stichting
mathematisch
centrum

MC

AFDELING NUMERIEKE WISKUNDE (DEPARTMENT OF NUMERICAL MATHEMATICS) NW 123/82 FEBRUARI

E.J. VAN ASSELT

THE MULTI-GRID METHOD AND ARTIFICIAL VISCOSITY

Preprint

kruislaan 413 1098 SJ amsterdam

Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: 65F10, 65N10

The Multi-Grid Method and Artificial Viscosity*)

by

E.J. van Asselt

ABSTRACT

The convergence behaviour of the multi-grid method is examined for different choices of artificial viscosity in the coarse-grid operators.

The two- and multi-level algorithms are studied when they are applied to the convection-diffusion equation in two dimensions with small diffusion coefficient.

KEY WORDS & PHRASES: *multi-grid methods, artificial viscosity, convection-diffusion equation, two-level analysis*

*) This report will be submitted for publication elsewhere.

0. INTRODUCTION

We consider the convection-diffusion equation in two dimensions:

(0.1) $$L_\varepsilon u = f,$$

with

$$L_\varepsilon = \varepsilon\Delta + b_1 \frac{\partial}{\partial x} + b_2 \frac{\partial}{\partial y},$$

$$b_1^2 + b_2^2 = 1, \quad u : \mathbb{R}^2 \to \mathbb{R}.$$

A well-known method to avoid unstable discretizations of (0.1) when the diffusion coefficient ε is small in comparison with the meshwidth h is the addition of artificial viscosity to ε.

Let $\beta = C_1 h$ be the amount of artificial viscosity added to ε on a fine grid, with mesh size $\bar{h} = (h,h)$, and $\bar{\beta}$ be the artificial viscosity added to ε on a corresponding coarse grid with mesh size $\bar{H} = (H,H) = (2h,2h)$ in a two-level algorithm (TLA).

In this paper by local mode analysis we analyse two different choices of $\bar{\beta}$: $\bar{\beta} = \beta = C_1 h$ and $\bar{\beta} = 2\beta = C_1 H$. For this purpose in section 1 we give the definitions of amplification and convergence factors.

In section 2 we show that in the TLA, $\bar{\beta} = \beta$ gives a smaller convergence factor than $\bar{\beta} = 2\beta$. Further it is proved that the choice $\bar{\beta} = \beta$ corresponds with the Galerkin Approximation for the coarse-grid operator up to order h^2.

In section 3, three variants of the choice of artificial viscosity on the coarse grids in a multi-level algorithm (MLA) are examined.

1. THE AMPLIFICATION FACTOR AND THE CONVERGENCE FACTOR IN THE TWO-LEVEL ALGORITHM AND THE COARSE-GRID CORRECTION

In this section we give definitions of the amplification factor and the convergence factor of the TLA and the coarse-grid correction (CGC).

Consider the linear partial differential equation:

(1.1) $$Lu(x) = f(x), \; x = (x_1,x_2) \in \mathbb{R}^2, \; u : \mathbb{R}^2 \to \mathbb{R}.$$

Let G_h and G_H be uniform grids with mesh size $\bar{h} = (h,h) \in \mathbb{R}^2$ and $\bar{H} = (H,H) = (2h,2h) \in \mathbb{R}^2$:

$$G_h = \{(jh,kh) \mid j,k \in \mathbb{Z}\},$$

$$G_H = \{(jH,kH) \mid j,k \in \mathbb{Z},\; H = 2h\}.$$

Let GF_h and GF_H be the spaces of grid functions:

(1.2) $$GF_h = \{u_h \mid u_h : G_h \to \mathbb{R}\}$$
$$GF_H = \{u_H \mid u_H : G_H \to \mathbb{R}\},$$

provided with norm

$$\|u_h\|_h = \sup_{j,k\in\mathbb{Z}} |u_h(jh,kh)|,$$

and

$$\|u_H\|_H = \sup_{j,k\in\mathbb{Z}} |u_H(jH,kH)|$$

respectively.
Let L_h and L_H be discretizations of L on G_h and G_H:

$$L_hu_h = f_h; \; L_Hu_H = f_H,$$

with $u_h,f_h \in GF_h$ and $u_H,f_H \in GF_H$.
The amplification matrix M of one cycle of the TLA is given by:

(1.3) $$M = S^qCS^p,$$

where the number of pre- and post-relaxations is p and q respectively; S denotes the amplification matrix of the smoothing process; and C of the CGC.

With prolongation $P : GF_H \to GF_h$ and restriction $R : GF_h \to GF_H$, we have

(1.4) $$C = I - PL_H^{-1}RL_h.$$

In order to express the rate of convergence of the TLA in terms of local mode analysis, we use the following notations:

$$\hat{u}_h(\omega) = \frac{h^2}{2\pi} \sum_{j\in\mathbb{Z}^2} e^{-ijh\omega} u_h(jh), \; \omega \in LF_h \cup HF_h;$$

$$LF_h = \{(\omega_1,\omega_2) \mid \omega_1,\omega_2 \in [-\frac{\pi}{2h}, \frac{\pi}{2h}]\},$$ the range of low frequencies;

$$HF_h = \{(\omega_1,\omega_2) \mid \omega_1,\omega_2 \in [-\frac{\pi}{h}, \frac{\pi}{h}], \; (\omega_1,\omega_2) \notin LF_h\},$$ the range of high frequencies.

<u>REMARK</u>.

$\hat{u}_h : LF_h \cup HF_h \to \mathbb{C}$ is the Fourier transform of u_h.
The backtransformation formula reads

$$u_h(jh) = \frac{1}{2\pi} \int_{\omega\in LF_h\cup HF_h} e^{ijh\omega}\hat{u}_h(\omega)d\omega$$ (cf. HEMKER [4]).

$\hat{u}_h(\omega)$ is called the amplitude, and $e^{ijh\omega}$ the mode of frequency ω.

