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STABILITY ANALYSIS OF NUMERICAL METHODS FOR VOLTERRA INTEGRAL EQUATIONS WITH POLYNOMIAL CONVOLUTION KERNELS

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Stability analysis of numerical methods for Volterra integral equations with polynomial convolution kernels*

by

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ABSTRACT

Direct quadrature methods and classical and modified Runge-Kutta methods yield structured systems of equations when applied to a class of test equations with polynomial convolution kernels. Exploiting the structure of the results obtained in the stability analysis for a basic test equation, we derive finite recurrence relations which enable us to relate the stability properties of the numerical methods to the location of the zeros of appropriate stability polynomials.

KEY WORDS & PHRASES: *Numerical analysis, Volterra integral equations of the second kind, stability, test equations with polynomial convolution kernels*

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*This report will be submitted for publication elsewhere.

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1. INTRODUCTION

1.1 Background and outline of the paper

We shall discuss the stability of a general class of methods, and certain modified forms, for the numerical solution of the Volterra integral equation

$$(1.1) \qquad f(x) - \int_0^x H(x,y)f(y)dy = g(x) \qquad (x \geq 0),$$

where H(x,y) and g(x) are given and continuous. Convergence of the numerical methods has been discussed in the literature and will not be considered here. The literature also contains a number of contributions to stability theory, which vary in the class of methods considered and the special case of (1.1) adopted as a test equation.

A test equation which has received particular emphasis is the basic equation

$$(1.2) \qquad f(x) - \lambda \int_0^x f(y)dy = g(x).$$

Results for this equation can be deduced from those for the equation

$$(1.3) \qquad f(x) - \int_0^x \sum_{s=0}^{S} \lambda_s (x-y)^s f(y)dy = g(x)$$

which we study here. We denote by K(x-y) the polynomial convolution kernel

$$(1.4) \qquad K(x-y) = \sum_{s=0}^{S} \lambda_s (x-y)^s$$

occurring in (1.3).

Stability of *quadrature methods* and *classical Runge-Kutta methods* applied to (1.2) has been discussed in [6], and [2,6] respectively. The stability of quadrature methods with a finite repetition factor (see [4]) was discussed in [14], in the context of (1.3). The case S = 1 in (1.3) was investigated for a special class of quadrature methods in [17] and for a large class of Runge-Kutta methods in [13] (including Beltyukov methods and the modified

Runge-Kutta methods introduced in [10,11]).

The institute report [3] was devoted to a study of stability of a rather wide class of quadrature and classical Runge-Kutta methods, applied to to (1.3) with arbitrary S. Our aim in the present work is to convey the structure of certain results in [3] and their extension to the modified Runge-Kutta methods. Our study thus covers a wide range of numerical methods.

REMARK. The authors of [3] and [12] emphasize that the behaviour of the kernel which they rely on, in their studies, is the polynomial dependence on x of H(x,y). Although the convolution property yields simplification and structure, their techniques yield results for kernels such as

(1.5) $$H(x,y) = \lambda_0 + \lambda_1^0 x + \lambda_1^1 y$$

investigated in [1,10]. (Both (1.5) and (1.4) are particular cases of kernels of the form

(1.6) $$H(x,y) = \sum_{s=0}^{S} X_s(x)Y_s(y)$$

studied in [5,12] by less direct means.) □

1.2 Preliminaries

The general form of the equations obtained by applying the *quadrature methods* and the *classical Runge-Kutta methods* (of the type considered by POUZET [9]) is given by $f_0 = g(0)$ and

(1.7) $$f_j - h \sum_{k\geq 0} \Omega_{jk} H(\tau_j,\tau_k) f_k = g(\tau_j), \qquad j = 1,2,\ldots ,$$

where $\{\tau_j\}$ are grid-points to be specified below.

The quadrature methods are defined by a family of quadrature rules

(1.8) $$\int_0^{nh} \phi(y)dy \simeq h \sum_{k\geq 0} \omega_{nk}\phi(kh), \qquad n = 1,2,3,\ldots,$$

with

(1.9) $\omega_{nk} = 0$ for $k > n$.

We then take $\Omega_{nk} = \omega_{nk}$, $\tau_k = kh$ in (1.7).

Our Runge-Kutta methods are defined by a tableau

$$(1.10)\qquad \underset{\sim}{\theta}|\underset{\approx}{A} = \begin{array}{c|ccccc} \theta_0 & A_{00} & A_{01} & \cdots & A_{0p} \\ \theta_1 & A_{10} & A_{11} & \cdots & A_{1p} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \theta_{p-1} & A_{p-1,0} & A_{p-1,1} & \cdots & A_{p-1,p} \\ \hline \theta_p & A_{p0} & A_{p1} & \cdots & A_{pp} \end{array}$$

where $\theta_p = 1$. (Such arrays with $A_{rp} = 0$ for $r = 0,1,\ldots,p$, occur in Runge-Kutta methods for differential equations.)

The extended Runge-Kutta methods depend upon (1.10) whilst the mixed quadrature-Runge-Kutta methods involve also the rules (1.8). In each we take $\tau_0 = 0$ and

(1.11) $\tau_j = ih + \theta_r h$, $\qquad j = i(p+1)+r+1$.

Thus

(1.12) $i = [(j-1)/(p+1)]$, $\qquad r \equiv (j-1)\bmod(p+1)$.

We shall *reserve* the integers i,j,r for use according to (1.12) henceforth, in this paper. The extended Runge-Kutta method results from the choice

$$(1.13a)\qquad \Omega_{jk} = \begin{cases} A_{pt}, & 0<k\le i(p+1) \\ A_{rt}, & i(p+1)<k\le(i+1)(p+1) \\ 0 & \text{otherwise;} \end{cases}$$

the mixed quadrature-Runge-Kutta method from the choice

$$(1.13b)\qquad \Omega_{jk} = \begin{cases} \omega_{im}, & k=m(p+1),\ \ m\le i \\ A_{rt}, & i(p+1)<k\le(i+1)(p+1) \\ 0 & \text{otherwise}; \end{cases}$$

with $t \equiv (k-1)\bmod(p+1)$ in (1.13a,b).

Observe that for both cases

$$(1.14)\qquad \Omega_{jk} = \Omega_{i(p+1)+1,k} \qquad \text{for } k \le i(p+1);$$

$$(1.15)\qquad \Omega_{jk} = 0 \text{ for } k > (i+1)(p+1);$$

these we assume in what follows.

