stichting
mathematisch
centrum

MC

AFDELING NUMERIEKE WISKUNDE NW 119/82 JANUARI
(DEPARTMENT OF NUMERICAL MATHEMATICS)

H. WOLFF

MULTIPLE GRID METHOD FOR THE CALCULATION OF POTENTIAL FLOW AROUND 3-D BODIES

Preprint

kruislaan 413 1098 SJ amsterdam

Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: 65R20, 65N20, 45B05, 76B05, 76U65

Multiple grid method for the calculation of potential flow around 3-D bodies [*)]

by

H. Wolff

ABSTRACT

In this report we apply a multiple grid method to the calculation of potential flow around a 3-D body. We use the classical approach of representation of the solution by means of a doublet distribution on the boundary of the domain. From the boundary condition one obtains a Fredholm integral equation for the doublet distribution μ. We approximate μ by a piecewise constant function. This numerical method results in a non-sparse system, that is solved by a multiple grid iterative process. We discuss the convergence rate of this process and compare its performance with the Jacobi iterative process. For flow around an ellipsoid the multiple grid process turns out to be much more efficient than the Jacobi iterative process.

KEY WORDS & PHRASES: *Potential flow, Fredholm Integral Equations of the second kind, Multiple Grid Methods*

---

*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

In a recent paper, HEMKER & SCHIPPERS [3], have shown that multiple grid methods can be used advantageously for solving non-sparse linear systems that occur in numerical methods for Fredholm integral equations of the second kind. In the present paper we use a multiple grid method for the calculation of the potential flow around 3-D bodies. This method can be very useful on applications in aerodynamics. The undisturbed flow at infinity is assumed to be uniform. For a basic work on potentail theory, see KELLOGG [4].

We consider the case of incompressible, irrotational flow. For this type of flow there exists a velocity potential satisfying Laplace's equation.

$$(1.1) \qquad \Delta\Phi = 0.$$

The velocity potential Φ can be obtained as the superposition of the potential ϕ_∞ due to the uniform onset flow $\vec{U}$ and a perturbation potential ϕ due to the presence of the body: $\Phi = \phi_\infty + \phi$. Since our problem is linear, the perturbation potential also satisfies (1.1). The perturbation potential may be represented by the potential due to a doublet distribution along the boundary S of the body,

$$(1.2) \qquad \phi(A) = -\frac{1}{4\pi}\iint_S \mu(Q)\frac{\partial}{\partial n_Q}\left(\frac{1}{|\vec{r}_{AQ}|}\right)d\sigma, \qquad A \notin S.$$

Here $\partial/\partial n_Q$ denotes the derivative in the direction of the outward normal $\vec{n}_Q$ to S, at $Q \in S$; $\vec{r}_{AQ}$ is the vector from A to the integration point Q.

At the surface the velocity potential must satisfy the boundary condition

$$(1.3) \qquad \left.\frac{\partial\Phi}{\partial n}\right|_e = 0,$$

where e denotes the exterior side of S.

If the doublet strength μ in (1.2) and the surface S, are sufficiently smooth (i.e. $S \in L^{1,\alpha}$ and $\mu \in H^{1,\alpha'}(S)$; see section 2), then the derivatives of the potential ϕ due to the doublet distribution, in the direction of the outward normal to S, have equal inner and outer limits (GÜNTER

[2,p.73]),

$$(1.4)\qquad \left.\frac{\partial\phi}{\partial n}\right|_e = \left.\frac{\partial\phi}{\partial n}\right|_i,$$

where the subscript i denotes the inner side of S. As a consequence

$$(1.5)\qquad \left.\frac{\partial\Phi}{\partial n}\right|_e = \left.\frac{\partial\Phi}{\partial n}\right|_i = 0.$$

From Green's theorem (KELLOGG [4,p.212])

$$(1.6)\qquad \iiint\limits_D v\Delta u\, dw = \iint\limits_{\partial D} v\,\frac{\partial u}{\partial n}\, d\sigma - \iiint\limits_D \text{grad } u.\ \text{grad } v\, dw,$$

applied with $u = v = \Phi$ and $\partial D = S$, follows that Φ_i = constant. We take $\Phi_i = 0$, which yields the boundary condition

$$(1.7)\qquad \phi_i(P) = -\phi_\infty(P), \quad P \in S.$$

Under more general conditions than for (1.4) it is shown (GÜNTER [2,p.49]) that for $P \in S$:

$$(1.8a)\qquad \phi_e(P) = -\tfrac{1}{2}\mu(P) - \frac{1}{4\pi}\iint\limits_S \mu\,\frac{\partial}{\partial n_Q}\left(\frac{1}{|\vec{r}_{PQ}|}\right)d\sigma,$$

$$(1.8b)\qquad \phi_i(P) = \tfrac{1}{2}\mu(P) - \frac{1}{4\pi}\iint \mu\,\frac{\partial}{\partial n_Q}\left(\frac{1}{|\vec{r}_{PQ}|}\right)d\sigma.$$

The interior Dirichlet problem given by (1.1) and (1.7), leads with (1.2) and (1.8b) to a Fredholm integral equation of the second kind for the doublet distribution μ:

$$(1.9)\qquad (I-K)\mu = g \text{ on } S,$$

where the operator K is given by

$$(1.10)\qquad K\mu(P) = -\frac{1}{2\pi}\iint\limits_S \mu(Q)\,\frac{\cos(\vec{n}_Q,\vec{r}_{PQ})}{|\vec{r}_{PQ}|^2}\, d\sigma$$

since

$$(1.11) \qquad \frac{\partial}{\partial n}\left(\frac{1}{|\vec{r}|}\right) = -\frac{\cos(\vec{n},\vec{r})}{|\vec{r}|^2} .$$

Since $\phi_\infty(P) = \vec{U}.\vec{r}_P$, the righthandside g is given by

$$(1.12) \qquad g(P) = -2\vec{U}.\vec{r}_P,$$

where $\vec{U}$ the uniform onset flow and $\vec{r}_P$ the vector from the origin to the point $P = (x_P,y_P,z_P)$ on S.