Let $\omega^{(1)} = (\omega_1^{(1)},\omega_2^{(1)}) \in LF_h$, then we define its *harmonics* by

$$\omega^{(2)} = (\omega_1^{(1)}, \omega_2^{(1)} \pm \tfrac{\pi}{h}),$$
$$\omega^{(3)} = (\omega_1^{(1)} \pm \tfrac{\pi}{h}, \omega_2^{(1)}),$$
$$\omega^{(4)} = (\omega_1^{(1)} \pm \tfrac{\pi}{h}, \omega_2^{(1)} \pm \tfrac{\pi}{h}),$$

where the + or − sign are chosen such that $\omega^{(k)} \in HF_h$, $k = 2,3,4$. (see figure 1).

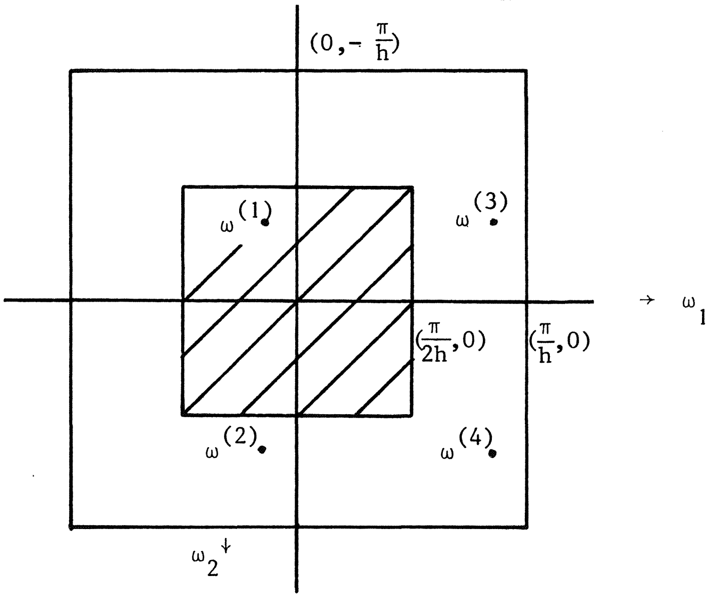


figure 1. $\omega^{(1)}$ and its harmonics.
LF_h is the shaded area.

Suppose that R and P are invariant for translations, then the four frequencies $\omega^{(k)}$, $k = 1,2,3,4$ are coupled by R and P.
For each $\omega^{(1)} \in LF_h$, we denote $\hat{u}_h(\omega^{(k)})$, $k=1,2,3,4$ in vector notation simply by $\hat{\bar{u}}_h(\omega^{(1)})$.
For all $\omega \in LF_h$ we can define a matrix $\hat{R}(\omega):\mathbb{R}^4 \to \mathbb{R}$ such that

$$\widehat{Ru_h}(\omega) = \hat{R}(\omega)\hat{\overline{u_h}}(\omega).$$

Similarly, for the prolongation we can define a $\widehat{P(\omega)}:\mathbb{R} \to \mathbb{R}^4$ such that

$$\widehat{Pu_h}(\omega) = \hat{P}(\omega)\hat{\overline{u_h}}(\omega).$$

For all $\omega \in LF_h$ with harmonics $\omega^{(k)}$, $k = 2,3,4$ we can introduce a 4×4 matrix $\hat{M}(\omega):\mathbb{R}^4 \to \mathbb{R}^4$ which relates the error e_h before to the error Me_h after one cycle of the TLA.

This $\hat{M}(\omega)$ reads:

$$(1.5) \qquad \hat{M}(\omega) = \tilde{S}(\omega)^q \, \hat{C}(\omega) \, \tilde{S}(\omega)^p,$$

with

$$\hat{C}(\omega) = I - \hat{P}(\omega)\,\widehat{L_H}(\omega)^{-1}\,\widehat{R}(\omega)\,\widetilde{L_h}(\omega),$$
$$\tilde{L}_h(\omega) = \mathrm{diag}(\hat{L}_h(\omega),\ \hat{L}_h(\omega^{(2)}),\ \hat{L}_h(\omega^{(3)}),\ \hat{L}_h(\omega^{(4)}));$$
$$\tilde{S}(\omega) = \mathrm{diag}(\hat{S}(\omega),\ \hat{S}(\omega^{(2)}),\ \hat{S}(\omega^{(3)}),\ \hat{S}(\omega^{(4)}));$$

where $\hat{L}_h$ and $\hat{S}$ are the characteristic forms (or symbols) of the operator L_h and the smoothing operator S.
The matrices $\hat{C}$ and $\hat{M}$ are called the *characteristic matrices* of the CGC and the TLA respectively.
The eigenvalues ρ of $\hat{C}(\omega)$ are:

(1.6) $$\rho_1(\omega) = 1 - \widehat{L_H}(\omega)^{-1}\,\widehat{R}(\omega)\,\widetilde{L_h}(\omega)\,\widehat{P}(\omega)$$
$$\rho_{2,3,4}(\omega) \equiv 1.$$

This leads us to the following definitions:

(1.7) DEFINITION. The eigenvalue $\rho_1(\omega)$, $\omega \in LF_h$ in (1.6) is called the *CGC amplification factor.*

(1.8) DEFINITION. The *CGC convergence factor* $\bar{\lambda}$ *is:*

$$\bar{\lambda} = \sup_{\substack{\omega \in LF_h \\ \omega \neq 0}} |\lambda(\omega)|,$$

with $\lambda(\omega)$ the CGC amplification factor.

(1.9) DEFINITION. (cf. BRANDT, DINAR [1]). The *two-level (TL) amplification factor* $\mu(\omega)$, $\omega \in LF_h$, is the eigenvalue of $\hat{M}(\omega)$ with largest modulus. $\bar{\mu} = \sup_{\substack{\omega \in LF_h \\ \omega \neq 0}} |\mu(\omega)|$ is called the *TL convergence factor.*

2. TWO-LEVEL ANALYSIS OF THE CONVECTION-DIFFUSION EQUATION

In section 2.1 we describe the addition of artificial viscosity to the diffusion coefficient ε when it is small in comparison with the meshwidth h.

