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A fourth order ADI method for semidiscrete parabolic equations

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Abstract: A fourth order fourstep ADI method is described for solving the systems of ordinary differential equations which are obtained when a (nonlinear) parabolic initial-boundary value problem in two dimensions is semi-discretized. The local time-discretization error and the stability conditions are derived. By numerical experiments it is demonstrated that the (asymptotic) fourth order behaviour does not degenerate if the time step increases to relatively large values. Also a comparison is made with the classical ADI method of Peaceman and Rachford showing the superiority of the fourth order method in the higher accuracy region, particularly in nonlinear problems.

Keywords: Numerical analysis, parabolic equations, method of lines, ADI methods.

1. Introduction

In a few recent papers [1,4,5] *multistep splitting methods* were analysed for solving two-dimensional parabolic initial-boundary value problems. By using the method of lines the problem is first reduced to a (usually very large) system of ODE's of the explicit form

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0, \tag{1.1}$$

and then an implicit linear multistep method (LMM) is applied to obtain at each integration step an *implicit* equation for the numerical solution y_{n+1} at t_{n+1}:

$$y_{n+1} - b_0 \tau f(t_{n+1}, y_{n+1}) = \sum_{l=1}^{k} \left[a_l y_{n+1-l} + b_l \tau f(t_{n+1-l}, y_{n+1-l}) \right]. \tag{1.2}$$

Here, τ is the integration step and $\langle a_l, b_l \rangle$ are coefficients defining the LMM. The papers mentioned above describe methods for approximating the solution of (1.2) by using a splitting of the right-hand side function $f(t, y)$, e.g. $f(t, y) = f_1(t, y) + f_2(t, y)$ where f_1 and f_2 have 'simply structured' Jacobian matrices. More generally, one may use splitting functions $F(t, u, v)$ such that $F(t, y, y) = f(t, y)$ and $\partial F/\partial u$, $\partial F/\partial v$ are again 'simply structured'.

The method analysed and tested in this paper is a special case of a class of methods described in [5]. These methods explicitly use the information that (1.1) originates from a parabolic problem so that the eigenvalues of $\partial f/\partial y$ will be located in a long narrow strip along the *negative axis*. At the same time, this is also a restriction in the applicability of these methods.

An outline of the construction of the method is as follows. The system of equations (1.2) is solved by a (nonlinear) splitting method (e.g. ADI) and this iteration process is accelerated by using *Chebyshev polynomials* [9, p. 344]. The relaxation parameter in the splitting method is chosen such that the approximation obtained for the solution of (1.2) has a maximal order of accuracy as $\tau \to 0$ for a given LMM and a given initial approximation used for starting the iteration process. The iteration parameters in the Chebyshev polynomials are chosen such that the *lower frequencies* in the initial error are strongly

damped. As a consequence we obtain a rather fast convergence to the solution of (1.2) in problems where the low frequencies are *dominant* in the solution of the initial-boundary value problem. For details of the construction we will refer to [5].

In Section 2 the specification of the method will be given, its local error will be derived and the characteristic equation for a class of model problems will be analysed. In Section 3 the results of Section 2 will be applied to the case where the LMM (1.2) is identified with the fourth order backward differentiation formula (BDF4) and where the initial approximation used in the iteration process is a 'smoothed' extrapolation formula. It will be shown that the resulting fourth order, fourstep splitting method has a real stability boundary bounded below by cm^4, m being the number of iterations and c some constant (numerical verification reveals that $c \equiv 4$). Finally, in Section 4 the method proposed in this paper is compared with the classical ADI method of Peaceman and Rachford [7] showing the superiority of the present method (particularly in nonlinear problems) if higher accuracies are desired. This favourable behaviour is due to the property that the fourth order method really behaves as a fourth order method even for relatively large integration steps (this behaviour is not shared by the high order splitting methods analysed in [4], the order of which degenerates for larger values of the time step).

2. Multistep splitting methods for nonlinear equations

2.1. Specification of the method

The method constructed in [5] and more fully analysed in this paper is defined by

$$y^{(0)} = \text{some predictor formula for } y_{n+1}, \tag{2.1a}$$

$$y^{(j+1)} = (\mu_j - \lambda_j) y^{(j)} + (1 - \mu_j) y^{(j-1)} + \lambda_j y^{**}, \quad j = 0, 1, \ldots, m-1, \tag{2.1b}$$

$$y_{n+1} = y^{(m)},$$

where y^{**} is determined by the two equations

$$\begin{aligned} &\omega y^{**} + (1-\omega) y^{*} - b_0 \tau F(t_{n+1}, y^{**}, y^{*}) = \Sigma, \\ &\omega y^{*} + (1-\omega) y^{(j)} - b_0 \tau F(t_{n+1}, y^{(j)}, y^{*}) = \Sigma, \end{aligned} \tag{2.2}$$

with

$$\Sigma = \sum_{l=1}^{k} [\, a_l y_{n+1-l} + b_l \tau f(t_{n+1-l}, y_{n+1-l})] \tag{2.3}$$

and $F(t, u, v)$ a splitting function such that $F(t, y, y) \equiv f(t, y)$.

The coefficients μ_j and λ_j are defined by

$$\begin{aligned} &\mu_0 = 1, \qquad \mu_j = 2w_0 \frac{T_j(w_0)}{T_{j+1}(w_0)}, \qquad w_0 = \frac{b+a}{b-a}, \\ &\lambda_0 = \frac{2}{b+a}, \qquad \lambda_j = \frac{2\mu_j}{b+a}, \quad j = 1, 2, \ldots, m-1, \\ &a = \frac{(2\omega - 1)(2S^* + 1)}{(S^* + \omega)^2}, \qquad b = \frac{2\omega - 1}{\omega}, \end{aligned} \tag{2.4}$$

where T_j is the Chebyshev polynomial of degree j, ω is the largest real solution of the equation

$$(2S^* + 1)\left(\cos \frac{\pi}{2m} + 1\right)\omega^2 = \left[2 + \omega\left(\cos \frac{\pi}{2m} - 1\right)\right](S^* + \omega)^2, \tag{2.5}$$

and $S^* (\geq 0)$ is a free parameter to be used for maximizing the stability interval. The method is completely

defined if we specify $y^{(0)}$ by some predictor formula, Σ by choosing an appropriate LMM$\langle a_j, b_j\rangle$, and (m, S^*) on basis of stability considerations. Notice that we do not need $y^{(-1)}$ because $\mu_0 = 1$.

We remark that for a class of model problems (cf. Section 2.2) the interval $[a, b]$ corresponds to the eigenvalue interval of those eigenvectors which are strongly damped by the Chebyshev iteration. Furthermore, as we will see in the next section, the relaxation parameter ω defined by (2.5) decreases the magnitude of the local error as $\tau \to 0$.