In section 2 we collect some properties of the operator K and discuss the solvability of (1.9).

In section 3 we describe the discretization method to compute an approximate solution μ_N of (1.9). We use a collocation method applied to the space of piecewise constant functions, on a partition of the surface S. In this case $K\mu_N$ can be easily approximated and amounts to the calculation of solid angles. We analyse the discretization error.

In section 4 we describe a multiple grid method and with the aid of the paper by HEMKER & SCHIPPERS [3], we discuss the rate of convergence. In section 5 we apply the multiple grid method described in the previous sections. We give numerical examples related to the potential flow around a sphere and an ellipsoid.

§2. PROPERTIES OF THE INTEGRAL OPERATOR

In this section we summarize some properties of the integral operator, that lead to the unique solvability of (1.9). We begin by defining some function spaces. Let D be an open connected set in the 3-dimensional Euclidian space, with boundary ∂D and closure $\bar{D}$. (See Günter [2, section I]).

DEFINITION 2.1. $C^k(D)$ will denote the space of functions which are k-times continuously differentiable in D.

DEFINITION 2.2. By $C^{k,\alpha}(D)$ we denote the subclass in $C^k(D)$ of functions u, whose derivatives of order k satisfy a uniform Hölder condition with exponent α, $0 < \alpha \leq 1$. This means for $u(P) = u(x,y,z)$:

$$(2.1)\qquad \left|\frac{\partial^k u}{\partial x^k}(P_1) - \frac{\partial^k u}{\partial x^k}(P_2)\right| \le B|P_1-P_2|^\alpha, \qquad \forall P_1,P_2 \in D$$

and the same bound is to hold for $\partial^k u/\partial y^k$ and $\partial^k u/\partial z^k$. B is called the Hölder constant, α the Hölder exponent.

DEFINITION 2.3. The surface S is a Lyapunov surface of order k, $k \ge 1$, S belongs to $L^{k,\alpha}$, if:

1^o. At each $Q \in S$, there exists a tangent plane to S. We introduce a local rectangular coordinate system, where this tangent plane is the $\xi\eta$-plane, Q is the origin, and the ζ-axis the normal to S in Q.

2^o. There exists some small number $\varepsilon > 0$, such that the intersection of S with the spherical neighbourhood B_ε of Q of radius ε(fig. 1), can be represented by a function

$$(2.2)\qquad \zeta = F(\xi,\eta),\ (\xi,\eta) \in D_\varepsilon ,$$

where $F \in C^{k,\alpha}(D_\varepsilon)$, D_ε the portion of the tangent plane within B_ε.

3^o. The final condition is, that the numbers B (Hölder constant), α and ε are independent of the choice of the point $Q \in S$.

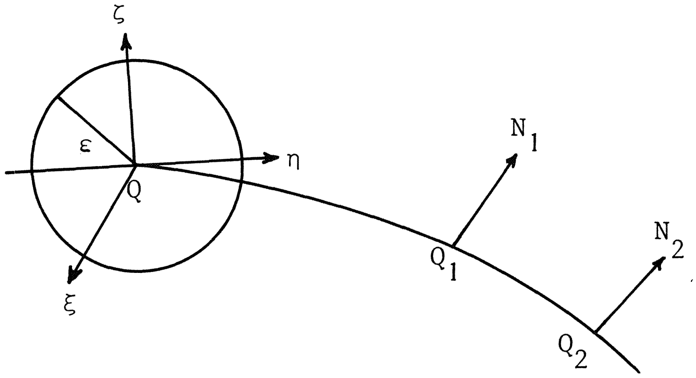

Fig. 1.

DEFINITION 2.4. The function f belongs to the space $H^{k,\alpha}(S)$, if the function $\hat{f}$, defined on each region D_ε of (2.2) by

$$\hat{f}(\xi,\eta) = f(\xi,\eta,F(\xi,\eta)),$$

belongs to the Hölder space $C^{k,\alpha}(D_\varepsilon)$.

Following Sloan [6, p.22], we define

DEFINITION 2.5. Z(S) denotes the complete closure (in the supremum norm $\|\cdot\|$) of the space of piecewise continuous functions f on S, which satisfy for $P \in S$

$$f(P) = \frac{1}{k} \sum_{i=1}^{k} \lim_{\substack{|P_i - P| \to 0 \\ P_i \in \Delta^{(i)}}} f(P_i), \qquad P \in \bigcap_{i=1}^{k} \bar{\Delta}^{(i)}$$

$\bar{\Delta}^{(i)}$ the closure of the open surface element $\Delta^{(i)}$.