In section 2.2 we express the CGC amplification factor in terms of the artificial viscosity on the fine ($\beta = C_1 h$) and on the coarse grid ($\bar{\beta}$).

In section 2.3 we show that the choice $\bar{\beta} = \beta$ gives a smaller CGC convergence factor than $\bar{\beta} = C_1 H$.

In section 2.4 by local mode analysis of a TLA with Symmetric Gauss Seidel (SGS) relaxation we obtain the same result for the TL convergence factor.

Finally in section 2.5 we show that the coarse-grid discretization with $\bar{\beta} = \beta$ corresponds with the Galerkin Approximation of $L_{\varepsilon+\beta,h}$ up to terms of order h^2.

2.1. The convection-diffusion equation

We study the convection-diffusion equation (0.1) in two dimensions. Stability of the discretization is considered in the following sense:

(2.1.1) DEFINITION. Let Lu = f be a linear PDE with constant coefficients.

Let L_h be a discretization of L, with characteristic form $\hat{L}_h$.

The *stability of* L_h *with respect to the mode* $e^{\underline{i}jh\omega}$ is the quantity $|\hat{L}_h(\omega)|$.

(2.1.2) DEFINITION. Let $L_{\varepsilon,h}$ be a discretization of (0.1) with characteristic form $\hat{L}_{\varepsilon,h}$.

The *asymptotic stability of* $L_{\varepsilon,h}$ *with respect to the mode* $e^{\underline{i}jh\omega}$ is the quantity $\lim_{\varepsilon\downarrow 0} |\hat{L}_{\varepsilon,h}(\omega)|$.

Discretization of (0.1) by central differences gives the following scheme:

$$(2.1.3) \qquad (L_{\varepsilon,h}u_h)_{i,j} \equiv \left(\frac{\varepsilon}{h^2} - \frac{b_2}{2h}\right)u^h_{i,j-1} + \left(\frac{\varepsilon}{h^2} + \frac{b_2}{2h}\right)u^h_{i,j+1} + \left(\frac{\varepsilon}{h^2} - \frac{b1}{2h}\right)u^h_{i-1,j} + \left(\frac{\varepsilon}{h^2} + \frac{b1}{2h}\right)u^h_{i+1,j} - \frac{4\varepsilon}{h^2}u^h_{i,j} = f^h_{i,j},$$

with $u_h = (\ldots,u^h_{i,j},\ u^h_{i+1,j},\ldots)$, $u^h_{i,j} = u(ih,jh)$, $f^h_{i,j} = f(ih,jh)$.

We consider $\varepsilon = O(h)$.

For all $\omega = (\omega_1,\omega_2)$ with $b_1 \sin \omega_1 h + b_2 \sin \omega_2 h = 0$ we find: $\lim_{\varepsilon\downarrow 0}|\hat{L}_{\varepsilon,h}(\omega)| = 0$. Hence the asymptotic stability of $L_{\varepsilon,h}$ with respect to the modes of these frequencies is zero.

The scheme is consistent of order 2, i.e. $\| J^1_h L_\varepsilon u - L_{\varepsilon,h} J^3_h u \|_h = O(h^2)$, with J^k_h the injection $C^k(\mathbb{R}^2) \to GF_h$, $k = 1,3$.

If we use artificial viscosity β for the discretization of (0.1), i.e. if we use $L_{\alpha,h}u_h = f_h$; $\alpha = \varepsilon + \beta = \varepsilon + h/2$, as a discretization of $L_\varepsilon u = f$, then this discretization has zero asymptotic stability: $\lim_{\varepsilon\downarrow 0} \hat{L}_{\alpha,h}(\omega) = 0$, only with respect to the mode of frequency $\omega = (\omega_1,\omega_2) = (0,0)$, and the consistency is of order 1.

2.2. The coarse-grid correction amplification factor.

In this section we give an explicit expression for the CGC amplification factor.

For prolongation P we take linear interpolation and for restriction R we take transposed linear interpolation. (7-point restriction and prolongation, cf. HEMKER [4], WESSELING [6]).

The characteristic forms read:

$$\hat{P}(\omega) = \hat{R}(\omega) = \tfrac{1}{4}(1 + \cos \omega_1 h + \cos \omega_2 h + \cos(\omega_1 - \omega_2)h),$$

The characteristic form of $L_{\alpha,h}$ reads

$$\hat{L}_{\alpha,h}(\omega) = \frac{2\alpha}{h^2}(\cos \omega_1 h + \cos \omega_2 h) - \frac{4\alpha}{h^2} + \underline{i}\,\frac{1}{h}(b_1 \sin \omega_1 h + b_2 \sin \omega_2 h);$$

An analogous form exists for the coarse-grid discretization $L_{\bar{\alpha},H}$.

Now we consider two choices for the amount of artificial viscosity $\bar\beta$ on the coarse grid:

$$\bar\beta = \beta = h/2, \qquad \text{i.e. } \bar\alpha = \alpha = \varepsilon + \beta = \varepsilon + h/2,$$

$$\bar\beta = H/2, \qquad \text{i.e. } \bar\alpha = \varepsilon + H/2 = \varepsilon + h.$$

We study the behaviour of the discretization in the limit for $\varepsilon \to 0$.