The methods given above are classical. The modified methods [11] depend upon a choice of $\gamma_r \varepsilon [0,1]$, $r = 0,1,\ldots,p$, defining

$$(1.16)\qquad \underset{\sim}{\gamma} = [\gamma_0,\gamma_1,\ldots,\gamma_p]^T.$$

Here, the equations defining the values

$$(1.17)\qquad f_j,\ j = i(p+1)+r+1 \qquad (r = 0,1,\ldots,p;\ i = 0,1,2,\ldots)$$

in terms of $f_0 = g(0)$ and the values $f_1, f_2,\ldots, f_{i(p+1)}$ previously computed by the modified method involve the "lag term", or "history term",

$$(1.18)\qquad \eta_i(x) = g(x) + h\sum_{k=0}^{i(p+1)} \Omega_{i(p+1)+1,k}H(x,\tau_k)f_k.$$

With our convention $i = [(j-1)/(p+1)]$, $r \equiv (j-1)\bmod(p+1)$, we define

$$(1.19)\qquad \eta_j = \eta_i(\tau_j), \qquad \hat{\eta}_i = \eta_i(ih).$$

(Observe that if $\theta_0 = 0$ then $\hat{\eta}_i = \eta_{i(p+1)+1}$.) Then we set

$$(1.20)\qquad f_j = \eta_j + \gamma_r\{f_{i(p+1)}-\hat{\eta}_i\}+h\sum_{k>i(p+1)} \Omega_{jk}H(\tau_j,\tau_k)f_k.$$

The last term in (1.20) is the "Runge-Kutta part"

$$h \sum_{t=0}^{p} A_{rt} H(ih+\theta_r h, ih+\theta_t h) f_{i(p+1)+t+1}.$$

When Ω_{jk} are determined by (1.13b) we have

$$\eta_i(x) = g(x) + h \sum_{k=0}^{i} \omega_{ik} H(x,kh) f_{k(p+1)}. \tag{1.21}$$

REMARK. The modified methods collapse to the classical methods if $\underset{\sim}{\gamma} = \underset{\sim}{0}$, if $p = 0$, or if Ω_{jk} are defined by (1.13a). Following [10,11,13] we regard $\eta_j^* = \eta_j + \gamma_r\{f_{i(p+1)} - \hat{\eta}_i\}$ as a revised lag term determined by (1.16) and by the residual

$$f_{i(p+1)} - \hat{\eta}_i \equiv f_{i(p+1)} - g(ih) - \sum_{k=0}^{i(p+1)} \Omega_{i(p+1)+1,k} H(ih,\tau_k) f_k. \quad \square$$

In both classical and modified Runge-Kutta methods the values f_j, $j = i(p+1)+r+1$, are associated in blocks defining

$$\underset{\sim}{f}_{i+1} = [f_{i(p+1)+1}, f_{i(p+1)+2}, \ldots, f_{(i+1)(p+1)}]^T . \tag{1.22}$$

It is convenient to write also

$$\underset{\sim}{\eta}_{i+1} = [\eta_{i(p+1)+1}, \eta_{i(p+1)+2}, \ldots, \eta_{(i+1)(p+1)}]^T \tag{1.23}$$

and to denote by

$$\underset{\sim}{e}_0, \underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_p, \underset{\sim}{e} \tag{1.24}$$

the successive columns of the identity matrix of order (p+1) and their sum. The principal purpose of the Runge-Kutta methods may be considered to be the generation of the "*full-step*" *values* $f_{i(p+1)} \equiv \underset{\sim}{e}_p^T \underset{\sim}{f}_i$ approximating $f(ih)$ $(i = 1,2,3,\ldots)$.

1.3. Stability

Stability properties, both of the integral equation (1.3) and the numerical methods discussed here, can be related to the location of zeros of appropriate polynomials.

For the integral equation, we may obtain the root conditions by reducing (1.3) to a system of differential equations. Thus (1.3) is, for $\underset{\sim}{\lambda} \in \mathbb{R}^{S+1}$, asymptotically stable if and only if $\Xi_S(\mu) := \mu^{S+1} - \sum_{s=0}^{S} \lambda_s s! \mu^{S-s}$ has its zeros in $\mathbb{C}_- := \{\mu \in \mathbb{C} | \mathrm{Re}(\mu) < 0\}$ and stable if and only if its zeros lie in the closure of $\mathbb{C}_-$, $\bar{\mathbb{C}}_- := \{\mu \in \mathbb{C} | \mathrm{Re}(\mu) \leq 0\}$ the zeros having $\mathrm{Re}(\mu) = 0$ being required to be simple. The requirements can be expressed in terms of conditions [1,2] on $\lambda_0, \lambda_1, \ldots, \lambda_S$ using the Routh-Hurwitz criteria.

We can reduce (1.3) to a system of differential equations. Likewise, our analysis relies on the reduction of the corresponding "summation" equations (1.7) to a finite term recurrence, of the form $\sum_{\ell=0}^{m} \underset{\sim}{X}_\ell\, \underset{\sim}{\phi}_{n-\ell} = \underset{\sim}{\delta}_n$, where $\underset{\sim}{X}_\ell \equiv X_\ell(h; \lambda_0, \lambda_1, \ldots, \lambda_S)$ and the components of $\underset{\sim}{\phi}_{n-\ell}$ are values f_j. Such a recurrence is "strictly stable" if and only if the zeros of the *stability polynomial* det $[\sum_{\ell=0}^{m} \underset{\sim}{X}_\ell \mu^{m-\ell}]$ satisfy $|\mu| < 1$ and stable if and only if they satisfy $|\mu| \leq 1$, those with $|\mu| = 1$ being semi-simple.

The relationship between the conditions on (1.4) and the conditions on the numerical methods is of practical interest but is not pursued here. Our aim is to show how finite term recurrence relations can be obtained, and to derive the appropriate stability polynomials for the methods applied to (1.3).

In what follows the advancement operator is denoted by E:

$$E\underset{\sim}{\phi}_n = \underset{\sim}{\phi}_{n+1}$$

for vectors $\underset{\sim}{\phi}_n$ and scalars alike. Relations $\sum_{\ell=0}^{m} \underset{\sim}{X}_\ell\, \underset{\sim}{\phi}_{n-\ell} = \underset{\sim}{\delta}_n$ thus give

$$\{\sum_{\ell=0}^{m} \underset{\sim}{X}_\ell\, E^{m-\ell}\}\underset{\sim}{\phi}_{n-m} = \underset{\sim}{\delta}_n$$

2. QUADRATURE METHODS

The classical quadrature method defined by (1.8) yields $f_0 = g(0)$ and, when applied to (1.1),

$$(2.1) \qquad f_n - h \sum_{k \geq 0} \omega_{nk} H(nh, kh)\, f_k = g(nh) \quad (n = 1, 2, \ldots)$$

wherein $f_k \simeq f(kh)$ and

$$\omega_{nk} = 0, \qquad k > n. \tag{2.2}$$

2.1. Reducible quadrature methods

First we consider the case where parameters $\{\alpha_\ell, \beta_\ell\}_{\ell=0}^{m}$ exist such that [8]

$$\sum_{\ell=0}^{m} \alpha_\ell \, \omega_{n-\ell,j} = \beta_{n-j} \qquad (j = 0,1,\ldots,n) \tag{2.3}$$

with the convention that $\beta_\ell = 0$ if $\ell \notin \{0,1,\ldots,m\}$, $\alpha_0 \neq 0$.