Some important results from GÜNTER [2, p.106], are the following lemmas:

LEMMA 2.1. *Assume* $S \in L^{1,\alpha}$. *If* μ *is bounded and integrable on* S, *then* $K\mu$ *is a Hölder-continuous function on* S, i.e. $K\mu \in H^{0,\alpha'}(S)$, *where* $\alpha' = \alpha$ *if* $\alpha < 1$, *and* $0 < \alpha' < 1$ *arbitrary if* $\alpha = 1$.

LEMMA 2.2. $S \in K^{k+2,\alpha}$ *and* $\mu \in H^{k,\alpha}(S)$, $0<\alpha<1$, $k \geq 0$, *imply* $K\mu \in H^{k+1,\alpha'}(S)$, *for an arbitrary* $0 < \alpha' < \alpha$.

These lemmas imply

LEMMA 2.3. *Let the righthandside of* (1.9), $g \in H^{k,\alpha}(S)$ *and the surface* $S \in L^{k+1,\alpha}$, $0 < \alpha \leq 1$, $k \geq 0$, *then the solution of* (1.9), $\mu \in H^{k,\alpha'}(S)$, *for an arbitrary* $0 < \alpha' < \alpha$.

REMARK. In section 1 we have shown that $\mu \in H^{1,\alpha'}(S)$ is a necessary condition for the reformulation of the problem of potential flow around a 3-D body into a Fredholm equation of the second kind. By lemma 2.3 this condition is satisfied if we assume $S \in L^{2,\alpha}$ and $g \in H^{1,\alpha}(S)$.

Once this condition is satisfied, we prove the unique solvability of (1.9), when μ and g are in Z(S). We need the following lemma.

LEMMA 2.4. K *is a compact operator on* Z(S).

PROOF. A bounded element of Z(S) is integrable on S. So, by lemma 2.1, K is an operator from Z(S) into Z(S), since $H^{0,\alpha'}(S) \subset Z(S)$. The image of the unit ball in Z(S),

$$(2.3) \qquad K(\beta) = \{Kf \,|\, \|f\| \leq 1, \quad f \in Z(S)\},$$

is uniformly bounded on S, since K is a bounded operator. $K(\beta) \subset H^{0,\alpha'}(S)$, so for $f \in \beta$ arbitrary:

$$(2.4) \qquad |Kf(P) - Kf(Q)| < B|P-Q|^{\alpha'}, \quad P,Q \in S,$$

B the Hölder constant. It follows that $K(\beta)$ is an equicontinuous set. The Arzela-Ascoli theorem (see KELLOGG [4, p.265]) implies that K is a compact operator on Z(S). ⊠

LEMMA 2.5. (*Fredholm alternative theorem, see* ATKINSON [1, p.26]). *Let* X *be a Banach space, let* K *be a compact operator on* X *into* X. *Then the equation* $(I-K)x = y$ *has a solution for each* $y \in X$, *if and only if the homogeneous equation* $(I-K)x = 0$ *has only the trivial solution.*

THEOREM 2.6. *Let* S *be a Lyapunov surface and let* $g \in Z(S)$. *Then* (1.9) *has a unique solution* $\mu \in Z(S)$.

PROOF. It is well-known that 1 is not an eigenvalue of K. The proof follows directly from lemmas 2.4 and 2.5. ⊠

§3. NUMERICAL APPROACH

The numerical method to find an approximate solution of (1.9) is connected with the shape of the kernel function, as defined by the operator K (1.10). Application of the collocation method in the space of piecewise constant functions leads to moment-integrals, which are obtained by computing solid angles.

Let X_N denote the space of piecewise constant functions on a partition Δ_N of S. We define the restriction operator T_N by piecewise constant interpolation at the collocation points $\{Q_i\}_{i=1}^{N}$. Let $Q, Q_i \in \Delta_N^{(i)}$, the i-th element of the partition of S.

$$(3.1)\qquad \begin{aligned} &T_N : Z(S) \to X_N \\ &T_N f(Q) = f(Q_i), \quad i = 1,2,\dots,N \end{aligned}$$

For KT_N we get:

$$(3.2)\qquad KT_N f(P) = -\frac{1}{2\pi} \sum_{i=1}^{N} f(Q_i) \iint_{\Delta_N^{(i)}} \frac{\cos(\vec{n}_Q, \vec{r}_{PQ})}{|\vec{r}_{PQ}|^2}\, d\sigma$$

Now

$$\frac{\cos(\vec{n}_Q, \vec{r}_{PQ})}{|\vec{r}_{PQ}|^2}\, d\sigma$$

is just the solid angle subtended at P by $d\sigma$. And so

$$\iint_{\Delta_N(i)} \frac{\cos(\vec{n}_Q, \vec{r}_{PQ})}{|\vec{r}_{PQ}|^2}\, d\sigma,$$

is the solid angle subtended at P by $\Delta_N^{(i)}$.

We define $h_N = \max\limits_{i=1,2,\dots,N} (\sup|x-y|,\ x,y \in \Delta_N^{(i)})$ and denote by $\|\cdot\|_{k,\alpha}$ the usual Hölder norm for the space $H^{k,\alpha}(S)$.

LEMMA 3.1. *Let* $S \in L^{k,\alpha}$, *and* $f \in H^{k,\alpha}(S)$. *Then*

$$\|(I-T_N)f\| \le C.h_N^{\ell}\|f\|_{k,\alpha} \quad \textit{for } h_N \to 0,$$

where $\begin{cases} \ell = 1, & \textit{for } k \ge 1 \\ \ell = \alpha & \textit{for } k = 0, \end{cases}$ *and* C *is a constant.*

PROOF. For $k = 0$ the lemma follows directly from the definition of Hölder-continuity. For $k = 1$ the proof follows with the Taylor-expansion of f. ☒

We have seen that $KT_N f$ can be easily obtained by computing solid angles. In general, these solid angles cannot be evaluated directly, but must be approximated in a numerical way. We approximate each element of Δ_N by one or more flat planes. The solid angles subtended by such planes can be evaluated directly.