From (1.6) it follows that

$$\begin{aligned} |\lambda(\omega)| = {} & |[\{p^2\bar\alpha(\alpha-\bar\alpha) - \tfrac{h^2}{2}qr(b_2-b_1)\} \\ & + \underline{i}\ hp\{\tfrac{\bar\alpha}{2}\ r(b_2-b_1) + (\alpha-\bar\alpha)q\}] \\ & /(p^2\bar\alpha^2 + h^2q^2)|, \end{aligned}$$

with

$$p = S_1^{\ 2} + S_2^{\ 2},$$
$$q = b_1S_1C_1 + b_2S_2C_2,$$
$$r = S_1S_2S_{12},$$

and

$$S_i = \sin \omega_i h, \ C_i = \cos \omega_i h, \ i = 1,2;$$
$$S_{12} = \sin(\omega_1 - \omega_2)h, \ C_{12} = \cos(\omega_1 - \omega_2)h.$$

For the choice $\bar\beta = \beta = h/2$ in the limit for $\varepsilon \to 0$ we find:

(2.2.1) $$|\lambda_{\bar\beta = \beta}(\omega)| = |b_2 - b_1||r|/(p^2 + 4q^2)^{\frac{1}{2}}.$$

For the choice $\bar\beta = 2\beta = H/2$ in the limit for $\varepsilon \to 0$ we find

(2.2.2) $$|\lambda_{\bar\beta = 2\beta}(\omega)| = \tfrac{1}{2}((|b_1 - b_2|^2|r|^2 + p^2)/(p^2 + q^2))^{\frac{1}{2}}.$$

These two CGC amplification factors are compared with each other in the following section.

2.3. The choice of artificial viscosity on the coarse grid.

Now we compare the two CGC amplification factors $\lambda_{\bar\beta = \beta}$ and $\lambda_{\bar\beta = 2\beta}$ for different values of the convection coefficients b_1 and b_2.

(2.3.1) LEMMA. *For all* b_1, b_2 *with* $b_1^{\ 2} + b_2^{\ 2} = 1$

a) $\lim\limits_{\omega\to 0} |\lambda_{\bar\beta = \beta}(\omega)| = 0$, *and in particular*

b) $\lim\limits_{\substack{\omega\to 0 \\ b_1\omega_1 + b_2\omega_2 = 0}} |\lambda_{\bar\beta = 2\beta}(\omega)| = \tfrac{1}{2}$

PROOF. Let $\theta_1 = \omega_1 h$, $\theta_2 = \omega_2 h$.

a) For $\theta_1 = 0$:

$$\lim_{\omega\to 0} |\lambda_{\bar\beta = \beta}(\omega)| = \lim_{\substack{\theta_2\to 0 \\ \theta_1 = 0}} |\lambda_{\bar\beta = \beta}(\omega)| = 0, \qquad \text{(cf. (2.2.1))}.$$

For $\theta_2 = \xi\theta_1$:

$$\lim_{\omega\to 0} |\lambda_{\bar\beta = \beta}(\omega)| = \lim_{\theta_1\to 0} |b_2 - b_1|\theta_1^2\xi(1-\xi)/\{\theta_1^2(1+\xi^2)^2 + 4(b_1+\xi b_2)^2\}^{\frac{1}{2}} = 0$$

independent of ξ, and a) is proved.

b) For $b_2 = 0$:

$$\lim_{\substack{\omega\to 0\\ b_1\omega_1+b_2\omega_2=0}} |\lambda_{\bar\beta = 2\beta}(\omega)| = \lim_{\substack{\theta_2\to 0\\ \theta_1=0}} |\lambda_{\bar\beta = 2\beta}(\omega)| = \tfrac{1}{2}, \text{(cf 2.2.2)}.$$

For $b_2 \neq 0$, with $\xi = -\frac{b_1}{b_2}$:

$$\lim_{\substack{\omega\to 0\\ b_1\omega_1+b_2\omega_2=0}} |\lambda_{\bar\beta = 2\beta}(\omega)| = \lim_{\substack{\theta_1\to 0\\ \theta_2=\xi\theta_1}} |\lambda_{\bar\beta = 2\beta}(\omega)| = \tfrac{1}{2},$$

(cf. 2.2.2),

and b) is proved. Q.E.D.

Remark that (2.3.1) b) implies $\bar\lambda_{\bar\beta = 2\beta} \geq \frac{1}{2}$. For the following lemma we use the abbreviations p,q and r as in section 2.2.

(2.3.2) <u>LEMMA</u>. *Let* $b_1, b_2 \in \mathbb{R}$ *with* $b_1^2 + b_2^2 = 1$ *be such that for all* $\omega \in LF_h$:

$$3|b_1 - b_1|^2\, r^2 \leq p^2 + 4q^2,$$

then

a) $\quad |\lambda_{\bar\beta = \beta}(\omega)| \leq |\lambda_{\bar\beta = 2\beta}(\omega)|$ *for all* $\omega \in LF_h$, $\omega \neq 0$, *and*

b) $\quad \bar\lambda_{\bar\beta = \beta} \leq \bar\lambda_{\bar\beta = 2\beta}$

<u>PROOF</u>.

a) For $\omega \neq 0$, $p^2 + 4q^2 \neq 0$, and $p^2 + q^2 \neq 0$, so $3|b_2 - b_1|^2 r^2 \leq p^2 + 4q^2$ implies

$$|b_2 - b_1|^2 r^2/(p^2+4q^2) \leq (|b_1 - b_2|^2 r^2 + p^2)/(4(p^2+q^2))$$

which proves a).

b) From a) and the continuity of $|\lambda_{\bar\beta}(\omega)|$ for $\bar\beta = \beta$ and $\bar\beta = 2\beta$ in the surrounding of the origin b) follows directly. Q.E.D.

From this lemma we can derive the following corollaries:

(2.3.3) <u>COROLLARY 1</u>. *For all* $b_1, b_2 \in \mathbb{R}$ *with* $b_1^2 + b_2^2 = 1$ *and* $|b_2 - b_1|^2 \leq 4/3$ $\bar\lambda_{\bar\beta=\beta} \leq \bar\lambda_{\bar\beta=2\beta}$.

<u>PROOF</u>. $|r|^2/|p^2| \leq \frac{1}{4}$, and $|b_2 - b_1|^2 \leq \frac{4}{3}$ imply $3r^2|b_2 - b_1|^2 \leq p^2 + 4q^2$.
Now apply lemma (2.3.2). Q.E.D.

