.J. VAN DER, *On the numerical solution of Volterra integral equations of the second kind*, I. *Stability*, Report NW 42/77, Mathematisch Centrum, Amsterdam (1977).

[12] HOUWEN, P.J. VAN DER, *Convergence and stability results in Runge-Kutta type methods for Volterra integral equations of the second kind*, BIT 20 (1980), 375-377.

[13] HOUWEN, P.J. VAN DER, P.H.M. WOLKENFELT & C.T.H. BAKER, *Convergence and stability analysis for modified Runge-Kutta methods in the numerical treatment of second kind Volterra integral equations*, Report NW 96/80, Mathematisch Centrum, Amsterdam (1980). (to appear in IMA Journal on Numerical Analysis)

[14] LAMBERT, J.D., *Computational methods in ordinary differential equations*, London, Wiley, 1973.

[15] LINZ, P., *Linear multistep methods for Volterra integro-differential equations*, J. Assoc. Comput. Mach. 16 (1969), 295-301.

[16] MATTHYS, J., *A-stable linear multistep methods for Volterra integro-differential equations*, Numer. Math. 27 (1976), 85-94.

[17] TAYLOR, P.J., *The solution of Volterra integral equations of the first kind using inverted differentiation formulae*, BIT 16 (1977), 312-320.

[18] WOLKENFELT, P.H.M., P.J. VAN DER HOUWEN & C.T.H. BAKER, *Analysis of numerical methods for second kind Volterra equations by imbedding techniques*, J. of Integral Eqns. 3 (1981), 61-82.

[19] WOLKENFELT, P.H.M., *Reducible quadrature methods for Volterra integral equations of the first kind*, BIT (1981), to appear.

[20] WOLKENFELT, P.H.M., *The construction of reducible quadrature rules for Volterra integral and integro-differential equations*, submitted for publication.

[21] WOLKENFELT, P.H.M., *Modified multilag methods for Volterra functional equations*, Report NW 108/81, Mathematisch Centrum, Amsterdam (1981). (preprint)

## APPENDIX *Proofs of the theorems*

In our proofs we shall apply the following well-known lemmas.

LEMMA 1. *If* $|v_n| \le hA\sum_{j=0}^{n} |v_j| + B$ *for* $n = m(1)N$, *where* $h > 0$, $A > 0$, $B > 0$ *and* $|v_j| \le V$ *for* $j = 0(1)m-1$, *then for* $h$ *sufficiently small*

$$\max_{m\le n\le N} |v_n| \le (hA^*mV + B^*)\exp(A^*x),$$

*where* $Nh = X$, $A^* = (1-hA)^{-1}A$ *and* $B^* = (1-hA)^{-1}B$.

PROOF. See e.g. BAKER [2, p.925]. □

LEMMA 2. *Let the sequence* $\{v_n\}_{n=m}^{N}$ *satisfy*

$$\sum_{i=0}^{k} a_i v_{n-i} = z_n \text{ for } n \ge m+k, \qquad (a_0 \ne 0)$$

*where* $v_m,\ldots,v_{m+k-1}$ *are given and where* $\{z_n\}_{n=m+k}^{N}$ *is an arbitrary sequence. Let the polynomial* $\rho(\zeta) = \Sigma_{i=0}^{k} a_i \zeta^{k-i}$ *satisfy the root condition, then*

$$|v_n| \le C \sum_{j=m+k}^{n} |z_j| + D \sum_{j=m}^{m+k-1} |v_j|, \qquad m+k \le n \le N,$$

*where* $C$ *and* $D$ *are uniformly bounded constants independent of* $N$.

PROOF. Along the lines indicated in HENRICI [8, p.243]. □

Note that $z_n$ may depend on $v_0,\ldots,v_n$, i.e. $z_n=z_n(v_0,\ldots,v_n)$. We shall frequently use this convenient property in our proofs.

To save space, we also introduce the following notation:

(A.1) $\quad e_n := f(x_n) - f_n;$

(A.2) $\quad \Delta K_{nj} := K(x_n,x_j,f(x_j)) - K(x_n,x_j,f_j);$

(A.3) $\quad \Delta\tilde{I}_n(x_m) := \psi_n(x_m) - \tilde{I}_n(x_m), \quad m = n-k(1)n-1;$

(A.4) $\quad \Delta F_n^{(1)} := F(x_n,f(x_n),\hat{I}_n) - F(x_n,f_n,\hat{I}_n);$

(A.5) $\quad \Delta F_n^{(2)} := F(x_n,f(x_n),\psi_n(x_n)) - F(x_n,f(x_n),\hat{I}_n);$