For h_N sufficiently small, that is for N sufficiently large, $(I-T_N k)^{-1}$ exists and is bounded on Z(S) (PRENTER [5]). Hence an approximate solution of (1.9) follows from

$$(3.3) \qquad (I-T_N K)\mu_N = T_N g, \qquad \mu_N \in X_N.$$

In the following theorem we give an error-bound for $\|\mu-\mu_N\|$.

THEOREM 3.2. *Let* $S \in L^{1,\alpha}$. *Then*

$$\|\mu-\mu_N\| \leq C.h_N^{\alpha'} \|\mu\|_{0,\alpha'}, \qquad \textit{for } h_N \to 0,$$

where $\alpha' = \alpha$ *if* $\alpha < 1$, *and* $0 < \alpha' < 1$ *arbitrary if* $\alpha = 1$.

PROOF. From (1.9) and (3.3) it follows that

$$(I-T_N K)(\mu-\mu_N) = \mu - T_N K\mu - T_N g = \mu - T_N\mu.$$

Hence

$$\|\mu-\mu_N\| \leq \|(I-T_N K)^{-1}\| \; \|\mu-T_N\mu\|.$$

Because $S \in L^{1,\alpha}$ it can be verified that $g \in H^{0,\alpha}(S)$. Use lemma 2.1 to obtain $\mu \in H^{0,\alpha'}(S)$. The proof follows from lemma 3.1. ⊠

§4. MULTIPLE GRID METHOD

Equation (1.9) can be written symbolically as

$$(4.1) \qquad A\mu = g, \qquad g \in H^{1,\alpha}(S),$$

with the surface $S \in L^{2,\alpha}$, $0 < \alpha \leq 1$ and $A = I - K$, I the identity operator and K the linear integral operator given by (1.10). According to section 2, A has a bounded inverse on Z(S). To obtain a sequence of approximations, converging to the unique solution of (4.1), we use a multiple grid method. This method uses a sequence of partitions of the surface S, called "grids", of increasing refinement. First we give some notations.

Let $\{N_p\}$ give the numbers of elements of the sequence of partitions of S. We write $\Delta_{N_p} \equiv \Delta_p$, $X_{N_p} \equiv X_p$, $T_{N_p} \equiv T_p$ and $h_{N_p} \equiv h_p$, $p = 0,1,2,\ldots$ We assume

$$N_0 < N_1 < N_2 < \ldots\ldots < N_p < \ldots.$$

and $\lim\limits_{p\to\infty} h_p = 0$.

Using the above partitions the assumptions A_1 and A_2 of HEMKER & SCHIPPERS [3, p.2] are satisfied, i.e.:

$$X_0 \subset X_1 \subset X_2 \subset \ldots\ldots \subset X_p \subset \ldots\ldots \subset Z(S)$$

and

$$\lim_{p\to\infty} \|f-T_pf\| = 0, \qquad \text{for all } f \in Z(S).$$

The following lemma is trivial (see also SLOAN [6, p.24]):

LEMMA 4.1. $\|T_p\| = 1, \quad p = 0,1,2\ldots$

In the context of multiple grid iteration, the subscript p is called "level".

For a fixed p, an approximate solution of (4.1) is obtained from

$$(4.2) \qquad A_p\mu_p = T_pg,$$

where $A_p = I-T_pK$.

From results given by PRENTER [5], for a general compact operator K, we have

LEMMA 4.2. *If* X_0 *is sufficiently large, then* $(I-T_pK)^{-1}$ *exists on* Z(S), *for* $p \geq 0$ *and*

$$C_1 = \sup_{p\geq 0} \|(I-T_pK)^{-1}\| < \infty.$$

For solving equation (4.2) we use a multiple grid technique (see HEMKER & SCHIPPERS [3]). From that paper we deduce our iterative method; considering only two levels we obtain the following iteration scheme:
Let g_p be the righthandside of (4.2) and let $\mu_p^{(0)} \equiv 0$.
Let $0 \leq \ell < p$.

$$(4.3) \qquad \mu_p^{(i+\frac{1}{2})} = T_pK\,\mu_p^{(i)} + g_p$$

$$(4.4) \qquad \mu_p^{(i+1)} = \mu_p^{(i+\frac{1}{2})} + (I-T_\ell K)^{-1}T_\ell(g_p-\mu_p^{(i+\frac{1}{2})} + T_pK\mu_p^{(i+\frac{1}{2})})$$

and eventually

$$(4.5) \qquad \tilde{\mu}_p^{(i+1)} = T_pK\mu_p^{(i+1)} + g_p.$$

We can show this in a diagram, where $d_p^{(i)}$ is the defect of $\mu_p^{(i)}$,

$$(4.6) \qquad d_p^{(i)} = g_p - \mu_p^{(i)} + T_pK\,\mu_p^{(i)}\ .$$

$$\begin{tikzcd}
\mu_p^{(i)} \arrow[r, "(4.3)"] & \mu_p^{(i+\frac{1}{2})} \arrow[r] & d_p^{(i+\frac{1}{2})} \arrow[d, "T_\ell"'] & v_p^{(i+\frac{1}{2})} \arrow[r, "(4.4)"] & \mu_p^{(i+1)} = \mu_p^{(i+\frac{1}{2})} + v_p^{(i+\frac{1}{2})} \arrow[r, "(4.5)"] & \tilde{\mu}_p^{(i+1)} \\
 & & d_\ell^{(i+\frac{1}{2})} \arrow[r] & v_\ell^{(i+\frac{1}{2})} = (I-T_\ell K)^{-1}d_\ell^{(i+\frac{1}{2})} \arrow[u, "X_\ell \subset X_p"'] & &
\end{tikzcd}$$