(2.3.5) <u>COROLLARY 2</u>. *For all* $b_1 > 0$, $b_2 \geq 0$ (*or* $b_1 < 0$, $b_2 \leq 0$) $\bar\lambda_{\bar\beta=\beta} \leq \frac{1}{2}$.

PROOF. $\max_{b_1,b_2\geq 0} |b_2 - b_1| = 1$, and $|S_{12}| \leq 1$, hence $4|b_2 - b_1|^2 r^2 \leq 4S_1{}^2 S_2{}^2 \leq p^2 + 4q^2$, and from this follows directly $|\lambda_{\bar{\beta}=\beta}(\omega)| \leq \frac{1}{2}$ for all $\omega \in LF_h$, $\omega \neq 0$. From the continuity of $|\lambda_{\bar{\beta}=\beta}(\omega)|$ in the surrounding of the origin it follows that $\bar{\lambda}_{\bar{\beta}=\beta} \leq \frac{1}{2}$. Q.E.D.

In corollary 1 we proved that for all b_1 and b_2 with $|b_2 - b_1|^2 \leq 4/3$ the amount of the fine-grid artificial viscosity on the coarse grid gives a smaller CGC convergence factor than the amount of artificial viscosity corresponding to the coarse-grid mesh size. We were not able to prove or disprove this for all b_1 and b_2.
Numerical computations of the CGC convergence factors and the CGC amplification factors on the set of frequencies:

(2.3.6) $$FG_h = \{(\omega_1 h, \omega_2 h) | \omega_1 h = j.\pi/32,\ \omega_2 h = k.\pi/32;\ j,k \in \mathbb{Z},\ -16 \leq j,\ k \leq 16,\ (j,k) \neq (0,0)\},$$

suggest that it is true for all b_1 and b_2 indeed.

Table 1 shows the maxima of the CGC amplification factors on FG_h for different values of the convection coefficients b_1 and b_2, and $\varepsilon = 10^{-6}$. Because of the symmetry of $|\lambda(\omega)|$ we considered only (b_1,b_2) on a quarter of the unit circle.

(b_1,b_2)	$\breve{\lambda}_{\bar{\beta}=\beta}$	$\breve{\lambda}_{\bar{\beta}=2\beta}$
$(\sqrt{\frac{1}{2}},\sqrt{\frac{1}{2}})$	$1.5.10^{-11}$	0.50
$(\frac{1}{2}\sqrt{3},\frac{1}{2})$	0.17	0.51
$(1,0)$	0.40	0.53
$(\frac{1}{2}\sqrt{3},-\frac{1}{2})$	0.47	0.55
$(\sqrt{\frac{1}{2}},-\sqrt{\frac{1}{2}})$	0.48	0.54

Table 1. Maxima of the CGC Amplification factors on FG_h with $\varepsilon = 10^{-6}$.

Figure 2 shows the CGC amplification factors on FG_h, multiplied by 10, and rounded to the nearest integer.

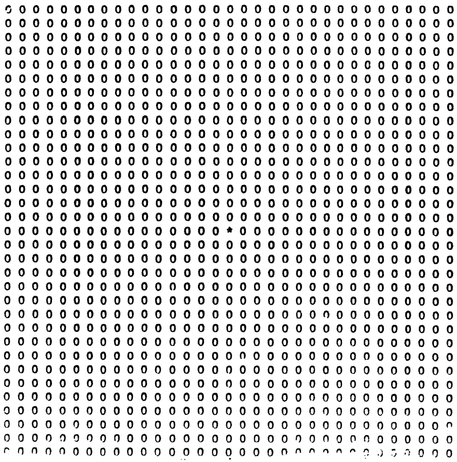


Fig. 2a. $b_1 = \sqrt{\frac{1}{2}}$, $b_2 = \sqrt{\frac{1}{2}}$, $\bar{\beta} = \beta$

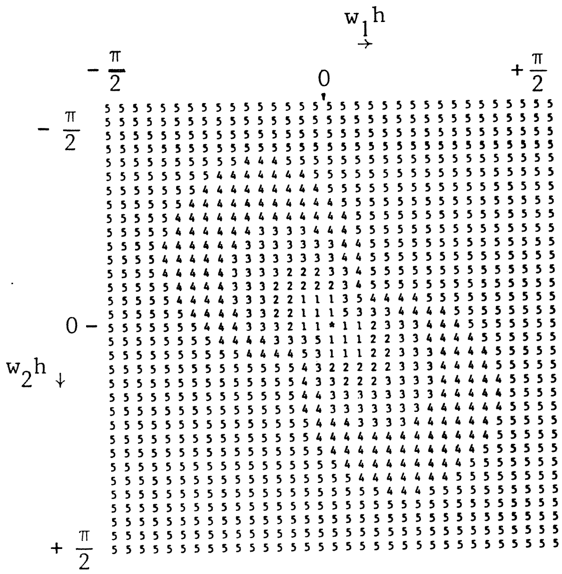


Fig. 2a'. $b_1 = \sqrt{\frac{1}{2}}$, $b_2 = \sqrt{\frac{1}{2}}$, $\bar{\beta} = 2\beta$

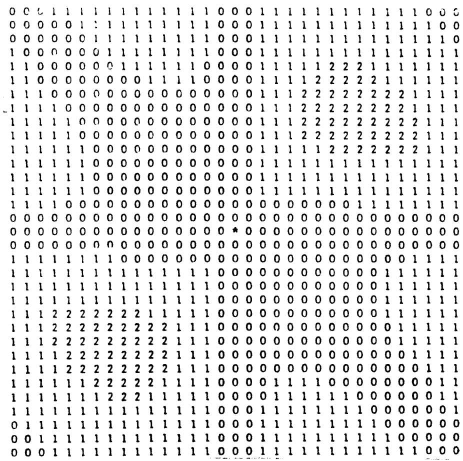


Fig. 2b. $b_1 = \frac{1}{2}\sqrt{3}$, $b_2 = \frac{1}{2}$, $\bar{\beta} = \beta$

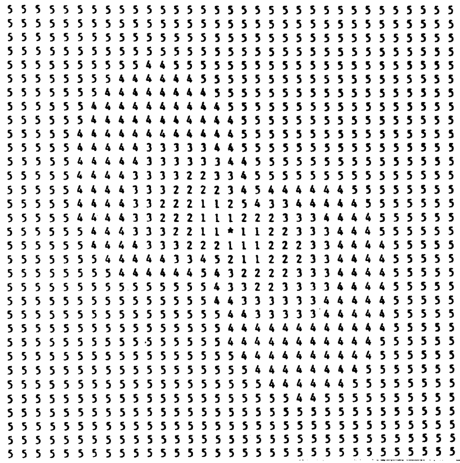


Fig. 2b'. $b_1 = \frac{1}{2}\sqrt{3}$, $b_2 = \frac{1}{2}$, $\bar{\beta} = 2\beta$