(A.6) $\quad \Delta\hat{I}_n := \psi_n(x_n) - \hat{I}_n$

For the quantities defined above, the following useful inequalities can be derived. In view of the Lipschitz conditions (3.3) and (3.7)

(A.7) $$|\Delta K_{nj}| \le L_1 |e_j|,$$

(A.8) $$|\Delta K_{nj} - \Delta K_{n-i,j}| \le L_1^* \, ih |e_j|.$$

Since $\Delta\tilde{I}_n(x_m) = \psi_n(x_m) - \tilde{I}_n(x_m) = \psi_n(x_m) - \tilde{\psi}_n(x_m) + \tilde{\psi}_n(x_m) - \tilde{I}_n(x_m)$, where $\tilde{\psi}_n(x)$ is defined in (2.5), we may write using (2.11) and (A.2)

$$\Delta\tilde{I}_n(x_m) = Q_n(h;x_m) + h \sum_{j=0}^{n} w_{nj} \Delta K_{mj}.$$

As a consequence

(A.9) $$|\Delta\tilde{I}_n(x_m)| \le Q_N(h) + hL_1 \bar{w} \sum_{j=0}^{n} |e_j|,$$

where $Q_N(h)$ is defined in (2.15) and $\bar{w}$ is the uniform bound of $|w_{nj}|$.

Analogously we can derive, using (A.8), that

(A.10) $$|\Delta\tilde{I}_{n-i}(x_n) - \Delta\tilde{I}_{n-i}(x_{n-i})| \le \Delta Q_N(h) + h^2 L_1^* \bar{w} \sum_{j=0}^{n-i} |e_j|,$$

where $\Delta Q_N(h)$ is defined in (2.16).

From the Lipschitz conditions (4.3) it follows that

(A.11) $$|\Delta F_n^{(1)}| \le L_2 |e_n|, \quad |\Delta F_n^{(2)}| \le L_3 |\Delta\hat{I}_n|.$$

We remark that $C_i$ occurring in the proofs below denotes a generic uniformly bounded constant.

<u>PROOF OF THEOREM 3.1</u>. The solution of the continuous problem (3.1) satisfies $f(x_n) = g(x_n) + \psi_n(x_n)$, or using (2.9),

(A.12) $$f(x_n) = g(x_n) - \sum_{i=1}^{k} a_i \psi_{n-i}(x_n) + h \sum_{i=0}^{k} b_i K(x_n, x_{n-i}, f(x_{n-i})) + T_n(h;x_n).$$

Subtract $f_n$ defined by (3.2) from (A.12) to obtain the equation for the global error $e_n$

$$\text{(A.13)}\qquad e_n = -\sum_{i=1}^{k} a_i \Delta\tilde{I}_{n-i}(x_n) + h\sum_{i=0}^{k} b_i \Delta K_{n,n-i} + T_n(h;x_n), \quad n_k \le n \le N,$$

where $\Delta\tilde{I}_n(x)$ and $\Delta K_{nj}$ are defined in (A.3) and (A.2). Using (A.9), (A.7) and (2.14) yields

$$\text{(A.14)}\qquad |e_n| \le hC_1 \sum_{j=0}^{n} |e_j| + C_2 Q_N(h) + C_3 T_N(h), \quad n_k \le n \le N.$$

Finally, application of Lemma 1 to (A.14) yields the result (3.4). □

PROOF OF THEOREM 3.3. Analogous to (A.12) the solution of the continuous problem (3.1) satisfies

$$\text{(A.15)}\qquad f(x_n) = g(x_n) - \sum_{i=1}^{k} a_i\{\psi_{n-i}(x_n) + f(x_{n-i}) - g(x_{n-i}) - \psi_{n-i}(x_{n-i})\}$$
$$+ h\sum_{i=0}^{k} b_i K(x_n, x_{n-i}, f(x_{n-i})) + T_n(h;x_n),$$

where we have used that $f(x_{n-i}) = g(x_{n-i}) + \psi_{n-i}(x_{n-i})$. Subtract $f_n$ defined by (3.6) from (A.15) to obtain after some manipulations

$$\text{(A.16)}\qquad \sum_{i=0}^{k} a_i\, e_{n-i} = -\sum_{i=1}^{k} a_i\{\Delta\tilde{I}_{n-i}(x_n) - \Delta\tilde{I}_{n-i}(x_{n-i})\}$$
$$+ h\sum_{i=0}^{k} b_i \Delta K_{n,n-i} + T_n(h;x_n), \quad n_k \le n \le N.$$