We determine the amplification-operator M of iteration (4.3)-(4.4).

$$\mu_p^{(i+1)} = T_pK\mu_p^{(i)} + g_p + (I-T_\ell K)^{-1}T_\ell\{g_p - T_pK\mu_p^{(i)} - g_p + T_pK(T_pK\mu_p^{(i)} + g_p)\}$$

$$\mu_p^{(i+1)} - \mu_p = T_pK(\mu_p^{(i)} - \mu_p) + (I-T_\ell K)^{-1}T_\ell T_pK\{(T_pK-I)(\mu_p^{(i)}-\mu_p)\}$$

So $\mu_p^{(i+1)} - \mu_p = M(\mu_p^{(i)}-\mu_p)$, with

$$(4.7) \qquad M = (I-T_\ell K)^{-1}T_\ell T_pK(T_pK-I) + T_pK.$$

This iteration corresponds in defect-correction formulation (STETTER [7]), with

$$(4.9) \qquad \tilde{B}_p = I + (I-T_\ell K)^{-1} T_\ell T_p K$$

as the approximate inverse of A_p.

When we apply (4.5), we get

$$\tilde{\mu}_p^{(i+1)} - \mu_p = T_p K \mu_p^{(i+1)} + g_p - \mu_p = T_p K(\mu_p^{(i+1)} - \mu_p).$$

So $\tilde{\mu}_p^{(i+1)} - \mu_p = T_p KM(\mu_p^{(i)} - \mu_p)$.

Before we discuss the convergence of the iteration process, we give the following lemma:

LEMMA 4.3. *Let* $S \in L^{1,\alpha}$. *Then*

$$\| (I-T_p)K \| \le C.h_p^{\alpha'}, \text{ for } h_p \to 0,$$

where $\alpha' = \alpha$ *if* $\alpha < 1$, *and* $0 < \alpha' < 1$ *arbitrary if* $\alpha = 1$.

PROOF. Let $\Psi = \{Kf \mid f \in Z(S) \text{ and } \|f\| < 1\}$. From lemma 2.1 it follows that $\Psi \subset H^{0,\alpha'}(S)$. Use lemma 3.1 to obtain

$$\| (I-T_p)K \| = \sup_{z \in \Psi} \| (I-T_p)z \| \le C.h_p^{\alpha'} \text{ for } h_p \to 0. \qquad \boxtimes$$

THEOREM 4.4. *For* X_ℓ *sufficiently large (i.e.* h_ℓ *sufficiently small), the iteration process* (4.3)-(4.4) *converges to the unique solution of* (4.2).

PROOF. The convergence of the process (4.3)-(4.4) depends on the norm of the amplification operator $M = I - \tilde{B}_p A_p$ given by (4.7), as a mapping from X_p to X_p. So its convergence rate is given by $\|MT_p\|$, and we prove $\|MT_p\| < 1$, if h_ℓ sufficiently small.

$$\begin{aligned} MT_p &= (I-T_\ell K)^{-1} T_\ell (I-I+T_p)K(T_pK-I)T_p + T_pKT_p = \\ &= (I-T_\ell K)^{-1} \{(I-T_\ell K)T_pK - T_\ell K(I-T_pK)\}T_p + \\ &+ (I-T_\ell K)^{-1} \{T_\ell T_p K T_p K - T_\ell K T_p K + T_\ell K - T_\ell T_p K\} T_p. \end{aligned}$$

Hence

$$\|MT_p\| \le \|(I-T_\ell K)^{-1}\|\{\|T_p\|\ \|(I-T_\ell)K\| +$$

$$+ \|T_\ell\|\ \|(I-T_p)K\|\ \|T_p\|\ \|K\| + \|T_\ell\|\ \|(I-T_p)K\|\}\ \|T_p\|,$$

where $\|(I-T_\ell)K\| \le C.h_\ell^\alpha$, $\|(I-T_p)K\| \le C.h_p^\alpha$, according to lemma 4.3. From lemma 4 we have $\|T_p\| = \|T_\ell\| = 1$. Boundedness of $\|K\|$ follows from the linearity and the continuity of K as operator on the space Z(S). Because $\alpha > 0$ we conclude that $\|MT_p\| < 1$ for h_ℓ sufficiently small. ☒

From the proof of theorem 4.4 it follows that the iteration process (4.3) - (4.5) has the same order of convergence. In practice it converges faster than iteration process (4.3)-(4.4).