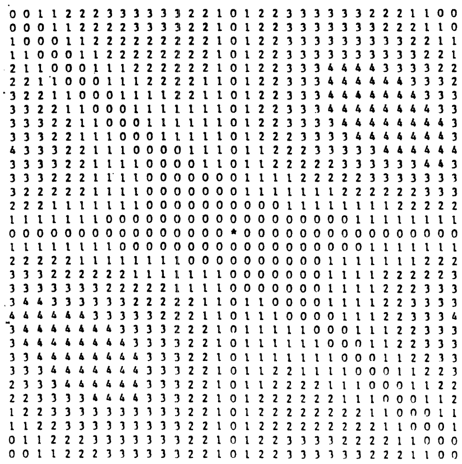


Fig. 2c. $b_1 = 1$, $b_2 = 0$, $\bar{\beta} = \beta$

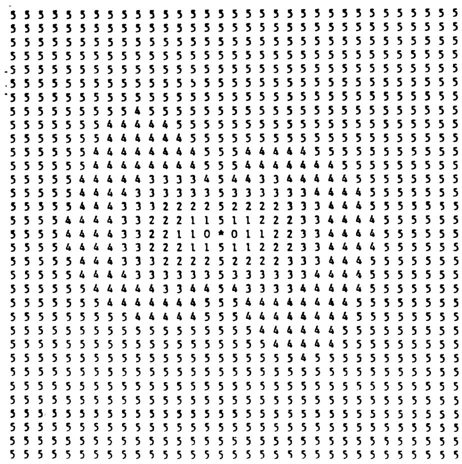
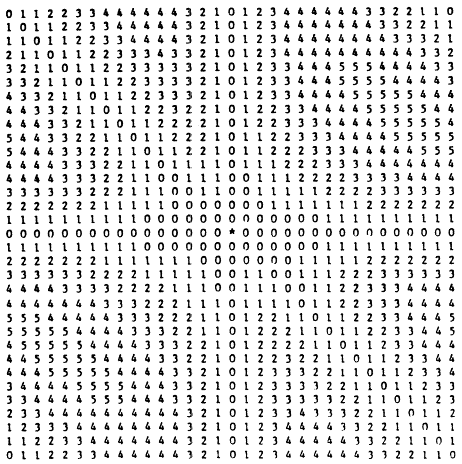
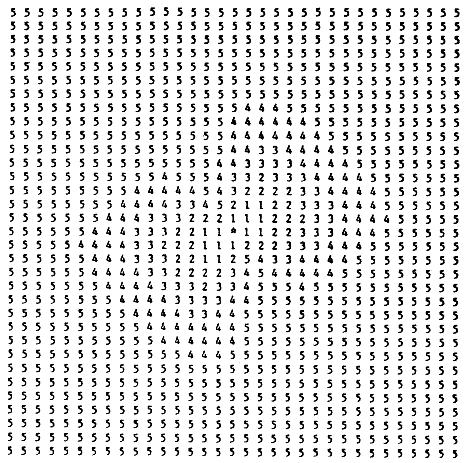


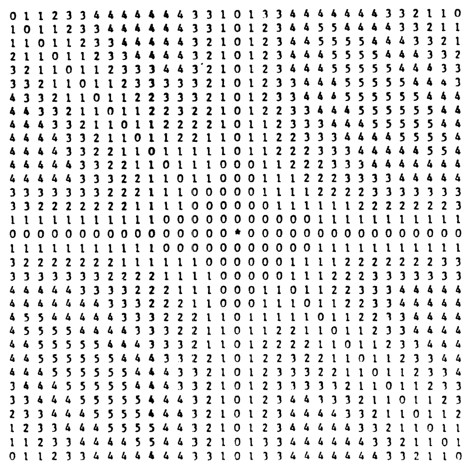
Fig. 2c'. $b_1 = 1$, $b_2 = 0$, $\bar{\beta} = 2\beta$



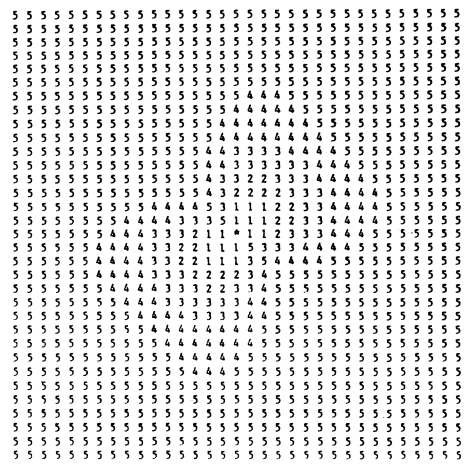
<u>Fig. 2d</u>. $b_1 = \frac{1}{2}\sqrt{3},\ b_2 = -\frac{1}{2},\ \bar{\beta} = \beta$



<u>Fig. 2d'</u>. $b_1 = \frac{1}{2}\sqrt{3},\ b_2 = -\frac{1}{2},\ \bar{\beta} = 2\beta$



<u>Fig. 2e</u>. $b_1 = \sqrt{\frac{1}{2}},\ b_2 = -\sqrt{\frac{1}{2}},\ \bar{\beta} = \beta$



<u>Fig. 2e'</u>. $b_1 = \sqrt{\frac{1}{2}},\ b_2 = -\sqrt{\frac{1}{2}},\ \bar{\beta} = 2\beta$

<u>Figure 2</u>.