2.2. The local error

Let η denote the solution of (1.2) and define the iteration error $\varepsilon_j = y^{(j)} - \eta$. Furthermore, we write $\varepsilon^* = y^* - \eta$ and $\varepsilon^{**} = y^{**} - \eta$. The local error is given by

$$y^{(m)} - y(t_{n+1}) = \varepsilon_m + \eta - y(t_{n+1}), \tag{2.6}$$

where we assume that $y_j = y(t_j)$ for $j \leqslant n$ (localizing assumption). Thus, when we are given the local error of the generating multistep method (1.2) and if we can find an estimate for the iteration error, then we have found an estimate for the local error of the splitting method (2.1)–(2.5).

In order to derive a recurrence relation for the iteration error as $\tau \to 0$ we first deduce from (2.2) the relations [1]

$$\begin{aligned} [\omega - Z_2]\varepsilon^* &= [\omega - 1 + Z_1]\varepsilon_j + \tau O\big(\|\varepsilon_j\|^2 + \|\varepsilon_j\|\|\varepsilon^*\| + \|\varepsilon^*\|^2\big), \\ [\omega - Z_1]\varepsilon^{**} &= [\omega - 1 + Z_2]\varepsilon^* + \tau O\big(\|\varepsilon^*\|^2 + \|\varepsilon^*\|\|\varepsilon^{**}\| + \|\varepsilon^{**}\|^2\big), \end{aligned} \tag{2.7}$$

as $\tau \to 0$. Here Z_1 and Z_2 are defined by $b_0\tau\partial F/\partial u$ and $b_0\tau\partial F/\partial v$ where the derivatives are evaluated at (t_{n+1}, η, η).

Let us assume that all iteration errors ε_j are $O(\tau^s)$ as $\tau \to 0$ for $s \geqslant 0$. Then it follows from (2.7) that

$$\varepsilon^{**} = [\omega - Z_1]^{-1}[\omega - 1 + Z_2][\omega - Z_2]^{-1}[\omega - 1 + Z_1]\varepsilon_j + O(\tau^{2s+1})$$

as $\tau \to 0$. From (2.1) we derive the recurrence relation

$$\varepsilon_{j+1} = \big[\mu_j - \lambda_j A\big]\varepsilon_j + (1 - \mu_j)\varepsilon_{j-1} + O(\tau^{2s+1}) \quad \text{as } \tau \to 0, \tag{2.1'}$$

where the matrix A is defined by

$$A = I - [\omega - Z_1]^{-1}[\omega - 1 + Z_2][\omega - Z_2]^{-1}[\omega - 1 + Z_1]. \tag{2.8}$$

Let us write

$$\varepsilon_j = P_j(A)\varepsilon_0 + c_j\tau^{2s+1} \quad \text{as } \tau \to 0, \tag{2.9}$$

where P_j is the polynomial which satisfies the recurrence relation

$$P_0 = I, \qquad P_{j+1} = (\mu_j - \lambda_j A)P_j + (1 - \mu_j)P_{j-1}, \quad j = 0, 1, \ldots \tag{2.10}$$

and c_j yet to be determined. Substitution of (2.9) into (2.1') reveals that the representation (2.9) is correct provided that c_j satisfies the recurrence relation

$$c_0 = 0, \qquad c_{j+1} = (\mu_j - \lambda_j A)c_j + (1 - \mu_j)c_{j-1} + O(1) \quad \text{as } \tau \to 0.$$

Evidently, for $j \leqslant m$ the coefficients c_j are bounded as $\tau \to 0$, hence

$$\varepsilon_j = P_j(A)\varepsilon_0 + O(\tau^{2s+1}) \quad \text{as } \tau \to 0. \tag{2.9'}$$

Finally, we have to determine s, that is the order in τ of ε_j as $\tau \to 0$. Suppose that the linear multistep method (1.2) and the predictor formula for $y^{(0)}$ have orders of accuracy p and q, respectively. Then

$$\varepsilon_0 = y^{(0)} - \eta = y^{(0)} - y(t_{n+1}) + y(t_{n+1}) - \eta = O(\tau^{q+1} + \tau^{p+1}).$$

[1] For the sake of simplicity we will omit the unit matrix whenever a scalar quantity ($\neq 1$) is multiplied by it, e.g. $\omega - Z_2$ is written instead of $\omega I - Z_2$.

From (2.9') it follows that apparently $\varepsilon_j = O(\tau^{q+1} + \tau^{p+1})$ provided that $P_j(A)$ is bounded as $\tau \to 0$ for $j = 1, 2, \ldots, m$. Thus $s = \min\{p+1, q+1\}$. Since it follows from (2.4) that $P_j(A)$ can be identified with the *shifted Chebyshev polynomial*

$$P_j(A) = \frac{T_j(w_0 + w_1 A)}{T_j(w_0)}, \quad w_1 = \frac{-2}{b-a},$$

so that $P_j(A)$ is bounded as $\tau \to 0$ for all finite j and S^*, we may summarize the results in the form of the following theorem.

Theorem 2.1. *The local error of the method* (2.1)–(2.5) *is given by*

$$y^{(m)} - y(t_{n+1}) = [I - P_m(A)](\eta - y(t_{n+1})) + P_m(A)(y^{(0)} - y(t_{n+1})) + O(\tau^{2p+3} + \tau^{2q+3}). \quad \square \tag{2.11}$$

In practice, we often have $p > q$ so that the term originating from the predictor formula will largely determine the magnitude of the local error. It is therefore of interest to estimate the norm of the amplification matrix $P_m(A)$ as $\tau \to 0$. In [5] an estimate is given which assumes the form

$$\|P_m(A)\| \leqslant D[T_m(w_0 + w_1\alpha_0) + O(\tau/D^{1/m})] \quad \text{as } \tau \to 0, \tag{2.12}$$

provided that $S^* \neq 0$ and where

$$D = T_m^{-1}(w_0), \qquad \alpha_0 = \frac{2\omega - 1}{\omega^2}. \tag{2.13}$$

From (2.4) and (2.5) it follows that $T_m(w_0 + w_1\alpha_0) = 0$, i.e. $P_m(\alpha_0) = 0$, hence by assuming that $D = O(\tau^r)$ as $\tau \to 0$ it follows from (2.12) and Theorem 2.1 that the order of consistency of the method (2.1)–(2.5) is given by

$$\tilde{p} = \min\{p, q + r + 1 - r/m\}, \quad S^* \neq 0. \tag{2.14}$$

In our experiments we used fixed S^* values for given m and therefore by virtue of (2.5) fixed values for ω. Since D is related to ω by the formula

$$D = T_m^{-1}\left(\frac{1 + \omega\cos(\pi/2m)}{\omega - 1}\right), \tag{2.13'}$$

we conclude that D does not depend on τ, i.e. $r = 0$, so that

$$\tilde{p} = \min(p, q+1), \quad S^* \neq 0. \tag{2.14'}$$

Thus, even for zero order predictor formulas ($q = 0$) the method (2.1)–(2.5) is still a consistent integration method provided (of course) that the generating LMM is consistent ($p \geqslant 1$).