Now we investigate the convergence of the multiple grid process defined by (see HEMKER & SCHIPPERS [3]):

$$\begin{cases} \mu_p^{(0)} = 0 \\ \mu_p^{(i+1)} = (I - B_pA_p)\mu_p^{(i)} + B_pg_p , \end{cases}$$

where B_p is recursively defined by

$$\begin{cases} B_0 = (I-T_0K)^{-1}T_0 = A_0^{-1}T_0 \\ B_p = I + Q_{p-1}T_{p-1}T_pK, \end{cases}$$

with

$$Q_p = \sum_{m=0}^{\gamma-1} (I-B_pA_p)^m B_p,$$

for some positive integer γ.

From now on with $\tilde{B}_p$ we denote the operator as defined in (4.9), with $\ell = p-1$.

DEFINITION 4.1. $M_p = (I-B_pA_p)$, $\quad n_p = \|M_pT_p\|_{X_p\to X_p}$

$\tilde{M}_p = (I-\tilde{B}_pA_p)$, $\quad \tilde{n}_p = \|\tilde{M}_pT_p\|_{X_p\to X_p}$

THEOREM 4.5. $\tilde{n}_p \le n_p + n_{p-1}^{\gamma}[\tilde{n}_p + \|T_p\|\|K\|\|T_p\|]$.

PROOF. We can write $Q_p = [I-(I-B_pA_p)^{\gamma}]A_p^{-1}$, so we have

$$M_pT_p = T_p-\{I+[I-(I-B_{p-1}A_{p-1})^{\gamma}]A_{p-1}^{-1}T_{p-1}(I-A_p)\}A_pT_p =$$

$$= (I-\tilde{B}_pA_p)T_p + (I-B_{p-1}A_{p-1})^{\gamma}T_{p-1}(\tilde{B}_p-I)A_pT_p =$$

$$= \tilde{M}_pT_p + (M_{p-1}T_{p-1})^{\gamma}\{(I-A_p)T_p-\tilde{M}_pT_p\}.$$

Hence

$$n_p \le \tilde{n}_p + n_{p-1}^{\gamma}(\tilde{n}_p+\|T_p\|\|K\|\|T_p\|). \quad \boxtimes$$

From lemma 4.3, and HEMKER & SCHIPPERS [3, theorem 4.3] follows:

THEOREM 4.6. *Let* $\gamma \ge 2$ *and let* $\tilde{n}_p$ *satisfy* $\tilde{n}_p \le v_p = d^pv_0$, *for some* $0<d<1$, *then if* $v_0 \le \frac{1}{2d}\{\sqrt{d^2+C_2^2} - C_2^2\}$, *where* $C_2 = \|K\|$, *it follows that* $n_p \le 2v_0d^p$.

§5. NUMERICAL RESULTS.

In this section we give some numerical results of the described method, applied to the calculation of the potential flow around a sphere and an ellipsoid respectively. We calculate the dipole density and compare the numerical results with the analytic values. However, we are mainly concerned in the convergence rate and the amount of computational work of our multiple grid method. We compare the performances of the multiple grid method with the Jacobi iterative process.

The discretization is carried out as follows: first dividing the surface of the body into $\bar{N}$ rings, by planes orthogonal to the z-axis, and then each ring into N^* trapeziform elements. The spherical caps are divided into N^* triangle-form elements. As collocation points are chosen the "midpoints" of these elements (see fig. 2).

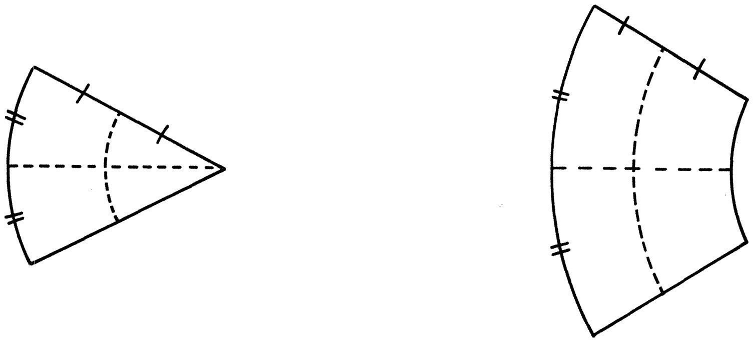

Fig. 2.

The different grid-levels are related by $\bar{N}_p = 2\bar{N}_{p-1}$ and $N^*_p = 2N^*_{p-1}$. The projection operator is defined analogous to definition 2.5: if $Q_{p-1,j} \in \Delta^{(j)}_{p-1}$, collocation point on level p-1, and $Q_{p,i} \in \Delta^{(i)}_p \subset \Delta^{(j)}_{p-1}$ for $i = 1,2,\ldots,k$, then $f_{p-1}(Q_{p-1,j}) = \frac{1}{k} \sum_{i=1}^{k} f_p(Q_{p,i})$.

Since $X_{p-1} \subset X_p$, interpolation from a coarse to a finer grid is defined by piecewise constant interpolation.

From theorem 4.4 we get $\tilde{n}_p = O(h^{\alpha}_{p-1})$. In case of a sphere and an ellipsoid we have $\alpha = 1$. Hence

$$\tilde{n}_p \leq v_p = C.h_{p-1}.$$

The successive mesh sizes are related by $h_p \approx h_0 2^{-p}$, so we have

$$\tilde{n}_p \leq 2C\, h_0\, 2^{-p}.$$

For the application of theorem 4.6,

$$v_0 = 2Ch_0 \leq \frac{1}{2d} \{\sqrt{d^2 + C_2^2} - C_2^2\}, \text{ with } d = 2,$$

must be valid. The validity of this condition for a certain h_0 only depends on the constant C, which depends on the Hölder-constant of the Lyapunov surface S.