In this case the quadrature rules (1.8) are called $\{\rho,\sigma\}$-*reducible* where

$$\rho(\mu) = \sum_{\ell=0}^{m} \alpha_\ell \, \mu^{m-\ell}, \quad \sigma(\mu) = \sum_{\ell=0}^{m} \beta_\ell \mu^{m-\ell} \tag{2.4}$$

and the quadrature method (2.1) will be called *a reducible quadrature method.*

REMARK. The polynomials $\rho(\mu), \sigma(\mu)$ are the first and second characteristic polynomials of an associated linear multistep method [7] for which the usual conditions are $\rho(1) = 0$, $\rho'(1) = \sigma(1)$. We assume that $\rho(\mu)$, $\sigma(\mu)$ have no common factors. It can happen that $\rho(\mu)$ has a root $\mu = 0$ of multiplicity ν in which case $\alpha_\ell = 0$ for $\ell = m, m-1, \ldots, m_0+1$ where $m_0 = m-\nu$ and (2.3) reduces to its minimal form

$$\sum_{\ell=0}^{m_0} \alpha_\ell \, \omega_{n-\ell,j} = \beta_{n-j}. \tag{2.5}$$

The assumption of reducibility (varied later) imposes a structure on the quadrature method which we may exploit, to obtain from (2.1) a finite-term recurrence between successive values f_n. The main tool is the repeated formation of linear combinations of (2.1) with varying n and corresponding weighting factors α_ℓ. □

In our present analysis we employ, in addition to (2.4), the polynomials

$$\text{(2.6a)} \qquad \sigma_t(\mu) = \sum_{\ell=0}^{m} \ell^t \beta_\ell \mu^{m-\ell} \qquad t = 0,1,2,\ldots;$$

$$\text{(2.6b)} \qquad \rho_t(\mu) = \sum_{\ell=0}^{m_0} \ell^t \alpha_\ell \mu^{m_0-\ell} \qquad t = 0,1,2,\ldots;$$

$$\text{(2.6c)} \qquad \hat{\rho}_{tr}(\mu) = \begin{cases} \binom{t}{r}\rho_{t-r}(\mu) & t = 0,1,2,\ldots;\ r = 0,1,\ldots t; \\ 0 & \text{otherwise} \end{cases}$$

Here $\rho_{tt}(\mu) \equiv \rho_0(\mu)$ and $m_0 = \max\{\ell | \alpha_\ell \neq 0,\ \ 0 \leq \ell \leq m\}$. Thus, $\rho(\mu) = \mu^{m-m_0}\rho_0(\mu)$, $\sigma_0(\mu) = \sigma(\mu)$.

2.2. A stability polynomial for reducible quadrature methods

Theorems 2.1 and 2.2 below state the required results for $\{\rho,\sigma\}$-reducible quadrature rules applied to (1.3) and will be obtained as a consequence of a sequence of lemmata which are useful later.

When $H(x,y)$ is replaced in (2.1) by $K(x-y) = \sum_{s=0}^{S} \lambda_s (x-y)^s$, equation (2.1) yields

$$\text{(2.7)} \qquad f_n = g(nh) + \sum_{s=0}^{S} \lambda_s a_n^{(s)}$$

where

$$\text{(2.8)} \qquad a_n^{(s)} = h \sum_{k \geq 0} \omega_{nk} (nh-kh)^s f_k.$$

LEMMA 2.1. *Let the quadrature rules* (1.8) *be* $\{\rho,\sigma\}$*-reducible. Then* $\{a_n^{(s)}\}_{n\geq 0}$ *satisfies a recurrence relation with constant coefficients of the form*

$$\text{(2.9)} \qquad \{\rho_0(E)\}^s \rho(E)\, a_n^{(s)} = h^{s+1} N_{s+1}(E)\, f_n$$

where the polynomial $N_{s+1}(\mu)$ *is defined recursively by*

$$N_{s+1}(\mu) = \{\rho_0(\mu)\}^s \sigma_s(\mu) - \sum_{t=0}^{s-1} \hat{\rho}_{st}(\mu)\{\rho_0(\mu)\}^{s-1-t} N_{t+1}(\mu)$$

with $N_1(\mu) = \sigma_0(\mu)$.

PROOF. The proof is by induction on s; details are given in the appendix. □

A useful adjunct to Lemma 2.1 is the following result.

LEMMA 2.2. *For* $s = 0,1,2,\ldots,$

$$N_{s+1}(\mu) = (-1)^s \det \begin{bmatrix} \sigma_0(\mu) & \rho_0(\mu) & & & \\ \sigma_1(\mu) & \rho_{1,0}(\mu) & \rho_0(\mu) & & \\ \vdots & & & \ddots & \rho_0(\mu) \\ \sigma_s(\mu) & \rho_{s,0}(\mu), & \ldots, & & \rho_{s,s-1}(\mu) \end{bmatrix} . \tag{2.10}$$

PROOF. Expand the determinant by the last row. □

LEMMA 2.3. *Let the quadrature rules* (1.8) *be* $\{\rho,\sigma\}$*-reducible and let*

$$\phi_n = h \sum_{k=0}^{n} \omega_{nk}\{\sum_{s=0}^{S} \Lambda_s(h)(nh-kh)^s\} f_k \tag{2.11}$$

for arbitrary functions $\{\Lambda_s(h)\}$ *defining a vector* $\underset{\sim}{\Lambda}(h) = [\Lambda_0(h),\Lambda_1(h),\ldots,\Lambda_S(h)]^T$. *Then* $\{\phi_n\}_{n\geq 0}$ *satisfies the recurrence relation*

$$\{\rho_0(E)\}^S \rho(E)\,\phi_n = h\, R_{S+1}(\underset{\sim}{\Lambda}(h);E) f_n \tag{2.12}$$

where $R_{S+1}(\underset{\sim}{\Lambda}(h);\mu)$ *is a polynomial in* μ *given by*

$$R_{S+1}(\underset{\sim}{\Lambda}(h);\mu) = \sum_{s=0}^{S} \Lambda_s(h) h^s \{\rho_0(\mu)\}^{S-s} N_{s+1}(\mu).$$

PROOF. Write $\phi_n = \sum_{s=0}^{S} \Lambda_s(h) a_n^{(s)}$ and apply $\{\rho_0(E)\}^S \rho(E)$ to both sides. Application of Lemma 2.1 yields the result. □

REMARK. $R_{S+1}(\underset{\sim}{\Lambda}(h);\mu)$ depends linearly upon $\underset{\sim}{\Lambda}(h)$. □

<u>THEOREM 2.1</u>. *Let* $H(x,y) = \sum_{s=0}^{S}\lambda_s(x-y)^s$, $\lambda_S \neq 0$, *and let* $\{f_n\}_{n\geq 0}$ *satisfy* (2.1). *If the rules* (1.8) *are* $\{\rho,\sigma\}$*-reducible then*

$$\{\rho_0(E)\}^S\rho(E)(f_n - g(nh)) = h\, R_{S+1}(\boldsymbol{\lambda};E)f_n \tag{2.13}$$

where $\boldsymbol{\lambda} = [\lambda_0,\lambda_1,\ldots,\lambda_S]^T$.

<u>PROOF</u>. Apply Lemma 2.3 with $\phi_n = f_n - g(nh)$ and $\boldsymbol{\Lambda}(h) = \boldsymbol{\lambda}$. □

The following result is now an immediate corollary of the previous theorem. Since our emphasis is on stability polynomials we state it as a theorem.