Apart from the estimate (2.12) it is of interest how the operator $P_m(A)$ damps the lower and higher frequencies in the predictor error $y^{(0)} - y(t_{n+1})$. In [5] a result is given for the following class of *model problems:*

(i) The matrices $Z_1 = b_0\tau(\partial F/\partial u)(t_{n+1}, \eta, \eta)$ and $Z_2 = b_0\tau(\partial F/\partial v)(t_{n+1}, \eta, \eta)$ have a *common eigensystem* $\{e_i\}$.

(ii) The eigenvalues $z_j^{(i)}$ of Z_j, $j = 1, 2$, are *negative.*

Theorem 2.2. *Let the local errors of* $y^{(0)}$ *and* $y^{(m)}$ *have the eigenvector expansions*

$$y^{(0)} - y(t_{n+1}) = \sum_i c_0^{(i)} e_i, \qquad y^{(m)} - y(t_{n+1}) = \sum_i c^{(i)} e_i,$$

then

$$|c^{(i)}| \leqslant \begin{cases} D|c_0^{(i)}| & \text{for } -S^* \leqslant z_1^{(i)}, z_2^{(i)} < 0, \\ |c_0^{(i)}| & \text{otherwise.} \end{cases} \quad \square$$

We will choose the splitting function $F(t, u, v)$ such that the eigenvectors e_i of low frequency correspond to eigenvalues $z_j^{(i)}$, $j = 1, 2$, on the right end of the eigenvalue interval (e.g. if F corresponds to ADI splitting). Then, Theorem 2.2 implies that the *low frequencies* in the local error of the predictor formula are *damped* by a *factor D*. Thus, if the problem is smooth so that no high frequencies are involved and if the LMM and the predictor formula themselves do not introduce high frequencies, we may expect a fast convergence to the solution of the LMM (1.2).

2.3. The characteristic equation

Here, we confine our considerations to the class of model problems specified in the preceding section. For such problems it was derived in [5] that the variational equation of the method (2.1)–(2.5) is given by

$$\Delta y_{n+1} = P_m(A)\Delta y^{(0)} + [I - P_m(A)][I - Z_1 - Z_2]^{-1}\Delta\Sigma, \tag{2.15}$$

where $\Delta y^{(0)}$ denotes a perturbation of $y^{(0)}$, and Δy_{n+1} and $\Delta\Sigma$ denote perturbations caused by perturbations of $y_n, y_{n-1}, \ldots, y_{n+1-k}$.

Let us assume that $\Delta y^{(0)}$ can also be expressed in terms of the perturbations $\Delta y_n, \ldots, \Delta y_{n+1-k}$, say

$$\Delta y^{(0)} = \sum_{l=1}^{k} c_l(Z_1, Z_2)\Delta y_{n+1-l}. \tag{2.16}$$

Then using the definition of Σ in (2.3) we find the characteristic equation

$$\zeta^k = \sum_{l=1}^{k} \left\{ P_m(\alpha)c_l(z_1, z_2) + \frac{1 - P_m(\alpha)}{1 - z_1 - z_2}\left[a_l + \frac{b_l}{b_0}(z_1 + z_2)\right]\right\}\zeta^{k-l}, \tag{2.17}$$

where α denotes an eigenvalue of A, i.e.

$$\alpha = (2\omega - 1)\frac{1 - z_1 - z_2}{(\omega - z_1)(\omega - z_2)}, \tag{2.18}$$

with z_1 and z_2 assuming values in the eigenvalue intervals of Z_1 and Z_2.

The method (2.1)–(2.5) will be called *stable* if (2.17) has its roots on the unit disk for all z_1 and z_2 in the eigenvalue intervals of Z_1 and Z_2, respectively. This condition will be called the *root condition*. If the functions c_l are constant and $b_l = 0$, $l > 0$, this root condition usually leads to a condition on $P_m(\alpha)$ of the type

$$-1 < -D_1 \leqslant P_m(\alpha(z_1, z_2)) \leqslant D_2 \leqslant 1, \tag{2.19}$$

where D_1 and D_2 are positive constants.

In the following stability theorem $T_{1/m}(\cdot)$ means $\cosh(\operatorname{arccosh}(\cdot)/m)$.

Theorem 2.3. *Let σ be the spectral radius of $\partial f/\partial y$ at (t_{n+1}, η) and let the root condition be satisfied if P_m satisfies* (2.19) *for all z_1 and z_2 in the eigenvalue intervals of Z_1 and Z_2. Then the method* (2.1)–(2.5) *is stable if*

$$\omega \leqslant \tilde{\omega}, \quad \tilde{\omega} = \frac{T_{1/m}(1/\tilde{D}) + 1}{T_{1/m}(1/\tilde{D}) - \cos(\pi/2m)}, \qquad \tilde{D} = \min\{D_1, D_2\}, \tag{2.20a}$$

$$\tau \leqslant \frac{\beta}{\sigma}, \quad \beta = \frac{2\omega(1 + \sqrt{1 - \tilde{a}}) - 2}{b_0(1 - \sqrt{1 - \tilde{a}})}, \tag{2.20b}$$

where $\tilde{a}$ is the point where $P_m(\alpha)$ assumes the value D_2, i.e.

$$\tilde{a} = \begin{cases} 0 & \text{if } D_2 = 1, \\ \dfrac{2\omega - 1}{\omega^2(\cos(\pi/2m) + 1)}\left[1 + \omega\cos(\pi/2m) - (\omega - 1)T_{1/m}(D_2/D)\right] & \text{if } D_2 < 1, \end{cases} \tag{2.21}$$

with D given by (2.13′).

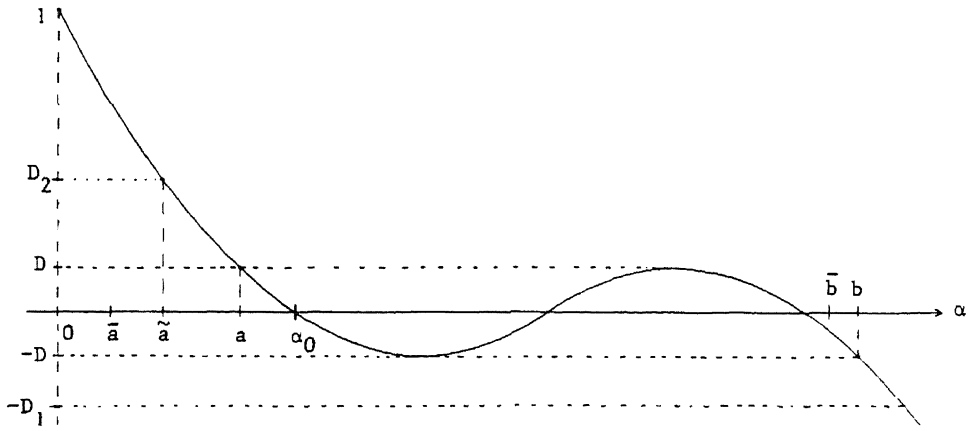

Fig. 2.1. The polynomial $P_m(\alpha)$ for $m = 3$.