Let $z_n$ denote the right-hand side of (A.16) then $|z_n|$ can be bounded by

$$\text{(A.17)}\qquad |z_n| \le h^2 C_1 \sum_{j=0}^{n-1} |e_j| + C_2 \Delta Q_N(h) + hC_3 \sum_{i=0}^{k} |e_{n-i}| + T_N(h),$$

where we have used (A.10), (A.7) and (2.14). Now equation (A.16) can be written as $\Sigma_{i=0}^{k} a_i e_{n-i} = z_n$ $(n \ge n_k = n_0 + k)$ and application of Lemma 2 yields the inequality

$$\text{(A.18)}\qquad |e_n| \le C_4 \sum_{j=n_0+k}^{n} |z_j| + C_5 \sum_{j=n_0}^{n_0+k-1} |e_j|, \quad n_k \le n \le N\ .$$

Substitution of (A.17) into (A.18) yields

$$(A.19)\qquad |e_n| \le hC_6 \sum_{j=0}^{n} |e_j| + C_7 h^{-1}\Delta Q_N(h) + C_8 h^{-1} T_N(h) + C_9 \delta_2(h)$$

where we have used that $nh \le X$ and where $\delta_2(h)$ is defined in (2.13). Finally, application of Lemma 1 to (A.19) yields the result (3.8). □

PROOF OF THEOREM 4.1. The solution of the continuous problem (4.1) satisfies

$$(A.20)\qquad \sum_{i=0}^{k} a_i^* f(x_{n-i}) = h \sum_{i=0}^{k} b_i^* F(x_{n-i}, f(x_{n-i}), \psi_{n-i}(x_{n-i})) + T_n^*(h;x_n)$$

where $T_n^*(h;x_n)$ denotes the local truncation error at $x=x_n$ of $(\rho^*,\sigma^*)$. Subtract (4.2a) from (A.20) to obtain

$$(A.21)\qquad \sum_{i=0}^{k} a_i^* e_{n-i} = h \sum_{i=0}^{k} b_i^*\{\Delta F_{n-i}^{(1)} + \Delta F_{n-i}^{(2)}\} + T_n^*(h;x_n), \quad n_k \le n \le N,$$

where we have used the notation (A.4) and (A.5). Let $z_n$ denote the right-hand side of (A.21), then $|z_n|$ can be bounded by

$$(A.22)\qquad |z_n| \le hC_1 \sum_{i=0}^{k} \{|e_{n-i}| + |\Delta\hat{I}_{n-i}|\} + T_N^*(h),$$

where we have used (A.11) and (4.6). Writing equation (A.21) as $\Sigma_{i=0}^{k} a_i^* e_{n-i} = z_n$ $(n_0+k \le n \le N)$ and applying Lemma 2 yields the inequality

$$(A.23)\qquad |e_n| \le C_2 \sum_{j=n_k}^{n} |z_j| + C_3 \sum_{j=n_0}^{n_k-1} |e_j|, \qquad n_k \le n \le N.$$

Substitution of (A.22) in (A.23) gives the inequality

$$(A.24)\qquad |e_n| \le hC_4 \sum_{j=n_0}^{n} \{|e_j| + |\Delta\hat{I}_j|\} + C_5 h^{-1} T_N^*(h) + C_6 \delta_2(h).$$

Next we derive an inequality for $\Delta\hat{I}_n := \psi_n(x_n) - \hat{I}_n$, where $\hat{I}_n$ is defined by (4.2b). Thus subtracting (4.2b) from (2.9) gives

$$\Delta\hat{I}_n = -\sum_{i=1}^{k} a_i \Delta\tilde{I}_{n-i}(x_n) + h \sum_{i=0}^{k} b_i \Delta K_{n,n-i} + T_n(h;x_n).$$

Use of (A.9) and (A.7) then yields the inequality

$$(A.25)\qquad |\Delta\hat{I}_n| \le C_7 Q_N(h) + hC_8 \sum_{j=0}^{n} |e_j| + T_N(h), \quad n_k \le n \le N.$$

Furthermore we have, in view of (4.2c), $\Delta\hat{I}_n = \Delta\tilde{I}_n(x_n)$ if $n_0 \le n \le n_k - 1$, which yields using (A.9)

$$(A.26)\qquad |\Delta\hat{I}_n| \le |Q_n(h;x_n)| + hC_9 \sum_{j=0}^{n} |e_j|, \qquad n_0 \le n \le n_k - 1.$$

Substitution of (A.25) and (A.26) into (A.24) gives

$$(A.27)\qquad |e_n| \le hC_{10} \sum_{j=0}^{n} |e_j| + C_{11} h\delta_3(h) + C_{12} Q_N(h) + C_{13} T_N(h)$$

$$+ C_5 h^{-1} T_N^*(h) + C_6 \delta_2(h), \qquad n_k \le n \le N,$$

where $\delta_3(h)$ is defined in (4.5). Application of Lemma 1 to (A.27) yields the result (4.4). □