<u>THEOREM 2.2</u>. *Under the assumptions of Theorem* 2.1 *a stability polynomial for the sequence* $\{f_n\}_{n\geq 0}$ *is*

$$Q_{S+1}(\boldsymbol{\lambda};\mu) = \{\rho_0(\mu)\}^S\rho(\mu) - h\, R_{S+1}(\boldsymbol{\lambda};\mu). \tag{2.14}$$

2.3. <u>Diagonally-block-reducible quadrature methods</u>

The assumption that the rules (1.8) are $\{\rho,\sigma\}$-reducible is somewhat restrictive. However, it is commonplace [6] to find that the array of weights ω_{nk} in (1.8) can be written in the form

$$\omega_{nk} = \begin{array}{c|c|c|c|c|c|c|c|c|c|}
\cdot & \cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot & & \\
\mathbf{W} & \mathbf{W}_0 & \cdot\;\cdot\;\cdot\;\cdot & \mathbf{W}_0 & \mathbf{W}_1 & \mathbf{W}_2 & \cdot\;\cdot\;\cdot & \mathbf{W}_P & & \\
\mathbf{W} & \mathbf{W}_0 & \cdot\;\cdot\;\cdot\;\cdot & \mathbf{W}_0 & \mathbf{W}_0 & \mathbf{W}_1 & \cdot\;\cdot\;\cdot & \mathbf{W}_{P-1} & \mathbf{W}_P & \\
\mathbf{W} & \mathbf{W}_0 & \cdot\;\cdot\;\cdot\;\cdot & \mathbf{W}_0 & \mathbf{W}_0 & \mathbf{W}_0 & \cdot\;\cdot\;\cdot & \mathbf{W}_{P-2} & \mathbf{W}_{P-1} & \mathbf{W}_P \\
\cdot & \cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot & \cdot
\end{array} \tag{2.15}$$

where $\mathbf{W}, \mathbf{W}_0,\ldots,\mathbf{W}_P$ are square matrices of order q, say, and we set $\mathbf{W}_{P+k} = \mathbf{0}$ $(k>0)$. Thus, defining the square matrices $\mathbf{V}_{nk}$ with elements

$$\mathbf{e}_\alpha^T\, \mathbf{V}_{nk}\, \mathbf{e}_\beta = \omega_{nq+\alpha,kq+\beta} \qquad (\alpha,\beta=0,1,\ldots,q-1). \tag{2.16}$$

and $\mathbf{W}_{-k} = \mathbf{W}_0$ for $k = 1,2,\ldots$, we can describe (2.15) by the relation $\mathbf{V}_{nk} = \mathbf{W}_{P-n+k}$ $(k=1,2,\ldots,n)$ and $\mathbf{V}_{n0} = \mathbf{W}$.

REMARK. Inconsequential changes in the above relations should be made when $\underset{\sim}{W}$ is rectangular rather than square and the pattern in (2.15) holds only for $n \geq n_0 \neq 0$. Observe that (2.15) implies that the rules have a finite "repetition factor" [3]. □

We may relate the rules with the above structure to a generalization of the $\{\rho,\sigma\}$-reducible rules. Quadrature rules are said to be *block-reducible* when the matrices $\underset{\sim}{V}_{nk}$ satisfy, for fixed matrices $\{\underset{\sim}{A}_\ell, \underset{\sim}{B}_\ell\}_{\ell=0}^{m}$, with $\sum_{\ell=0}^{m} \underset{\sim}{A}_\ell \underset{\sim}{e} = \underset{\sim}{0}$ and $\underset{\sim}{B}_\ell = \underset{\sim}{0}$ if $\ell \notin \{0,1,\ldots,m\}$ the relations

$$(2.17) \qquad \sum_{\ell=0}^{m} \underset{\sim}{A}_\ell \underset{\sim}{V}_{n-\ell,k} = \underset{\sim}{B}_{n-k} \qquad (k = 0,1,\ldots,n).$$

The stability polynomial for block-reducible methods applied to (1.2) can be shown to be $\det [\sum_\ell \{\underset{\sim}{A}_\ell - \lambda h \underset{\sim}{B}_\ell\} \mu^{m-\ell}]$. In order to achieve a simple generalization for (1.3) further assumptions will be made.

The rules in (2.15) are observed to be block-reducible on taking $m = P$, $\underset{\sim}{A}_0 = \underset{\sim}{I}$, $\underset{\sim}{A}_1 = -\underset{\sim}{I}$ and $\underset{\sim}{A}_\ell = \underset{\sim}{0}$, $\ell \neq 0,1$, $\underset{\sim}{B}_\ell = \underset{\sim}{W}_{P-\ell} - \underset{\sim}{W}_{P-\ell+1}$. Thus, they have the special feature that the corresponding matrices $\underset{\sim}{A}_\ell$ are diagonal; this proves a useful assumption when considering (1.3). When each matrix $\underset{\sim}{A}_\ell$ is diagonal we shall call the block-reducible rule *diagonally-block-reducible*; to emphasize we then write

$$(2.18) \qquad \underset{\sim}{A}_\ell = \underset{\sim}{\Delta}_\ell \qquad (\ell = 0,1,\ldots,m),$$

$\underset{\sim}{\Delta}_\ell$ being diagonal.

2.4. A stability polynomial for diagonally-block-reducible methods

To exploit (2.17), (2.18), we wish to write the equations (2.1) with $H(x,y) = K(x-y)$ in vector form involving the matrices $\underset{\sim}{V}_{nk}$. We define

$$(2.19) \qquad \underset{\sim}{\psi}_n = [f_{nq}, f_{nq+1}, \ldots, f_{(n+1)q-1}]^T$$

and the matrices $\underset{\sim}{K}_{nk}$ with entries

$$(2.20) \qquad \underset{\sim}{e}_\alpha^T \underset{\sim}{K}_{nk} \underset{\sim}{e}_\beta = K((nq+\alpha)h-(kq+\beta)h) \qquad (\alpha,\beta = 0,1,\ldots,q-1).$$

We also write

$$(2.21) \qquad \underset{\sim}{g}_n = [g(nqh),\ g((nq+1)h),\ldots,g((n+1)qh-h)]^T.$$

Since $K(x-y) = \sum_{s=0}^{S} \lambda_s (x-y)^s$,

$$(2.22) \qquad \underset{\sim}{K}_{nk} = \sum_{s=0}^{S} \lambda_s \underset{\sim}{M}_{n-k}^{(s)}$$

where elements of $\underset{\sim}{M}_{n-k}^{(s)}$ are

$$(2.23) \qquad \underset{\sim}{e}_\alpha^T \underset{\sim}{M}_{n-k}^{(s)} \underset{\sim}{e}_\beta = \{(n-k)qh + (\alpha-\beta)h\}^s$$

Now let $\underset{\sim}{G}$, $\underset{\sim}{H}$ be two matrices of the same size with elements $G_{\alpha\beta}$, $H_{\alpha\beta}$ respectively. Then the *pointwise* (or Schur) *product* denoted $\underset{\sim}{G} * \underset{\sim}{H}$ is the matrix with elements $G_{\alpha\beta}H_{\alpha\beta}$. With this notation,

$$(2.24) \qquad \underset{\sim}{\psi}_n = \underset{\sim}{g}_n + h \sum_{k=0}^{n} (\underset{\sim}{V}_{nk} * \underset{\sim}{K}_{nk})\underset{\sim}{\psi}_k.$$

Employing (2.22),

$$(2.25) \qquad \underset{\sim}{\psi}_n = \underset{\sim}{g}_n + \sum_{s=0}^{S} \lambda_s \underset{\sim}{a}_n^{(s)}$$

where

$$(2.26) \qquad \underset{\sim}{a}_n^{(s)} = h \sum_{k=0}^{n} \underset{\sim}{V}_{nk} * \underset{\sim}{M}_{n-k}^{(s)} \underset{\sim}{\psi}_k.$$