In tables 5.1-5.4, we give the residuals $\| d_p \|$ (4.6) and the observed rates of convergence of the iteration process:

$$\zeta_i = \| \mu_p^{(i+1)} - \mu_p^{(i)} \| / \| \mu_p^{(i)} - \mu_p^{(i-1)} \| .$$

We also give the mean convergence rates

$$\zeta = [\prod_{i=1}^{k} \zeta_i]^{1/k},$$

and the maximum errors between the numerical solutions and the analytic values.

We express the operation count in work units (WU), i.e. (total number of multiplications)/$(\bar{N}_p * N_p^*)^2$. We only take into account matrix-vector multiplications, and the direct solution on the coarsest grid, for which we count $\frac{1}{3}(\bar{N}_0 * N_0^*)^3$ multiplications.

We mention the results for $\gamma = 1$, because they hardly differ from those with $\gamma = 2$. Furthermore one pre-relaxation (4.3) and one post-relaxation (4.5) was carried out.

$p = 2$ $\bar{N}_p = 16$, $N_p^* = 16$, $\bar{N}_0 = 4$, $N_0^* = 4$					
Multiple grid			Jacobi-iteration		
Iter.	residue	conv. rate	Iter.	residue	conv. rate
1°	$9.97\ 10^{-3}$		1°	$2.26\ 10^{-1}$	
2°	$1.21\ 10^{-4}$	$1.08\ 10^{-2}$	2°	$4.97\ 10^{-2}$	$2.35\ 10^{-1}$
3°	$1.53\ 10^{-6}$	$1.23\ 10^{-2}$	3°	$1.10\ 10^{-2}$	$2.20\ 10^{-1}$
4°	$2.03\ 10^{-8}$	$1.30\ 10^{-2}$	4°	$2.44\ 10^{-3}$	$2.22\ 10^{-1}$
			5°	$5.41\ 10^{-4}$	$2.21\ 10^{-1}$
			6°	$1.20\ 10^{-4}$	$2.22\ 10^{-1}$
			⋮	⋮	⋮
			10°	$2.89\ 10^{-7}$	$2.22\ 10^{-1}$
mean conv. rate: $1.20\ 10^{-2}$ $\|\mu_p - T_p\mu\| = 8.03\ 10^{-3}$ operation count = 8.54 WU.			mean conv. rate: $2.23\ 10^{-1}$ $\|\mu_p - T_p\mu\| = 8.03\ 10^{-3}$ operation count = 10 WU		

Table 5.1. $x^2 + y^2 + z^2 = 1$, $\vec{U}$ parallel to the x-axis, $\gamma = 1$, $\|T_p\mu\| = 7.34\ 10^{-1}$.

$p = 3$ $\bar{N}_p = 32$, $N^*_p = 32$			$\bar{N}_0 = 4$, $N^*_0 = 4$		
Multiple grid			Jacobi-iteration		
Iter.	residue	conv. rate	Iter.	residue	conv. rate
1°	$2.69\ 10^{-3}$		1°	$2.82\ 10^{-1}$	
2°	$3.97\ 10^{-5}$	$3.11\ 10^{-3}$	2°	$7.80\ 10^{-2}$	$2.84\ 10^{-1}$
3°	$5.54\ 10^{-7}$	$1.65\ 10^{-2}$	3°	$2.16\ 10^{-2}$	$2.77\ 10^{-1}$
			⋮		⋮
			7°	$1.28\ 10^{-4}$	$2.77\ 10^{-1}$
			8°	$3.55\ 10^{-5}$	$2.78\ 10^{-1}$
			⋮		⋮
			11°	$7.62\ 10^{-7}$	$2.78\ 10^{-1}$
mean conv. rate: $7.14\ 10^{-3}$ $\lVert \mu_p - T_p\mu \rVert = 1.29\ 10^{-2}$ operation count 6.40.			mean conv. rate: $2.78\ 10^{-1}$ $\lVert \mu_p - T_p\mu \rVert = 1.29\ 10^{-2}$ operation count: 11		

Table 5.2. $x^2 + y^2 + z^2 = 1$, $\vec{U}$ parallel to the x-axis, $\gamma = 1$, $\lVert T_p\mu \rVert = 7.46\ 10^{-1}$.

$p = 2 \quad \bar{N}_p = 16, \quad N_p^* = 16 \quad \bar{N}_0 = 4, \quad N_0^* = 4$

Multiple grid			Jacobi-iteration		
Iter.	residue	conv. rate	Iter.	residue	conv. rate
1^o	$1.17\ 10^{-1}$		1^o	1.73	
2^o	$2.04\ 10^{-3}$	$4.13\ 10^{-2}$	2^o	$8.05\ 10^{-1}$	$4.51\ 10^{-1}$
3^o	$7.75\ 10^{-5}$	$1.40\ 10^{-2}$	3^o	$3.82\ 10^{-1}$	$4.68\ 10^{-1}$
4^o	$1.89\ 10^{-6}$	$4.63\ 10^{-2}$	4^o	$1.83\ 10^{-1}$	$4.75\ 10^{-1}$
5^o	$6.54\ 10^{-8}$	$2.36\ 10^{-2}$	5^o	$8.75\ 10^{-2}$	$4.78\ 10^{-1}$
			6^o	$4.20\ 10^{-2}$	$4.79\ 10^{-1}$
			7^o	$2.01\ 10^{-2}$	$4.80\ 10^{-1}$
			⋮	⋮	⋮
			21^o	$6.94\ 10^{-7}$	$4.80\ 10^{-1}$
mean conv. rate: $2.83\ 10^{-2}$ $\|\mu_p - T_p\mu\| = 9.48\ 10^{-3}$ operation count = 10.68			mean conv. rate: $4.77\ 10^{-1}$ $\|\mu_p - T_p\mu\| = 9.48\ 10^{-3}$ operation count = 21		

Table 5.3. $\frac{x^2}{4} + y^2 + z^2 = 1$, $\vec{U}$ parallel to the x-axis,

$\gamma = 1$, $\|T_p\mu\| = 2.37$.