PROOF OF THEOREM 4.3. The error equation for $e_n$ is the same as in the proof of Theorem 4.1. (equation (A.16)), so that we arrive at the inequality (A.24). The error equation for $\Delta\hat{I}_n$, however, is different and is derived as follows. Write (2.9) as

$$(A.28)\qquad \psi_n(x_n) = -\sum_{i=1}^{k} a_i\{\psi_{n-i}(x_n) + \psi_{n-i}(x_{n-i}) - \psi_{n-i}(x_{n-i})\}$$

$$+ h \sum_{i=0}^{k} b_i K(x_n, x_{n-i}, f(x_{n-i})) + T_n(h;x_n), \quad n_k \le n \le N.$$

Substitute $r_n$ defined by (4.8c) into (4.8b) and subtract the resulting equation from (A.28). We then obtain

$$\sum_{i=0}^{k} a_i \Delta\hat{I}_{n-i} = -\sum_{i=1}^{k} a_i\{\Delta\tilde{I}_{n-i}(x_n) - \Delta\tilde{I}_{n-i}(x_{n-i})\}$$

$$(A.29)\qquad + h \sum_{i=0}^{k} b_i \Delta K_{n,n-i} + T_n(h;x_n), \quad n_k \le n \le N.$$

Let $z_n$ denote the right-hand side of (A.29), then

$$\text{(A.30)} \qquad |z_n| \le C_7 \Delta Q_N(h) + h^2 C_8 \sum_{j=0}^{n-1} |e_j| + hC_9 \sum_{i=0}^{k} |e_{n-i}| + T_N(h),$$

where we have used (A.10) and (A.7). Writing (A.29) as $\Sigma_{i=0}^{k} a_i \Delta\hat{I}_{n-i} = z_n$ ($n_k \le n \le N$) and applying Lemma 2 yields

$$\text{(A.31)} \qquad |\Delta\hat{I}_n| \le C_{10} \sum_{j=n_k}^{n} |z_j| + C_{11} \sum_{j=n_0}^{n_k-1} |\Delta\hat{I}_j|, \qquad n_k \le n \le N.$$

Since $\hat{I}_n = \tilde{I}_n(x_n)$ for $n = n_0(1)n_k-1$, $\Delta\hat{I}_n$ equals $\Delta\tilde{I}_n(x_n)$ which yields

$$\text{(A.32)} \qquad |\Delta\hat{I}_n| \le |Q_n(h;x_n)| + hC_{12} \sum_{j=0}^{n} |e_j|, \qquad n_0 \le n \le n_k-1.$$

Substitution of (A.30) and (A.32) into (A.31) yields after some manipulations

$$\text{(A.33)} \qquad |\Delta\hat{I}_n| \le C_{13} h^{-1} \Delta Q_N(h) + hC_{14} \sum_{j=0}^{n} |e_j| + C_{15} h^{-1} T_N(h) + \delta_3(h),$$
$$n_k \le n \le N.$$

Next we substitute (A.32) and (A.33) into (A.24) to obtain

$$\text{(A.34)} \qquad |e_n| \le hC_{16} \sum_{j=0}^{n} |e_j| + C_{17} h^{-1} \Delta Q_N(h) + C_{18} h^{-1} T_N(h) +$$
$$+ C_{19}\delta_3(h) + C_5 h^{-1} T_N^*(h) + C_6 \delta_2(h), \qquad n_k \le n \le N.$$

Finally, application of Lemma 1 to (A.34) yields the result (4.11). □

PROOF OF THEOREM 5.1. The solution of the continuous problem (5.2) satisfies

$$\text{(A.35)} \qquad -\sum_{i=1}^{k} a_i \{\psi_{n-i}(x_n) + g(x_{n-i}) - \psi_{n-i}(x_{n-i})\} + hb_0 K(x_n, x_n, f(x_n)) +$$
$$T_n(h;x_n) = g(x_n).$$

Subtract (5.1) to obtain

$$hb_0\Delta K_{nn} = \sum_{i=1}^{k} a_i\{\Delta\tilde{I}_{n-i}(x_n) - \Delta\tilde{I}_{n-i}(x_{n-i})\} - T_n(h;x_n).$$

which yields the inequality

(A.36) $$h|b_0|L_4|e_n| \le C_1\Delta Q_N(h) + C_2h^2 \sum_{j=0}^{n-1} |e_j| + T_N(h), \quad n_k \le n \le N,$$

where we have used (A.10) and (2.14) and the fact that $|\Delta K_{nn}| \ge L_4|e_n|$ (see condition (5.3)). Dividing through by $h|b_0|$ and applying Lemma 1 yields the result (5.4). □