To proceed, we introduce

$$(2.27) \qquad \underset{\sim}{P}(\mu) = \sum_{\ell=0}^{m} \underset{\sim}{\Delta}_\ell \mu^{m-\ell}$$

$$(2.28) \qquad \underset{\sim}{P}_t(\mu) = \sum_{\ell=0}^{m_0} (\ell q)^t \underset{\sim}{\Delta}_\ell \mu^{m_0-\ell}$$

where m_0 denotes the largest integer such that $\underset{\sim}{\Delta}_{m_0} \neq \underset{\sim}{0}$, and

(2.29) $$\underset{\sim}{\Sigma}_t(\mu) = \sum_{\ell=0}^{m} h^{-t}(\underset{\sim}{B}_\ell * \underset{\sim}{M}_\ell^{(t)})\mu^{m-\ell}$$

LEMMA 2.4. *Let the rules* (1.8) *be diagonally-block-reducible. Then* $\{\underset{\sim}{a}_n^{(s)}\}_{n\geq 0}$ *satisfies a recurrence relation with constant coefficients of the form*

(2.30) $$\{\underset{\sim}{P}_0(E)\}^s \, \underset{\sim}{P}(E)\underset{\sim}{a}_n^{(s)} = h^{s+1} \, \underset{\sim}{N}_{s+1}(E) \, \underset{\sim}{\psi}_n,$$

where $\underset{\sim}{N}_{s+1}(E)$ *is defined recursively by*

(2.31) $$\underset{\sim}{N}_{s+1}(\mu) = \{\underset{\sim}{P}_0(\mu)\}^s \underset{\sim}{\Sigma}_s(\mu) - \sum_{t=0}^{s-1} \binom{s}{t} \underset{\sim}{P}_{s-t}(\mu)\{\underset{\sim}{P}_0(\mu)\}^{s-1-t}\underset{\sim}{N}_{t+1}(\mu)$$

with $N_1(\mu) = \Sigma_0(\mu)$.

PROOF. See Appendix. □

LEMMA 2.5. *Let the rules* (1.8) *be diagonally-block-reducible and let*

(2.32) $$\underset{\sim}{\phi}_n = h \sum_{k=0}^{n} (\underset{\sim}{V}_{nk} * \{\sum_{s=0}^{S} \Lambda_s(h)\underset{\sim}{M}_{n-k}^{(s)}\})\underset{\sim}{\psi}_k.$$

Then $\{\phi_n\}_{n\geq 0}$ *satisfies the recurrence*

(2.33) $$\{\underset{\sim}{P}_0(E)\}^S \, \underset{\sim}{P}(E) \, \underset{\sim}{\phi}_n = h \, \underset{\sim}{R}_{S+1}(\underset{\sim}{\Lambda}(h);E)\underset{\sim}{\psi}_n$$

where

(2.34) $$\underset{\sim}{R}_{S+1}(\underset{\sim}{\Lambda}(h);\mu) = \sum_{s=0}^{S} \Lambda_s(h)h^s\{\underset{\sim}{P}_0(\mu)\}^{S-s}\underset{\sim}{N}_{s+1}(\mu).$$

PROOF. Apply Lemma 2.4. □

Our principal results for this section now follow.

THEOREM 2.3. *Let* $H(x,y) = \sum_{s=0}^{S} \lambda_s(x-y)^s$. *If the vectors* (2.19) *satisfy* (2.24) *and the quadrature rules are diagonally-block-reducible, then*

(2.35) $$\{\underset{\sim}{P}_0(E)\}^S \, \underset{\sim}{P}(E)\{\underset{\sim}{\psi}_n - \underset{\sim}{g}_n\} = h \, \underset{\sim}{R}_{S+1}(\underset{\sim}{\lambda};E)\underset{\sim}{\psi}_n$$

PROOF. Use Lemma 2.5 (compare with the proof of Theorem 2.1). □

As a corollary we find

THEOREM 2.4. *Under the assumptions of Theorem* 2.3, *a stability polynomial for the sequence* $\{\underset{\sim}{\psi}_n = [f_{nq}, f_{nq+1}, \ldots, f_{(n+1)q-1}]^T\}_{n\geq 0}$ *is*

$$(2.36) \qquad \det[\{\underset{\sim}{P}_0(\mu)\}^S \underset{\sim}{P}(\mu) - h\, \underset{\sim}{R}_{S+1}(\underset{\sim}{\lambda};\mu)].$$

3. MIXED QUADRATURE-RUNGE-KUTTA METHODS

We now consider the classical and modified mixed methods defined by (1.7), (1.13b) and (1.19), (1.20), (1.21) respectively. The analysis presented for the quadrature methods can be adapted to treat these methods applied to (1.3). We rely in particular on Lemma 2.3 and Lemma 2.5. Thus, for the classical mixed method employing $\{\rho,\sigma\}$-reducible rules, we can find (see [3]) a relation of the form of (2.11), namely

$$(3.1) \qquad f_{(n+1)(p+1)} = h \sum_{k=0}^{n} \omega_{nk} \left\{ \sum_{s=0}^{S} \Lambda_s(h)(nh-kh)^s \right\} f_{k(p+1)} + \hat{g}(nh).$$

Thus, see (2.12), $\{\rho_0(E)\}^S \rho(E) f_{(n+1)(p+1)} - h\, R_{S+1}(\underset{\sim}{\Lambda}(h);E) f_{n(p+1)} = \delta_n$ (for some term δ_n depending on g(x)) and obtaining a stability polynomial is immediate when the values of $\Lambda_s(h)$ are determined from $\{\lambda_s\}$.

Here, we shall treat the classical and modified mixed methods assuming either $\{\rho,\sigma\}$-reducibility or diagonal-block-reducibility of the rules (1.8).

When H(x,y) is replaced in (1.21) by K(x-y) we obtain

$$(3.2) \qquad \eta_i(x) = g(x) + h \sum_{k=0}^{i} \omega_{ik}\, K(x-kh)\, f_{k(p+1)}$$

Thus, writing $\underset{\sim}{g}^*_{i+1} = [g(\tau_{i(p+1)+1}), g(\tau_{i(p+1)+2}), \ldots, g(\tau_{(i+1)(p+1)})]^T$ and employing the notation (1.23) we find from (3.2)

$$(3.3) \qquad \underset{\sim}{\eta}_{i+1} = \underset{\sim}{g}^*_{i+1} + h \sum_{k=0}^{i} \omega_{ik}\, f_{k(p+1)}\, \underset{\sim}{\kappa}_{i-k}$$

where

(3.4) $$\underset{\sim}{k}_{i-k} = [K((i-k)h+\theta_0 h),\ K((i-k)h+\theta_1 h),\ldots,K((i-k)h+\theta_p h)]^T .$$

Further, (1.20) then yields for

$$\underset{\sim}{f}_{i+1} = [f_{i(p+1)+1},\ f_{i(p+1)+2},\ldots,f_{(i+1)(p+1)}]^T$$

the relation

(3.5) $$\underset{\sim}{f}_{i+1} = \underset{\sim}{\eta}^*_{i+1} + h(\underset{\sim}{A} * \underset{\sim}{K}_0)\ \underset{\sim}{f}_{i+1}$$

where $\underset{\sim}{K}_0 \equiv \underset{\sim}{K}_0(\underset{\sim}{\lambda};h)$ has elements