Proof. It may be helpful to consider the behaviour of $P_m(\alpha)$ as a function of α (see Fig. 2.1). $P_m(\alpha)$ assumes the value 1 at $\alpha = 0$ and it has a zero at $\alpha = \alpha_0$ where α_0 is defined in (2.13). Furthermore, $P_m(\alpha)$ assumes a minimal maximum norm of magnitude D defined in (2.13) over the interval $[a, b]$. We observe that the eigenvalue interval of the matrix A is given by (cf. (2.4) and (2.18))

$$[\bar{a}, \bar{b}], \qquad \bar{a} = (2\omega - 1)\frac{S+1}{\left(\frac{1}{2}S + \omega\right)^2}, \quad \bar{b} = \frac{2\omega - 1}{\omega}\frac{S+1}{S+\omega}, \tag{2.22}$$

where $S = b_0\tau\sigma$.

It is now evident from (2.19) that $D \leqslant \tilde{D}$, hence from (2.13') it easily follows that ω should satisfy (2.20a). Thus, (2.20a) implies that $P_m(\alpha) \geqslant -D_1$ for all eigenvalues $\alpha \in [\bar{a}, \bar{b}]$.

The condition $P_m(\alpha) \leqslant D_2$ for all $\alpha \in [\bar{a}, \bar{b}]$ is satisfied if $\bar{a} \geqslant \tilde{a}$ where $\tilde{a}$ is the (first) point where $P_m(\alpha) = D_2$ (see Fig. 2.1), i.e. the point defined by (2.21). From (2.22) it follows that $\bar{a} \geqslant \tilde{a}$ if S satisfies the inequality

$$(2\omega - 1)\frac{S+1}{\left(\frac{1}{2}S + \omega\right)^2} \geqslant \tilde{a}.$$

Replacing S by $b_0\tau\sigma$ leads to condition (2.20b). □

From this theorem it follows that (2.20a) presents an upperbound for ω and by observing that S^* is an increasing function of ω (cf. (2.5)) an upperbound for S^*. This means that only a limited number of low frequencies in the local error of $y^{(0)}$ can be strongly damped (cf. Theorem 2.2). Of course, by decreasing τ the number of strongly damped eigenvectors can be increased.

The condition (2.20b) on the integration step τ is illustrated in the following subsections.

2.3.1. Stability boundaries for the method of successive corrections

Suppose we choose $S^* = 0$, then $\omega = \lambda_j = \mu_j = 1$ and the method reduces to the *method of successive corrections* analysed in [4]. For this method we have $D = 0$ so that the equation for $\tilde{a}$ reduces to

$$P_m(\tilde{a}) = (1 - \tilde{a})^m = D_2.$$

Solving this equation and substitution into (2.20b) yields

$$\tau \leqslant \frac{2D_2^{1/2m}}{b_0\sigma\left(1 - D_2^{1/2m}\right)} = \frac{4m}{b_0\sigma \ln(1/D_2)}\left(1 + O(1/m^2)\right) \quad \text{as } m \to \infty. \tag{2.20'b}$$

Table 2.1
Stability boundaries for the method of successive corrections based on BDF4 and extrapolation

Predictor formula	q	$-D_1 \leqslant P_m(\alpha) \leqslant D_2$	$\beta(m=2)$	$\beta(m \gg 1)$
$y^{(0)} = y_n$	0	$-0.7493 \leqslant P_m(\alpha) \leqslant 1$	∞	∞
$y^{(0)} = 2y_n - y_{n-1}$	1	$-\frac{1}{3} \leqslant P_m(\alpha) \leqslant 1$	∞	∞
$y^{(0)} = 3y_n - 3y_{n-1} + y_{n-2}$	2	$-\frac{1}{7} \leqslant P_m(\alpha) \leqslant 0.4951$	21.7	$11.9m$
$y^{(0)} = 4y_n - 6y_{n-1} + 4y_{n-2} - y_{n-3}$	3	$-\frac{1}{15} \leqslant P_m(\alpha) \leqslant 0.1999$	8.4	$5.2m$

Let us in particular consider the BDF4 formula defined by (cf. [6, p. 242])

$$b_0 = \tfrac{12}{25}, \qquad \Sigma = \tfrac{1}{25}[48y_n - 36y_{n-1} + 16y_{n-2} - 3y_{n-3}], \tag{2.23}$$

and let the predictor $y^{(0)}$ be defined by extrapolation. In order to satisfy the root condition we obtain the bounds for $P_m(\alpha)$ listed in Table 2.1 (for a proof we refer to the Appendix).

From (2.20'b) the *stability boundaries* $\beta = \beta(m)$ can now be derived. Evidently, they are infinitely large if $D_2 = 1$, that is in the case of zero order and first order extrapolation. For higher order extrapolation predictors we have conditional stability. In Table 2.1 the values of β are listed for $m = 2$ and for $m \gg 1$. From these values we may conclude that the method of successive corrections based on BDF4 is of less practical value if higher order predictors are used.

2.3.2. *Optimal stability boundaries for the iterated BDF4*

Again we consider the BDF4 with the second and third order extrapolation predictors listed in Table 2.1, but now with the maximal value for ω allowed by condition (2.20a). In Table 2.2 the corresponding S^* and β values are listed for various values of m.

We notice that the stability boundary obtained for $m = 2$ and for optimal (maximal) S^* is already considerably larger than that for $m = 2$ and $S^* = 0$.

2.3.3. *Behaviour of the stability boundary for large values of m*

In order to get an impression of the stability boundary for large values of m we prove the following corollary of Theorem 2.3.

Corollary 2.1. *If the conditions of Theorem* 2.3 *are satisfied, then*

$$\beta = \frac{64\omega^2 m^2}{b_0[16m^2 - \pi^2\omega - 4\omega d_2^2]}\left[1 + O(1/m^2)\right] \quad \text{as } m \to \infty, \tag{2.24a}$$

$$\beta \leqslant \frac{256}{b_0[\pi^2 + 4d_1^2][d_1^2 - d_2^2]} m^4\left[1 + O(1/m^2)\right] \quad \text{as } \omega = O(m^2), \tag{2.24b}$$

Table 2.2
Stability boundaries for BDF4 combined with second and third order extrapolation as predictor formula

		$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$
$q=2$	$S^* \cong$	0.98	9.4	43	131	316	649
	$\beta \cong$	13.8	98	413	1224	2898	5908
$q=3$	$S^* \cong$	0.48	4	18	54	129	264
	$\beta \cong$	4	26	109	319	751	1526

where

$$d_1 = \operatorname{arccosh} \frac{1}{\tilde{D}}, \qquad d_2 = \operatorname{arccosh} \frac{D_2}{\tilde{D}},$$

with $\tilde{D} = \min\{D_1, D_2\}$ *and* D *given by* (2.13').