CGC amplification factors, on FG_h, as a function of $\omega \in [-\frac{\pi}{h}, \frac{\pi}{h}]^2$, multiplied by 10, and rounded to the nearest integer.

2.4. <u>Values of the two-level convergence factor for Symmetric Gauss Seidel relaxation.</u>

Now we consider the TLA with SGS relaxation.
Table 2 shows the maxima of the TL amplification factors on FG_h (cf. 2.3.6) for SGS relaxation, $\varepsilon = 10^{-6}$, and different values of the convection coefficients b_1 and b_2. These maxima are approximations of the TL convergence factors. It indicates that the choice $\bar{\beta} = \beta$ gives a smaller TL convergence factor than $\bar{\beta} = 2\beta$.

For $\bar{\beta} = \beta$ the maximum amplification factor occurs for frequencies away from zero, and SGS damps the corresponding amplitudes. Therefore a second SGS sweep improves the convergence factor. However for $\bar{\beta} = 2\beta$ the very low frequencies give high values of the CGC amplification factor (cf. Figure 2) whereas SGS has no influence on them.

	$\tilde{\lambda}_{\bar{\beta}} = \beta$		$\tilde{\lambda}_{\bar{\beta}} = 2\beta$	
number of SGS-sweeps (b_1,b_2)	1 SGS-sweep	2 SGS-sweeps	1 SGS-sweep	2 SGS-sweeps
$(\sqrt{\frac{1}{2}},\sqrt{\frac{1}{2}})$	0.15	0.06	0.50	0.50
$(\frac{1}{2}\sqrt{3},\frac{1}{2})$	0.17	0.08	0.48	0.47
$(1,0)$	0.23	0.10	0.50	0.50
$(\frac{1}{2}\sqrt{3},-\frac{1}{2})$	0.28	0.11	0.49	0.48
$(\sqrt{\frac{1}{2}},-\sqrt{\frac{1}{2}})$	0.24	0.11	0.50	0.50

Table 2. Maxima of TL amplification factors on FG_h as approximation to the TL convergence factor; $\varepsilon = 10^{-6}$.

2.5. Relation of $L_{\bar{\alpha},h}$ to the Galerkin Approximation of $L_{\alpha,h}$.

Here we show that the operator $L_{\bar{\alpha},H}$, $\bar{\alpha} = \varepsilon + \beta$ corresponds to the Galerkin Approximation of $L_{\alpha,h}$, $\alpha = \varepsilon + \beta$ (cf. FREDERICSON [2], HACKBUSCH [3], WESSELING [6]), up to terms of order h^2.

2.5.1. DEFINITION. $C^k(\mathbb{R}^2)$ is the space of real functions f with continuous partial derivatives $\dfrac{\partial^j f}{\partial^m x \partial^n y}$, $j = m+n = 0,\dots,k$; $m,n \geq 0$, and with norm $\|f\|_k = \max\limits_{j=0,\dots,k} |f|_j$, where

$$|f|_j = \max_{j=m+n} \{ \sup_{(x,y)\in\mathbb{R}^2} |\frac{\partial f^j(x,y)}{\partial x^m \partial y^n}| \}.$$

2.5.2. DEFINITION. J_h^k is the injection $C^k(\mathbb{R}^2) \to GF_h$.

The mapping $\|\cdot\|_{3,h}$: $GF_h \to \mathbb{R}$ defined by

$$\begin{aligned}\|u_h\|_{3,h} = \max\{\sup_{i,j}|u^h_{i,j}|,\ &\sup_{i,j} |\Delta_{xxx}\, u^h_{i,j}| \\ ,\ &\sup_{i,j} |\Delta_{xxy}\, u^h_{i,j}| \\ ,\ &\sup_{i,j} |\Delta_{xyy}\, u^h_{i,j}| \\ ,\ &\sup_{i,j} |\Delta_{yyy}\, u^h_{i,j}|\},\end{aligned}$$

where

Δ_{xxx}, Δ_{xxy},... are third order differences,

e.g.

$$\Delta_{xxx}\, u^h_{i,j} = \frac{1}{h^3}\,(u^h_{i+2,j} - 3u^h_{i+1,j} + 3u^h_{i,j} - u^h_{i-1,j}),$$

$$\Delta_{xxy}\, u^h_{i,j} = \frac{1}{h^3}\,(u^h_{i+1,j+1} - 2u^h_{i,j+1} + u^h_{i-1,j+1} - u^h_{i+1,j} + 2u^h_{i,j} - u^h_{i-1,j}),$$

is a norm on GF_h.

(2.5.3) THEOREM. *Let* P *be the 7-point prolongation* $P: GF_H \to GF_h$, *and* $\bar{R}$ *be the 7-point restriction*: $\bar{R}: GF_h \to GF_H$. *Let* R *be the injection*: $GF_h \to GF_H$. *Let* $G_{\bar{C}} = \{v \mid v \in GF_h,\ h \in \mathbb{R},\ \exists u \in C^4(\mathbb{R}^2),\ \|u\|_4 \le \bar{C},\ J^4_h u = v\}$, *then for all* $v_H \in GF_H$ *with* $Pv_H \in G_{\bar{C}}$

$$\| L_{\alpha,H} v_H - \bar{R} L_{\alpha,h} P v_H \|_H \le h^2 \tilde{C} \| Pv_H \|_{3,h},$$

where the constant $\tilde{C}$ *depends on* $\bar{C}$, b_1 *and* b_2.