(3.6) $$\underset{\sim}{e}_r\ \underset{\sim}{K}_0^T\ \underset{\sim}{e}_t = K((\theta_r - \theta_t)h) \qquad (r,t = 0,1,\ldots,p),$$

and

(3.7) $$\underset{\sim}{\eta}^*_{i+1} = \underset{\sim}{\eta}_{i+1} + (f_{i(p+1)} - \hat{\eta}_j)\underset{\sim}{\gamma}$$

and $\hat{\eta}_i$ is defined in (1.19). In consequence

(3.8) $$\underset{\sim}{f}_{i+1} = (\underset{\sim}{I} - h\ \underset{\sim}{A}*\underset{\sim}{K}_0)^{-1}\underset{\sim}{\eta}^*_{i+1}.$$

Since $\underset{\sim}{e}_p^T\ \underset{\sim}{f}_{i+1} = f_{(i+1)(p+1)}$, (3.8) permits us to derive the following Lemma. Lemma 3.1 requires no assumptions on the quadrature rules (1.8). Setting $\underset{\sim}{\gamma} = \underset{\sim}{0}$ gives a result for the classical mixed method.

LEMMA 3.1. *Suppose* $\{f_j\}_{j\geq 0}$ *to be defined by the modified mixed method applied to* (1.3). *Let*

$$\gamma^{\#} = \underset{\sim}{e}_p^T(\underset{\sim}{I} - h\underset{\sim}{A}*\underset{\sim}{K}_0)^{-1}\ \underset{\sim}{\gamma}$$

and denote by ϕ_i *the expression*

(3.9) $$\phi_i = f_{(i+1)(p+1)} - \gamma^{\#} f_{i(p+1)} - \hat{g}_{i+1},$$

where $\hat{g}_{i+1} = \underset{\sim}{e}_p^T(I - hA^*K_0)^{-1}\underset{\sim}{g}^*_{i+1} - \gamma^{\#}g(ih)$. *Then*

$$\phi_i = h \sum_{k=0}^{i} \omega_{ik} \hat{K}(ih-kh) f_{k(p+1)}, \tag{3.10}$$

where

$$\hat{K}(ih-kh) = \underset{\sim}{e}_p^T(\underset{\sim}{I} - h\underset{\sim}{A}^*\underset{\sim}{K}_0)^{-1} \underset{\sim}{\kappa}_{i-k} - \gamma^{\#}K(ih-kh). \tag{3.11}$$

PROOF. Applying $\underset{\sim}{e}_p^T$ to (3.8) and using (3.7) and (3.3) gives $f_{(i+1)(p+1)} \equiv \underset{\sim}{e}_p^T \underset{\sim}{f}_{i+1} = (f_{i(p+1)} - \hat{\eta}_i)\gamma^{\#} + \underset{\sim}{e}_p^T(\underset{\sim}{I} - h\underset{\sim}{A}*\underset{\sim}{K}_0)^{-1} \underset{\sim}{g}^*_{i+1} + h\sum_{k\geq 0} \omega_{ik} f_{k(p+1)} \underset{\sim}{e}_p^T(\underset{\sim}{I} - h\underset{\sim}{A}*\underset{\sim}{K}_0)^{-1}\underset{\sim}{\kappa}_{i-k}$ and setting $\hat{\eta}_i = \eta_i(ih)$ in (1.21) yields the result. □

3.1. Mixed methods with $\{\rho,\sigma\}$-reducible rules

We now consider the case where the rules (1.8) are $\{\rho,\sigma\}$-reducible. To proceed we require a convenient expression for $\hat{K}(ih-kh)$.

LEMMA 3.2. *Let* $\mu_s = \underset{\sim}{e}_p^T(\underset{\sim}{I}-h\underset{\sim}{A}*\underset{\sim}{K}_0)^{-1}\underset{\sim}{\theta}^s$ *where* $\underset{\sim}{\theta}^s = [\theta_0^s,\theta_1^s,\ldots,\theta_p^s]^T$. *Then*

$$\hat{K}(ih - kh) = \sum_{s=0}^{S} \hat{\Lambda}_s(h)(ih-kh)^s, \tag{3.12}$$

where

$$\hat{\Lambda}_s(h) = \Lambda_s(h) - \gamma^{\#}\lambda_s \tag{3.13}$$

with

$$\Lambda_s(h) = \sum_{t=s}^{S} \lambda_t \binom{t}{s} h^{t-s} \mu_{t-s} \tag{3.14}$$

PROOF. Expand $\hat{K}(ih-kh)$ in powers of ih − kh. □

THEOREM 3.1. *If the quadrature rules in the mixed methods defined by* $\underset{\sim}{\gamma}$ *in* (1.16) *are* $\{\rho,\sigma\}$*-reducible then the "full-step" values* $\{f_{n(p+1)}\}_{n\geq 0}$ *satisfy the constant term recurrence*

$$(3.15)\qquad \{\rho_0(E)\}^S\ \rho(E)(Ef_{n(p+1)}-\gamma^{\#}f_{n(p+1)}-\hat{g}_{n+1}) = hR_{S+1}(\hat{\Lambda}(h);E)f_{n(p+1)}.$$

PROOF. Employ (3.9), (3.10) and (3.13) in Lemma 2.3; note that $Ef_{n(p+1)} = f_{(n+1)(p+1)}$. □

A consequence of Theorem 3.1 is the following result.

THEOREM 3.2. *Under the assumptions of Theorem* 3.1 *a stability polynomial for the sequence* $\{f_{n(p+1)}\}_{n\geq 0}$ *is*

$$(3.16)\qquad \{\mu\{\rho_0(\mu)\}^S\rho(\mu) - hR_{S+1}(\Lambda(h);\mu)\} - \gamma^{\#}Q_{S+1}(\lambda;\mu)$$

using the notation (2.14).

PROOF. $R_{S+1}(\hat{\Lambda}(h);\mu) \equiv R_{S+1}(\Lambda(h)-\gamma^{\#}\lambda;\mu) = R_{S+1}(\Lambda(h);\mu) - \gamma^{\#}R_{S+1}(\lambda;\mu)$ and the result follows from Theorem 3.1. The term in $Q_{S+1}(\lambda;\mu)$ drops out for the classical methods ($\gamma = 0$). □

3.2. Mixed methods with diagonally block-reducible rules

The relations (3.9), (3.10) continue to be applicable when the quadrature rules of the mixed methods satisfy the assumptions (2.17) and (2.18) of section 2.3.