Proof. From (2.21) it follows that for $m \gg 1$

$$\tilde{a} = \frac{2\omega - 1}{\omega^2(c+1)}\left[1 + \omega c - (\omega - 1)\cosh(d_2/m)\right]$$

$$= \frac{2\omega - 1}{\omega^2(c+1)}\left[2 + \omega c - \omega - \tfrac{1}{2}(\omega - 1)\frac{d_2^2}{m^2}\left(1 + \frac{1}{12}\frac{d_2^2}{m^2} + O\left(\frac{1}{m^4}\right)\right)\right]$$

where we have written $c = \cos(\pi/2m)$. Expansion of c yields for $\tilde{a}$

$$\tilde{a} = \frac{1}{\omega}\left[2 - \frac{\omega(\pi^2 + 4d_2^2)}{8m^2} + O\left(\frac{1}{\omega}\right) + O\left(\frac{1}{m^2}\right) + O\left(\frac{\omega}{m^4}\right)\right].$$

From (2.20b) it follows that

$$\beta = \frac{8\omega}{b_0\tilde{a}}\left[1 + O(1/\omega) + O(\tilde{a})\right].$$

Substitution of $\tilde{a}$ and using $\omega = O(m^2)$ yields (2.24a).

The upperbound (2.24b) is obtained by substituting

$$\omega = \tilde{\omega} = \frac{16m^2}{\pi^2 + 4d_1^2}\left(1 + O\left(\frac{1}{m^2}\right)\right). \qquad \square$$

Thus, under the conditions of Theorem 2.3 the stability boundary $\beta = \beta(m)$ has at least an $O(m^4)$ behaviour as $m \to \infty$. In this connection, we remark that explicit Runge–Kutta methods with maximal real stability interval have a stability boundary $\beta(m)$ of $O(m^2)$ as $m \to \infty$ (see [8]). The partial implicitness of the splitting method (2.1)–(2.5) is apparently compensated by a considerably larger stability interval (we recall that $\beta = \infty$ if $D_2 = 1$). However, if we use higher order predictors we still need a relatively large number of iterations if $\tau\sigma$ is large.

As an example we consider BDF4 with third order extrapolation for $y^{(0)}$. Choosing S^* maximal (i.e. $\omega = \tilde{\omega}$) we find from (2.24) and table 2.1 ($D_1 = \frac{1}{15}$, $D_2 = 0.1999$)

$$\beta(m) \equiv 1.12\, m^4 \quad \text{as } m \gg 1.$$

Hence, for given τ and σ we need at least

$$m = \left(\frac{\tau\sigma}{1.12}\right)^{1/4}$$

iterations in order to have stability. For instance, if $\tau\sigma = 1000$ we need 6 iterations which is rather expensive.

3. A fourth order splitting method with large stability boundaries

The stability boundaries derived in the preceding section for the fourth order method based on BDF4 and third order extrapolation as predictor formula, are relatively low. This is caused by a too fast increase of $P_m(\alpha)$ as $\alpha \to 0$, that is the *high frequencies* are not damped sufficiently. We can not correct this by choosing S^* larger because S^* is limited to the values given in Table 2.2. In this section we investigate the

effect of performing an *adjusted Jacobi iteration* on an extrapolation predictor in order to remove the high frequencies. This iteration leads to the following smoothed predictor

$$\tilde{y}^{(0)} = \Sigma + b_0\tau\tilde{F}\big(t_{n+1}, \tilde{y}^{(0)}, y^{(0)}\big), \tag{3.1a}$$

$$y^{(0)} = \sum_{l=1}^{k} \hat{a}_l y_{n+1-l}, \tag{3.1b}$$

where $\tilde{F}(t, u, v)$ denotes a Jacobi type splitting function which is required to satisfy the conditions

$$\tilde{F}(t, y, y) \equiv f(t, y), \qquad \frac{\partial \tilde{F}}{\partial u} = -\theta\tilde{\sigma}I, \tag{3.2}$$

with I the identity matrix, $\tilde{\sigma}$ an estimate of the spectral radius of $\partial f/\partial y$, and θ a positive parameter to be determined below.

Let us define $\tilde{F}$ by specifying its ith component $\tilde{F}_i$ according to

$$\tilde{F}_i(t, u, v) = f_i(t, v_1, \ldots, v_{i-1}, \gamma_i u_i + (1-\gamma_i)v_i, v_{i+1}, \ldots, v_N), \tag{3.3}$$

$$\gamma_i = -\theta\tilde{\sigma}\left[\frac{\partial f_i}{\partial y_i}\right]^{-1}, \quad i = 1, 2, \ldots, N, \tag{3.4}$$

where $v_1, v_2, \ldots$; $u_1, u_2, \ldots$ and $y_1, y_2, \ldots$ denote the components of the vectors v, u and y, respectively, and N is the number of vector components.

Then using the definition of Σ in (2.3) we prove the following comparison theorem.

Theorem 3.1. *Let the method* (2.1)–(2.5) *be stable if* $y^{(0)}$ *is defined by* (3.1b) *and if* $P_m(\alpha)$ *satisfies* (2.19). *Then this method is also stable if the predictor is given by* $\tilde{y}^{(0)}$ *(defined by* (3.1)) *and* $P_m(\alpha)$ *satisfies the condition*

$$-D_1 \leqslant \frac{z + \theta b_0\tau\tilde{\sigma}}{1 + \theta b_0\tau\tilde{\sigma}} P_m\big(\alpha(z_1, z_2)\big) \leqslant D_2, \quad z = z_1 + z_2. \tag{3.5}$$

Proof. The variational equation for the predictor $\tilde{y}^{(0)}$ is given by

$$\Delta\tilde{y}^{(0)} = \sum_{l=1}^{k} \frac{a_l I + (b_l/b_0)(Z_1 + Z_2) + \hat{a}_l(Z_1 + Z_2 + \theta b_0\tau\tilde{\sigma}I)}{1 + \theta b_0\tau\tilde{\sigma}} \Delta y_{n+1-l}.$$

Substitution into (2.17) reveals that the characteristic equation assumes the form

$$\zeta^k = \sum_{l=1}^{k} \left\{ P_m(\alpha)\frac{a_l + (b_l/b_0)z + \hat{a}_l(z + \theta b_0\tau\tilde{\sigma})}{1 + \theta b_0\tau\tilde{\sigma}} + \left(a_l + \frac{b_l}{b_0}z\right)\frac{1 - P_m(\alpha)}{1 - z} \right\} \zeta^{k-l}. \tag{3.6}$$

It is easily verified that this equation can be written in the form

$$\zeta^k = \sum_{l=1}^{k} \left\{ \hat{a}_l \tilde{P}_m(z_1, z_2) + \left(a_l + \frac{b_l}{b_0}z\right)\frac{1 - \tilde{P}_m(z_1, z_2)}{1 - z} \right\} \zeta^{k-l} \tag{3.6'}$$

where

$$\tilde{P}_m(z_1, z_2) = \frac{z + \theta b_0\tau\tilde{\sigma}}{1 + \theta b_0\tau\tilde{\sigma}} P_m\big(\alpha(z_1, z_2)\big), \quad z = z_1 + z_2. \tag{3.7}$$

We observe that replacing the predictor $\tilde{y}^{(0)}$ (defined by (3.1)) by the predictor $y^{(0)}$ (defined by (3.1b)) implies that in the characteristic equation one should replace $\tilde{P}_m$ by P_m. Since this latter equation satisfies the root condition if P_m satisfies (2.19), equation (3.6') will also satisfy the root condition if $-D_1 \leqslant \tilde{P}_m \leqslant D_2$, that is if (3.5) is satisfied. □