PROOF. Let $v_H \in GF_H$ with $Pv_H \in G_{\bar{C}}$, hence there exists an $u \in C^4(\mathbb{R}^2)$ with $J^4_h u = Pv_H$. Application of Taylor expansion, and the mean value theorem, for $\bar{\alpha} = \alpha$ yields (cf. 1.2):

$$\| L_{\alpha,H} R J^4_h u - \bar{R} L_{\alpha,h} J^4_h u \|_H \le C_1 h^2 |u|_3;$$

C_1 depends on b_1 and b_2.

Application of Taylor expansion and the mean value theorem yields:

$$\sup_{i,j} \left| \frac{\partial^3 u}{\partial x^3}(ih,jh) - \Delta_{xxx}(J^4_h u)_{i,j} \right| \le C.\bar{C}.h,$$

$$\sup_{i,j} \left| \frac{\partial^3 u}{\partial x^2 \partial y}(ih,jh) - \Delta_{xxy}(J^4_h u)_{i,j} \right| \le C.\bar{C}.h,$$

$$\sup_{i,j} \left| \frac{\partial^3 u}{\partial x \partial y^2}(ih,jh) - \Delta_{xyy}(J^4_h u)_{i,j} \right| \le C.\bar{C}.h,$$

$$\sup_{i,j} \left| \frac{\partial^3 u}{\partial y^3}(ih,jh) - \Delta_{yyy}(J^4_h u)_{i,j} \right| \le C.\bar{C}.h.$$

Hence

$$\| L_{\alpha,H} R P v_H - \bar{R} L_{\alpha,h} P v_H \|_H \le \tilde{C} h^2 \| Pv_H \|_{3,h},$$

where $\tilde{C}$ depends on $\bar{C}$, b_1 and b_2.

Since RP = I, the identity on GF_H the theorem is proved. Q.E.D.

REMARK. The operator $\bar{R} L_{\alpha,h} P$ is called the *Galerkin Approximation of* $L_{\alpha,h}$.

$$\| Pv_H \|_{3,h} \le \|P\| \, \|v_H\|_{3,H} \text{ with } \|P\| = \sup_{\|v_H\|_{3,H}=1} \|Pv_H\|_{3,h},$$

hence the right hand side of the inequality in theorem (2.5.4) can be replaced by $\hat{C} h^2 \|v_H\|_{3,H}$ where $\hat{C}$ depends on $\bar{C}$, b_1, b_2 and P.

3. THE CHOICE OF ARTIFICIAL VISCOSITY IN THE MULTI-LEVEL ALGORITHM

In this section we describe how the results of section 2 can be used in a MLA. We discuss three variants. Consider a MLA with n+1 levels: $\ell_0, \ell_1, \ldots, \ell_n$. To solve $L_{\alpha,h}\, u_h = f_h$ on level ℓ_n the MLA can be applied with different amounts of artificial viscosity on the levels $\ell_0, \ldots, \ell_{n-1}$. On each level the amount can be related either to the meshwidth of the finer or to that of the coarser grid.

Table 3 shows the three variants.
In variant 1 the artificial viscosity is the same on all levels.
In variant 2 the artificial viscosity on each level ℓ_k ($0 \le k < n+1$) corresponds to the meshwidth h_{k+1} on the level ℓ_{k+1}.
In variant 3 the viscosity on each level ℓ_k corresponds to the meshwidth h_k.

level	variant 1	variant 2	variant 3
ℓ_n	β	β	β
ℓ_{n-1}	β	β	2β
ℓ_{n-2}	β	2β	4β
$\vdots$	$\vdots$	$\vdots$	$\vdots$
ℓ_k	β	$2^{n-k-1}\cdot\beta$	$2^{n-k}\cdot\beta$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
ℓ_0	β	$2^{n-1}\cdot\beta$	$2^n\cdot\beta$

Table 3. Three variants for the choice of artificial viscosity on sublevels in a MLA.

Many authors applied the Galerkin Approximation with success (cf. MOL [5]). So according to section 2.5, this corresponds with variant 1. However if the number of levels becomes large, the asymptotic stability of the operators on the coarser grids descreases and numerical experiments show that (e.g. for SGS-relaxation) divergence may occur (cf. de ZEEUW, van ASSELT [7]).

With values of β even smaller than $h_n/2$ these experiments also show that the other variants still converge, and variant 2 has a better rate of convergence than variant 3.

Further, the asymptotic stability of the operators in variant 2 and in variant 3 are of the same order, and the amount of work for the three variants is the same.

4. CONCLUSIONS

For a two-level algorithm the choice $\bar{\beta} = \beta$ is better than $\bar{\beta} = 2\beta$. The choice $\bar{\beta} = \beta$ corresponds with the Galerkin Approximation of $L_{\alpha,h}$ up to terms of order h^2.

For a multi-level algorithm variant 2 is preferable to variant 1 and variant 3.

ACKNOWLEDGEMENT

The author wishes to thank Dr. P.W. Hemker, and Drs. P.M. de Zeeuw for the stimulating discussion on the subject and the assistence in the computational work.

REFERENCES

[1] BRANDT, A.; DINAR, N., *Multi-grid solutions to elliptic flow problems*, Report Number 79-15, July 16, ICASE (1979).

[2] FREDERICKSON, P.O., *Fast approximate inversion of large sparse linear systems*, Mathematics report 7-75, Lakehead University, Ontario, Canada (1975).

[3] HACKBUSCH, W., *On the multigrid method applied to difference equations*, Computing 20 (1978), 291-306.

[4] HEMKER, P.W., *Fourier Analysis of gridfunctions, prolongations and restrictions*, Report NW 93/80, Mathematical Centre, Amsterdam (1980).

[5] MOL, W.J.A., *On the choice of suitable operators and parameters in multigrid methods*, Report NW 107/81, Mathematical Centre, Amsterdam (1981).

[6] WESSELING, P., *Theoretical and practical aspects of a multigrid method*, Report NA-37, Delft University of Technology (1980).

[7] ZEEUW, P. de; ASSELT. E.J. van, *Numerical computation of the convergence rate of the multi-level algorithm applied to the convection-diffusion equation.* To be published as Report NW, Mathematical Centre, Amsterdam (1982).