If we write

$$(3.17)\qquad \phi_n = [\phi_{nq}, \phi_{nq+1},\ldots,\phi_{(n+1)q-1}]^T$$

then (3.10) may be re-expressed as

$$(3.18)\qquad \phi_n = h\sum_{k\geq 0}(V_{nk} * \hat{K}(n-k)\ \psi_k$$

where $e_\alpha^T\ \hat{K}_k e_\beta = \hat{K}((kq+\alpha-\beta)h)$ and

$$(3.19)\qquad \psi_k = [f_{kq(p+1)}, f_{(kq+1)(p+1)},\ldots,f_{(kq+q-1)(p+1)}]^T.$$

On the other hand, (3.9) may be re-expressed as

(3.20) $$\underset{\sim}{\phi}_n = \hat{\underset{\sim}{\psi}}_n - \gamma^{\#}\underset{\sim}{\psi}_n - \hat{\underset{\sim}{g}}_n$$

where

(3.21) $$\hat{\underset{\sim}{\psi}}_k = [f_{(kq+1)(p+1)}, f_{(kq+2)(p+1)}, \ldots, f_{(kq+q)(p+1)}]^T.$$

It follows from Lemma 2.5 that

(3.22) $$\{\underset{\sim}{P}_0(E)\}^S \underset{\sim}{P}(E)\{\hat{\underset{\sim}{\psi}}_n - \gamma^{\#}\underset{\sim}{\psi}_n - \hat{\underset{\sim}{g}}_n\} = h\, \underset{\sim}{R}_{S+1}(\hat{\underset{\sim}{\Lambda}}(h);E)\underset{\sim}{\psi}_n.$$

We wish to express either $\underset{\sim}{\psi}_n$ or $\hat{\underset{\sim}{\psi}}_n$ in terms of the other and introducing the matrices

(3.23) $$\underset{\sim}{J} = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & \cdots & & 0 \\ 0 & 1 & & & \cdot \\ 0 & 0 & \ddots & & \cdot \\ \cdot & & & \ddots & \\ \cdot & & & & \\ 0 & 0 & & 1 & 0 \end{bmatrix}, \quad \underset{\sim}{J}^{\#} = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ 0 & \cdots & 0 & 0 \end{bmatrix}$$

we write

(3.24) $$\underset{\sim}{\psi}_n = \underset{\sim}{J}\,\hat{\underset{\sim}{\psi}}_n + \underset{\sim}{J}^{\#}\hat{\underset{\sim}{\psi}}_{n-1}.$$

We may then deduce the following result.

<u>THEOREM 3.2</u>. *Assume that the quadrature rules are diagonally-block-reducible, and let* $\underset{\sim}{P}_0(\mu)$, $\underset{\sim}{P}(\mu)$ *and* $\underset{\sim}{R}_{S+1}(\underset{\sim}{\Lambda}(h);\mu)$ *be the matrix polynomials of Lemma* 2.5. *Then the sequence* $\{\hat{\underset{\sim}{\psi}}_n\}_{n\geq 0}$ *satisfies a recurrence relation whose stability polynomial is*

(3.25) $$\det[\{\underset{\sim}{P}_0(\mu)\}^S\underset{\sim}{P}(\mu)\{\mu(\underset{\sim}{I} - \gamma^{\#}\underset{\sim}{J}) - \gamma^{\#}\underset{\sim}{J}^{\#}\} - h\, \underset{\sim}{R}_{S+1}(\hat{\underset{\sim}{\Lambda}}(h);\mu)(\mu\underset{\sim}{J}+\underset{\sim}{J}^{\#})]$$

wherein

$$\hat{\Lambda}(h) = \Lambda(h) - \gamma^{\#}\lambda.$$

4. EXTENDED RUNGE-KUTTA METHODS

The analysis of extended Runge-Kutta methods defined by (1.13a) may be deduced from the analysis of section 2.4 by means of a systematic adaptation of our earlier technique.

For the extended methods discussed here modified methods reduce to the classical methods so we may assume $\gamma = 0$.

With (1.4) in place of H(x,y) in (1.7) and the choice (1.13a) the vectors (1.22) satisfy

$$f_{n+1} = g^*_{n+1} + h \sum_{k=0}^{n-1} A^{\dagger}_p * K^*_{nk} \, f_{k+1} + hA * K^*_{nn} \, f_{n+1} \tag{4.1}$$

wherein

$$A^{\dagger}_p = e \, e^T_p \, A \tag{4.2}$$

$$e^T_\alpha \, K^*_{nk} \, e_\beta = K((n-k)h + (\theta_\alpha - \theta_\beta)h) \tag{4.3}$$

and g^*_i is defined as in (3.3).

Equation (4.1) is of the form of (2.24) under the replacement of ψ_n by f_{n+1}, g_n by g^*_{n+1}, K_{nk} by K^*_{nk} and, further, V_{nn} by A and V_{nk} by $A^{\dagger}_p$ $(k = 0,1,\ldots,n-1)$. Evidently, (2.17) is then satisfied with m = 1, $A_0 = A_1 = I$, $B_0 = A$, and $B_1 = A^{\dagger}_p - A$. Since A_0, A_1 are diagonal it is possible to deduce from (2.36) a stability polynomial for the extended Runge-Kutta method.

Thus, we now set

$$P(\mu) = P_0(\mu) = (\mu - 1)I$$

but note that (2.28) is now replaced by

$$P_t(\mu) = -I \qquad (t = 1,2,\ldots).$$

(Replace q by 1 in (2.28)!). Further, we set

$$\underset{\sim}{\Sigma}_t(\mu) = h^{-t}\, \underset{\sim}{A} * M_0^{(t)}\mu + h^{-t}(\underset{\sim}{A}_p^{\dagger} - \underset{\sim}{A}) * \underset{\sim}{M}_1^{(t)}$$

where

$$h^{-t}\underset{\sim}{M}_1^{(t)} = \sum_{\ell=0}^{t} \binom{t}{\ell} h^{-\ell}\, \underset{\sim}{M}_0^{(\ell)}, \qquad \underset{\sim}{e}_r^T\, \underset{\sim}{M}_0^{(t)}\, \underset{\sim}{e}_s = h^t\, (\theta_r - \theta_s)^t.$$

The expression for $\underset{\sim}{\Sigma}_t(\mu)$ depends upon $\underset{\sim}{A} * \underset{\sim}{M}_0^{(t)}$ and $(\underset{\sim}{A}_p^{\dagger} - \underset{\sim}{A}) * \underset{\sim}{M}_0^{(\ell)}$, $\ell = 0,1,\ldots,t$. For an arbitrary matrix $\underset{\sim}{G}$,

$$\text{(4.4)} \qquad \underset{\sim}{G}[\![t]\!] \equiv h^{-t}\, \underset{\sim}{G} * \underset{\sim}{M}_0^{(t)}$$

can be determined on setting $\underset{\sim}{\Theta} = \operatorname{diag}(\theta_0, \theta_1, \ldots, \theta_p)$ from

$$\text{(4.5)} \qquad \underset{\sim}{G}[\![t]\!] = \begin{cases} \underset{\sim}{G} & \text{if } t = 0 \\ [\underset{\sim}{\Theta}, \underset{\sim}{G}[\![t-1]\!]] & t = 1,2,\ldots \end{cases}$$

where $[\underset{\sim}{\Theta}, \underset{\sim}{G}[\![t-1]\!]]$ is the commutator $\underset{\sim}{\Theta}\, \underset{\sim}{G}[\![t-1]\!] - \underset{\sim}{G}[\![t-1]\!]\, \underset{\sim}{\Theta}$.

In this notation we have the following result.