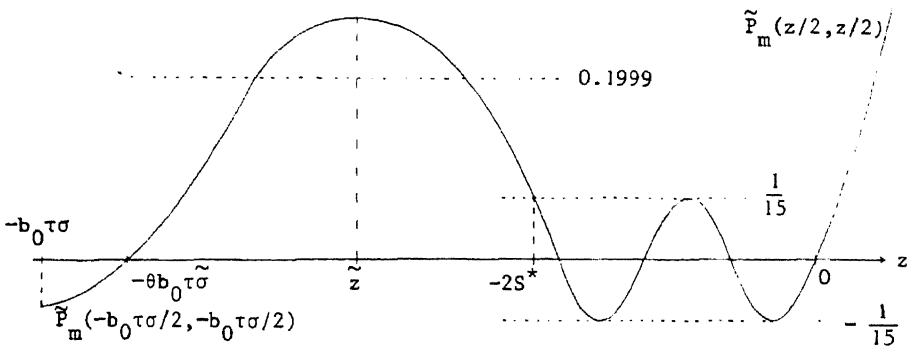

Fig. 3.1. The function $\tilde{P}_m(z/2, z/2)$ for $-\sigma < -\theta\tilde{\sigma}$.

We apply this theorem to the case where b_0 and Σ are defined by the BDF4 given by (2.23), and where $y^{(0)}$ is given by third order extrapolation. Using $\tilde{y}^{(0)}$ as defined by (3.1) as the predictor formula, Theorem 3.1 implies that the resulting method is stable if (see Table 2.1)

$$-\tfrac{1}{15} \leqslant \tilde{P}_m(z_1, z_2) \leqslant 0.1999. \tag{3.5'}$$

Let us assume that $P_m(\alpha) \geqslant -\frac{1}{15}$ for $z_i \geqslant -S^*$, that is S^* is bounded by the values listed in Table 2.2 for $q = 3$. Then $\tilde{P}_m(z_1, z_2)$ is certainly bounded below by $-\frac{1}{15}$ if $z + \theta b_0\tau\tilde{\sigma} \geqslant 0$. If $z + \theta b_0\tau\tilde{\sigma} < 0$ (this happens when $\theta < \sigma/\tilde{\sigma}$) then $\tilde{P}_m(z_1, z_2)$ is bounded below by $-\frac{1}{15}$ if

$$\frac{-b_0\tau\sigma + \theta b_0\tau\tilde{\sigma}}{1 + \theta b_0\tau\tilde{\sigma}} \geqslant -\tfrac{1}{15}.$$

By choosing

$$\theta = \tfrac{15}{16}, \qquad \tilde{\sigma} \geqslant \sigma - \frac{1}{15b_0\tau}, i \tag{3.8}$$

this inequality is satisfied.

In order to satisfy the inequality $\tilde{P}_m(z_1, z_2) \leqslant 0.1999$ it is sufficient to consider the case $z_i \leqslant -S^*$. Let us consider the function $\tilde{P}_m(z_1, z_2)$ along the line $z_1 + z_2 = \text{const}$. Since in this region $P_m(\alpha(z_1, z_2))$ is larger as $\alpha(z_1, z_2)$ is smaller, we find a maximal value of $\tilde{P}_m(z_1, z_2)$ at the point where $\alpha(z_1, z_2)$ is minimal. From the definition of α it follows that along the line $z_1 + z_2 = \text{const}$. the minimum is reached at $z_1 = z_2$.

Thus the function $\tilde{P}_m(z_1, z_2)$ is maximal in magnitude along the line $z_1 = z_2 = \frac{1}{2}z$. In Fig. 3.1 the behaviour is illustrated. If $\tilde{z}$ is the point where $\tilde{P}_m$ assumes a maximum value in the interval $[-b_0\tau\sigma, -2S^*]$, then we should choose $\tau\tilde{\sigma}$ such that $\tilde{P}_m(-\tilde{z}/2, -\tilde{z}/2) \leqslant 0.1999$. This yields an upper bound for $\tau\tilde{\sigma}$ which is just the stability boundary β of the method. In Table 3.1 these values are listed for $m = 1$ until 6 and S^* as large as allowed by (2.20a). A comparison with the maximal boundaries attainable for the third order extrapolation as listed in Table 2.2 reveals that we have gained a factor of about $3\frac{1}{2}$ by performing the Jacobi iteration (3.1). For large values of m the stability boundary tends to behave as $4m^4$.

Summmarizing, we conclude that the iterated BDF4 together with the predictor (3.1) where $y^{(0)}$ is

Table 3.1
Maximal stability boundaries for BDF4 combined with the fourth order predictor (3.1)

	m = 1	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$
S^*	0.48	4	18	54	129	264
$\beta(m)$	20	101	385	1095	2549	5150
β/m^4	20	6.3	4.8	4.3	4.1	4.0

defined by third order extrapolation, generate a fourth order accurate splitting method which is stable for the S^*-values listed in Table 3.1 and for integration steps satisfying the condition

$$\tau \leqslant \frac{\beta(m)}{\tilde{\sigma}}. \tag{3.9}$$

This method will be denoted by *SC method* since it can be considered as a variant of the method of successive corrections introduced in [4] (see also Section 2.3.1).

In actual application of the method we will choose for m the smallest integer such that (3.9) is satisfied when τ and $\tilde{\sigma}$ are prescribed.

4. Numerical experiments

4.1. Methods used

We tested the SC method by comparing it with the ADI method of Peaceman and Rachford [7]. This method is denoted by PR(ν) in the tables of results, where ν indicates the number of Newton iterations used for solving each implicit relation. In the SC method we performed only one Newton iteration in solving both (2.2) and (3.1).

The splitting function $F(t, u, v)$ used in the SC method was defined by the same ADI splitting function as used in the PR method. The Jacobian matrices $\partial F/\partial u$ and $\partial F/\partial v$ needed in the Newton iterations were updated at the beginning of each integration step. The examples are such that an analytical expression for the Jacobian matrices was available.

The estimate $\tilde{\sigma}$ needed for determining a safe number of iterations was either given in analytical form or computed by applying Gerschgorin's theorem to the matrix $\partial f/\partial y = \partial F/\partial u + \partial F/\partial v$ (this hardly requires additional effort).

The starting values needed by the SC method were obtained by computing them from the exact values prescribed at t_{-3}, t_{-2}, t_{-1} and t_0.

4.2. Numerical examples

The problems we chose were all of the form

$$\frac{\partial U}{\partial t} = d\left[\frac{\partial^2 U^r}{\partial x_1^2} + \frac{\partial^2 U^r}{\partial x_2^2}\right] + \left[\frac{\partial U}{\partial x_1}\right]^s + \left[\frac{\partial U}{\partial x_2}\right]^s + v, \quad 0 \leqslant t \leqslant 1, \tag{4.1}$$

where the diffusion coefficient d and the term v are functions of U, t, x_1 and x_2 to be specified below, and the integer r and s are nonlinearity parameters. The domain in the (x_1, x_2)-plane is given by the square $0 \leqslant x_1, x_2 \leqslant 1$; the Dirichlet boundary and the initial conditions at $t_0 = 0$ follow from the exact solution given in Table 4.1.