$p = 3 \qquad \bar{N}_p = 32,\ N_p^* = 32,\ \bar{N}_0 = 4,\ N_0^* = 4$					
Multiple grid			Jacobi-iteration		
Iter.	residue	conv. rate	Iter.	residue	conv. rate
1^o	$4.56\ 10^{-2}$		1^o	2.15	
2^o	$4.38\ 10^{-4}$	$1.67\ 10^{-2}$	2^o	1.20	$5.44\ 10^{-1}$
3^o	$8.48\ 10^{-6}$	$6.98\ 10^{-3}$	3^o	$6.72\ 10^{-1}$	$5.57\ 10^{-1}$
4^o	$9.93\ 10^{-8}$	$2.56\ 10^{-2}$	4^o	$3.79\ 10^{-1}$	$5.62\ 10^{-1}$
			5^o	$2.14\ 10^{-1}$	$5.64\ 10^{-1}$
			6^o	$1.21\ 10^{-1}$	$5.65\ 10^{-1}$
			7^o	$6.85\ 10^{-2}$	$5.65\ 10^{-1}$
			8^o	$3.88\ 10^{-2}$	$5.66\ 10^{-1}$
			⋮	⋮	⋮
			27^o	$7.79\ 10^{-7}$	$5.66\ 10^{-1}$
mean conv. rate: $1.44\ 10^{-2}$ $\|\mu_p - T_p\mu\| = 2.46\ 10^{-2}$ operation count = 8.53			mean conv. rate: $5.64\ 10^{-1}$ $\|\mu_p - T_p\mu\| = 2.46\ 10^{-2}$ operation count = 27		

Table 5.4. $\frac{x^2}{4} + y^2 + z^2 = 1$, $\vec{U}$ parallel to the x-axis, $\gamma = 1$,

$\|T_p\mu\| = 2.41$.

REMARK. The maximum errors between the approximate solutions and the analytic values in the collocation points always appeared in the spherical caps. Here we see that the error increases, as h_p decreases. However, in all of the other rings, the maximum error decreases as h_p decreases (see table 5.5). The explanation of this feature is found in the discretization: the solid angle of a triangle-form surface element of one of the caps cannot be approximated securely enough by the solid angle of a flat triangle.

	$x^2 + y^2 + z^2 = 1$		$x^2/4 + y^2 + z^2 = 1$	
	$p = 2$	$p = 3$	$p = 2$	$p = 3$
max. error	$8.03\ 10^{-3}$	$1.29\ 10^{-2}$	$9.48\ 10^{-3}$	$2.46\ 10^{-2}$
max. error excluding the caps	$2.71\ 10^{-3}$	$1.46\ 10^{-3}$	$7.38\ 10^{-3}$	$2.16\ 10^{-3}$

Table 5.5. Maximum errors between the approximate solution (iterated up to a residual smaller than 10^{-6}) and the analytic values in the collocation points.

Finally we remark that applications in aerodynamics will often deal with bodies which approximately have the shape of a very thin ellipsoid (for instance a symmetrical wing). We expect that for such surfaces the multiple grid method with a suitable relaxation scheme will have considerable advantage with regard to a common iterative method. Another advantage of our method is its robustness: it can be applied to a large class of bodies. Whenever the surface of the body is known, eventually only in a finite number of points, the method is applicable.

ACKNOWLEDGEMENT

This work is part of the author's master's thesis, prepared while he was a trainee at the Department of Numerical Mathematics at the Mathematical Centre. The author is grateful to the Mathematical Centre for publishing this work in the NW-series. He also wishes to thank Dr. H. Schippers for his many helpful suggestions and his continuous interest in this work.

REFERENCES

[1] ATKINSON, K.E., *A survey of numerical methods for the solution of Fredholm integral equations of the second kind,* SIAM, 1976.

[2] GUNTER, N.M., *Potential theory and its applications to basic problems of mathematical physics,* New York, F. Ungar Pub., 1967.

[3] HEMKER, P.W. & H. SCHIPPERS, *Multiple grid methods for the solution of the second kind,* Mathematics of Computation, 36 (1981), pp. 215-232.

[4] KELLOGG, O.D., *Foundations of Potential Theory,* New York, F. Ungar Pub., 1929.

[5] PRENTER, P.M., *A collocation method for the numerical solution of Integral Equations,* SIAM J. Numer. Anal. 10 (1973). pp. 570-581.

[6] SLOAN, I.M., E. NOUSSAIR & B.J. BURN, *Projection methods for equations of the second kind,* Journal of Math. Anal. and Appl. 69 (1979), pp. 84-103.

[7] STETTER, H.J., *The defect correction principle and discretization methods,* Numerische Mathematik, 29 (1978), pp. 425-443.

ONTVANGEN 1 5 FEB. 1982