THEOREM 4.1. *A stability polynomial for the extended Runge-Kutta method applied to* (1.3) *is*

$$\text{(4.6)} \qquad \det\, [(\mu-1)^{S+1}\, \underset{\sim}{I} - h\, \underset{\sim}{R}_{S+1}(\underset{\sim}{\lambda};\mu)]$$

where

$$\underset{\sim}{R}_{S+1}(\underset{\sim}{\lambda};\mu) = \sum_{s=0}^{S} \lambda_s h^s\, (\mu-1)^{S-s}\, \underset{\sim}{N}_{s+1}(\mu)$$

$$\underset{\sim}{N}_{s+1}(\mu) = (\mu-1)^s\, \underset{\sim}{\Sigma}_s(\mu) + \sum_{t=0}^{s-1} \binom{s}{t} (\mu-1)^{s-1-t}\, \underset{\sim}{N}_{t+1}(\mu)$$

$$\underset{\sim}{\Sigma}_t(\mu) = \mu\, \underset{\sim}{A}(t) + \sum_{\ell=0}^{t} \binom{t}{\ell} (\underset{\sim}{A}_p^{\dagger} - \underset{\sim}{A})[\![\ell]\!].$$

APPENDIX

Although Lemma 2.1 is a consequence of Lemma 2.4 (it is a special case) we shall indicate the proof of both results for clarity.

PROOF OF LEMMA 2.1. Our proof is by induction on S. Using the reducibility property (2.3) it is easily verified that the lemma is true for S = 0. As induction hypothesis, suppose that the result is true for s = 0,1,...,S-1. In order to establish the result for S we write

$$a_n^{(S)} = h^{S+1} \sum_{k=0}^{n} \omega_{nk}(n-k)^S f_k$$

and use the relation

$$\text{(A.1)} \qquad (n-\ell-k)^S = (n-k)^S - \sum_{s=0}^{S-1} \binom{S}{s} (n-\ell-k)^s \ell^{S-s}$$

to obtain

$$\text{(A.2)} \qquad \sum_{\ell=0}^{m_0} \alpha_\ell a_{n-\ell}^{(S)} = h^{S+1} \sum_{\ell=0}^{m_0} \alpha_\ell \sum_{k=0}^{n-\ell} \omega_{n-\ell,k}(n-k)^S f_k$$

$$- h^{S+1} \sum_{\ell=0}^{m_0} \alpha_\ell \sum_{k=0}^{n-\ell} \omega_{n-\ell,k} \sum_{s=0}^{S-1} \binom{S}{s} (n-\ell-k)^s \ell^{S-s} f_k .$$

Using the reducibility property (2.5) and the definition of $a_n^{(s)}$ for s=0(1)S-1, relation (A.2) can be written

$$\sum_{\ell=0}^{m_0} \alpha_\ell a_{n-\ell}^{(S)} = h^{S+1} \sum_{\ell=0}^{m} \beta_\ell \ell^S f_{n-\ell} - \sum_{s=0}^{S-1} \binom{S}{s} h^{S-s} \sum_{\ell=0}^{m_0} \alpha_\ell \ell^{S-s} a_{n-\ell}^{(s)} .$$

Equivalently, using the polynomials (2.6),

$$\text{(A.3)} \qquad \rho_0(E)E^{n-m_0} a_n^{(S)} = h^{S+1} \sigma_S(E) f_n - \sum_{s=0}^{S-1} h^{S-s} \hat{\rho}_{Ss}(E) E^{m-m_0} a_n^{(s)} .$$

In order to eliminate $E^{m-m_0} a_n^{(s)}$ in (A.3) for s = 0,1,...,S-1, apply Lemma

2.1 which is true by hypothesis. In particular, in order to eliminate $a_n^{(S-1)}$ we have to apply $\{\rho_0(E)\}^S$ to both sides of (A.3). (Note that by definition $\rho_0(E)\, E^{m-m_0} = \rho(E)$.). This yields

$$\{\rho_0(E)\}^S \rho(E) a_n^{(S)} = h^{S+1}\{\rho_0(E)\}^S \sigma_S(E) f_n$$

$$- \sum_{s=0}^{S-1} h^{S-s} \hat{\rho}_{Ss}(E)\{\rho_0(E)\}^{S-1-s}\{\rho_0(E)\}^s \rho(E) a_n^{(s)}$$

or

$$\{\rho_0(E)\}^S \rho(E)\ a_n^{(S)} = h^{S+1}[\{\rho_0(E)\}^S \sigma_S(E) -$$

$$- \sum_{s=0}^{S-1} \hat{\rho}_{Ss}(E)\{\rho_0(E)\}^{S-1-s} N_{s+1}(E)] f_n .$$

Hence, the result is true for S. □

<u>PROOF OF LEMMA 2.4</u>. Using (A.1) it is easily verified that

$$\text{(A.4)} \qquad \underset{\sim}{M}^{(S)}_{n-\ell-k} = \underset{\sim}{M}^{(S)}_{n-k} - \sum_{s=0}^{S-1} \binom{S}{s} (\ell q h)^{S-s}\ \underset{\sim}{M}^{(s)}_{n-\ell-k} .$$

Using this relation we derive

$$\sum_{\ell=0}^{m_0} \underset{\sim}{\Delta}_\ell \underset{\sim}{a}^{(S)}_{n-\ell} = h \sum_{\ell=0}^{m_0} \underset{\sim}{\Delta}_\ell \sum_{k=0}^{n-\ell} (\underset{\sim}{V}_{n-\ell,k} * \underset{\sim}{M}^{(S)}_{n-\ell-k}) \underset{\sim}{\psi}_k$$

$$= h \sum_{\ell=0}^{m_0} \underset{\sim}{\Delta}_\ell \sum_{k=0}^{n-\ell} (\underset{\sim}{V}_{n-\ell,k} * \underset{\sim}{M}^{(S)}_{n-k}) \underset{\sim}{\psi}_k$$

$$- \sum_{\ell=0}^{m_0} \underset{\sim}{\Delta}_\ell \sum_{k=0}^{n-\ell} (\underset{\sim}{V}_{n-\ell,k} * \sum_{s=0}^{S-1} \binom{S}{s} (\ell q h)^{S-s} \underset{\sim}{M}^{(s)}_{n-\ell-k}) \underset{\sim}{\psi}_k .$$

Using the diagonal-block-reducibility (2.17) and the property that in (2.18) the matrices $\underset{\sim}{\Delta}_\ell$ are diagonal yields

$$\sum_{\ell=0}^{m_0} \underset{\sim}{\Delta}_\ell\ \underset{\sim}{a}^{(S)}_{n-\ell} = h \sum_{\ell=0}^{m} (\underset{\sim}{B}_\ell * \underset{\sim}{M}^{(S)}_\ell) \underset{\sim}{\psi}_{n-\ell}$$

$$- \sum_{s=0}^{S-1} \binom{S}{s} \sum_{\ell=0}^{m_0} (\ell q h)^{S-s}\ \underset{\sim}{\Delta}_\ell\ \underset{\sim}{a}^{(s)}_{n-\ell} ;$$

equivalently, using the notation (2.27) to (2.29)

$$\{\underset{\sim}{P}_0(E)\}\, E^{m-m_0}\, \underset{\sim}{a}_n^{(S)} = h^{S+1}\, \underset{\sim}{\Sigma}_S(E)\, \underset{\sim}{\psi}_n - \sum_{s=0}^{S-1} \binom{S}{s} \underset{\sim}{P}_{S-s}(E)\, h^{S-s} E^{m-m_0}\, \underset{\sim}{a}_n^{(s)}.$$

The results (2.30) and (2.31) are now obtained by induction (compare the proof of Lemma 2.1). □

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