Table 4.1
Specification of the testproblems

Example	Solution	d	r	s	v	$\tilde{\sigma}$
I	$1+e^{-t}(x_1^2+x_2^2)$	1	1	0	$-e^{-t}(x_1^2+x_2^2+4)-2$	$8h^{-2}$
II	$1+e^{-t}(x_1^2+x_2^2)$	$\frac{1}{1+t}$	1	2	$-e^{-t}[4d+(1+4e^{-t})(x_1^2+x_2^2)]$	
III	$\frac{1}{2}(x_1+x_2)\sin 2\pi t$	$\frac{x_1+x_2}{2(1+t)}$	3	0	$-[\frac{3}{4}\frac{(x_1+x_2)^2}{1+t}\sin^3 2\pi t + 2 - \pi(x_1+x_2)\cos 2\pi t]$	$\frac{24\sin^2 2\pi t}{(1+t)h^2}$

The initial-boundary value problems were semi-discretized by using standard differences on a uniform grid with grid points (jh, lh). The components of the right-hand side function of the resulting system of ODE's (1.1) are evidently coupled according to a five-point molecule which allows the use of ADI splitting functions.

In Examples I and III we used for $\tilde{\sigma}$ the expression listed in Table 4.1. In Example II, a Gerschgorin estimate was used.

4.3. Numerical results

In the tables of results we listed the accuracy measured by the number *sd* of correct significant digits defined by

$$\text{sd} = -\log_{10}|\text{maximum absolute error at } t = 1| \tag{4.2}$$

and the computational effort *ce* measured by the total number of right-hand side evaluations needed in the integration process.

Problem I belongs to the class of model problems specified in Section 2.2, so that the stability theory can be rigorously applied. The results in Table 4.2 show that the SC method does behave in a stable way.

The second order and fourth order behaviour of the PR and SC method is also reflected in the sd-values (on halving the integration step the sd-value should increase with $\log_{10} 2^{\tilde{p}}$, $\tilde{p}$ being the order of the method). Due to the fourth order behaviour of the SC method this method is much more accurate than the PR method for the same step size. This makes the fourth order method much more efficient than the PR method if high accuracy is desired. For example, to get four correct digits the PR method needs 80 right-hand side evaluations, whereas the SC method requires only 45 evaluations.

Problem II is mildly nonlinear; as a consequence one Newton iteration for solving the implicit relations is not sufficient if larger integration steps are used (indicated by * in Table 4.3). For smaller steps we see a similar behaviour as exhibited by Problem I.

Problem III is rather nonlinear with a rapidly changing spectral radius. It turned out to be a more difficult problem for both methods than Problem II. For $\tau \geqslant 1/40$ both methods failed because the Newton

Table 4.2
sd/ce-values obtained for Problem I with $h = 1/24$

Method	$\tau = 1/2$	$\tau = 1/5$	$\tau = 1/10$	$\tau = 1/20$	$\tau = 1/40$	$\tau = 1/80$
PR(1)	1.1/4	2.0/10	2.6/20	3.2/40	3.9/80	4.5/160
SC	2.0/22	4.0/45	5.1/90	6.3/140	7.4/280	8.7/400

Table 4.3
sd/ce-values obtained for Problem II with $h = 1/24$

Method	$\tau = 1/5$	$\tau = 1/10$	$\tau = 1/20$	$\tau = 1/40$	$\tau = 1/80$
PR(1)	*	*	2.0/40	3.6/80	4.3/160
PR(2)	1.6/20	2.4/40	3.1/80	3.7/160	4.3/320
SC	*	*	6.1/140	7.5/212	8.7/400

Table 4.4
sd/ce-values obtained for Problem III with $h = 1/24$

Method	$\tau \geqslant 1/40$	$\tau = 1/80$	$\tau = 1/160$
PR(1)	*	2.1/160	2.7/320
PR(2)	*	3.0/320	4.1/640
SC	*	5.9/390	6.9/676

process did not converge (indicated by *). Again the SC method is superior to the PR method if high accuracies are desired.

4.4. Concluding remarks

In the high accuracy region, the experiments reported in the preceding section show the superiority of the SC method over the classical PR method because of the order four behaviour which is maintained for realistic integration steps. In this connection we remark that the method of successive corrections with $S^* = 0$, $\omega = \lambda_j = \mu_j = 1$, analysed in [4], shows its fourth order behaviour only for relatively small integration steps (relative to the spectral radius). For realistic integration steps the order of this method degenerates so that it is hardly more efficient than e.g. the PR method. By virtue of the effective fourth order behaviour of the SC method, a variable order splitting method, e.g. composed of the SC and the PR method, may turn out to be an efficient method for solving parabolic equations with an arbitrary degree of accuracy.

In the SC method described in this paper there are several choices which are not necessarily the best possible. For instance, the Jacobi iteration (3.1) may be replaced by Gauss–Seidel iteration (although this would complicate the theoretical analysis considerably). Furthermore, the relaxation parameter ω (or equivalently the parameter S^*) and the number of iterations m were chosen such that the effective stability boundary $\beta(m)/m$ is as large as possible. An alternative may be the use of another predictor formula (e.g. linear extrapolation) which yields an infinite stability boundary (cf. Table 2.1). This leaves ω and m free for minimizing the local error. Finally, the choice of the splitting function $F(t, u, v)$: Alternatives are odd–even hopscotch splitting which reduces the computational effort of solving the implicit relations, and line hopscotch which allows the integration of equations with mixed derivatives (for a treatment of hopscotch splittings we refer to [2,3]).

In the Appendix a lot of additional experiments are reported which given some insight into these questions.

Appendix A

For the BDF4 defined by (2.23) with $y^{(0)}$ defined by extrapolation we show how the bounds D_1 and D_2 (cf. (2.19)) for $P_m(\alpha)$ listed in Table 2.1 are obtained by applying Hurwitz's criterion to the characteristic equation

$$\zeta^4 + \left[-P_m(\alpha)\hat{a}_1 - \frac{48}{25}\frac{1-P_m(\alpha)}{1-z} \right]\zeta^3 + \left[-P_m(\alpha)\hat{a}_2 + \frac{36}{25}\frac{1-P_m(\alpha)}{1-z} \right]\zeta^2$$
$$+ \left[-P_m(\alpha)\hat{a}_3 - \frac{16}{25}\frac{1-P_m(\alpha)}{1-z} \right]\zeta + \left[-P_m(\alpha)\hat{a}_4 + \frac{3}{25}\frac{1-P_m(\alpha)}{1-z} \right] = 0. \qquad \text{(A1)}$$

Here, $z = z_1 + z_2$ and $y^{(0)} = \sum_{l=1}^{4} \hat{a}_l y_{n+1-l}$, where the coefficients $\hat{a}_l$ are specified in Table 2.1. Note that (A1) is a special case of (2.17).

We also describe a number of experiments for the linear problem I with the splitting method (2.1)–(2.5) based on BDF4. These results show the effect of the parameter S^* and the number of iterations m, the dependence of the accuracy on the mesh spacing h, the order of accuracy, the use of other predictor formulas and the effect of violating the stability conditions.

In order to test the theory developed for the splitting method (2.1)–(2.5) by performing a large number of experiments, it is convenient to denote this splitting method in a slightly different manner than in Section 3. The method (2.1)–(2.5) with a given value of m and S^* will be denoted by SC(q, m, S^*) where q indicates the predictor formula, i.e. $q = 1$, 2, 3 and 4 corresponding to the first order extrapolation, second order extrapolation, third order extrapolation and the smoothed predictor (3.1) where $y^{(0)}$ is defined by third order extrapolation, respectively. We recall that the generating multistep formula defining b_0 and Σ is the BDF4 defined by (2.23).

The results in the following sections have led us to the choice of the SC method defined in Section 3, i.e.

$$\mathrm{SC} \equiv \mathrm{SC}(4, m, S^*_{max}), \qquad m = \begin{cases} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{cases} \quad \text{if } \tau\tilde{\sigma} \in \begin{cases} (0, 20), \\ [20, 101), \\ [101, 385), \\ [385, 1095), \\ [1095, 2549), \\ [2549, 5150). \end{cases}$$

Here, $\tilde{\sigma}$ is an estimate of the spectral radius of $\partial f/\partial y$ as used in the predictor formula (3.1) and $S^*_{\max}$ is given in Table 3.1 ($S^*_{\max}$ is the maximal value of S^* allowed by condition (2.20a) for the third order extrapolation (see also Table 2.2)).

In our experiments for the linear example I (see Table 4.1) presented in this appendix, the starting values needed in the SC(q, m, S^*) method were prescribed by the exact solution.

In the tables of results we also listed the maximal step $\tau_{\max}$ allowed by the stability condition (2.20b) for $S^* > 0$ and (2.20'b) for $S^* = 0$. For more details concerning the implementation and notation we refer to Section 4.

A1. The stability interval

Writing (A1) as

$$\zeta^4 + c_1\zeta^3 + c_2\zeta^2 + c_3\zeta + c_4 = 0 \tag{A1'}$$

we find by applying Hurwitz's criterion [6, p. 80] that (A1') has its roots on the unit disk if

$$\gamma_i \geqslant 0, \quad i = 0(1)4, \qquad \gamma_1\gamma_2\gamma_3 - \gamma_1^2\gamma_4 - \gamma_0\gamma_3^2 \geqslant 0, \tag{A2}$$

where

$$\begin{aligned}
\gamma_0 &= 1 - c_1 + c_2 - c_3 + c_4 = 1 + P_m(\alpha)(\hat{a}_1 - \hat{a}_2 + \hat{a}_3 - \hat{a}_4) + \frac{103}{25}\frac{1 - P_m(\alpha)}{1 - z}, \\
\gamma_1 &= 4 - 2c_1 + 2c_3 - 4c_4 = 4 + 2P_m(\alpha)(\hat{a}_1 - \hat{a}_3 + 2\hat{a}_4) + \frac{52}{25}\frac{1 - P_m(\alpha)}{1 - z}, \\
\gamma_2 &= 6 - 2c_2 + 6c_4 = 6 + 2P_m(\alpha)(\hat{a}_2 - 3\hat{a}_4) - \frac{54}{25}\frac{1 - P_m(\alpha)}{1 - z}, \\
\gamma_3 &= 4 + 2c_1 - 2c_3 - 4c_4 = 4 + 2P_m(\alpha)(\hat{a}_3 - \hat{a}_1 + 2\hat{a}_4) - \frac{76}{25}\frac{1 - P_m(\alpha)}{1 - z}, \\
\gamma_4 &= 1 + c_1 + c_2 + c_3 + c_4 = 1 - P_m(\alpha)\sum_{l=1}^{4}\hat{a}_l - \frac{1 - P_m(\alpha)}{1 - z}.
\end{aligned} \tag{A3}$$

Note that for all predictor formulas listed in Table 2.1 $\sum_{l=1}^{4}\hat{a}_l = 1$, i.e.

$$\gamma_4 = -z\frac{1 - P_m(\alpha)}{1 - z}.$$

Evidently the conditions (A2) are satisfied for all negative z if we put $P_m(\alpha) \equiv 0$, because (A1) then corresponds to the BDF4. For nonzero $P_m(\alpha)$ we have the following theorem.

Theorem A1. *The characteristic equation* (A1) *has its roots on the unit disk for all negative* z_1 *and* z_2 *if* $P_m(\alpha)$ *satisfies the condition* (2.19), *i.e.*

$$-1 \leqslant -D_1 \leqslant P_m(\alpha) \leqslant D_2 \leqslant 1,$$

where for each extrapolation formula the bounds D_1 *and* D_2 *are already given in Table* 2.1.

Proof. (1) Consider the third order extrapolation formula (i.e. $q = 3$ in Table 2.1). The relations for γ_i with $i = 0, 1, 2, 3$ (see (A3)) can be simplified as follows:

$$\gamma_0 = 1 + 15P_m(\alpha) + \tfrac{103}{25}\frac{1 - P_m(\alpha)}{1 - z},$$

$$\gamma_1 = \frac{1 - P_m(\alpha)}{1 - z}(6\tfrac{2}{25} - 4z), \qquad \gamma_2 = \frac{1 - P_m(\alpha)}{1 - z}(3\tfrac{21}{25} - 6z). \qquad \gamma_3 = \frac{1 - P_m(\alpha)}{1 - z}(\tfrac{24}{25} - 4z).$$

It can be easily shown that if $z \leqslant 0$ and $-\frac{1}{15} \leqslant P_m(\alpha) \leqslant 1$, then $\gamma_i \geqslant 0$ for all i. Substituting γ_i in the nonlinear condition in (A2) gives after a tedious calculation the inequality

$$\frac{(1 - P_m(\alpha))^2}{(1 - z)^3} R(z) \geqslant 0, \tag{A4}$$

where

$$R(z) = r_3 z^3 + r_2 z^2 + r_1 z + r_0,$$

$$r_3 = 320\, P_m(\alpha) - 64, \qquad r_2 = 92.16 - 471.04\, P_m(\alpha),$$

$$r_1 = 203.5712\, P_m(\alpha) - 65.9456, \qquad r_0 = 17.69472 - 32.44032\, P_m(\alpha).$$

It is easily numerically verified that $R(z) \geqslant 0$ if $z \leqslant 0$ and $P_m(\alpha) \leqslant 0.1999$. Thus, the inequality (A4) holds for all $z \leqslant 0$ and $-1 \leqslant P_m(\alpha) \leqslant 0.1999$. From Hurwitz's criterion (see (A2)) it follows that the characteristic equation (A1) with the third order extrapolation formula has its roots on the unit disk if $z \leqslant 0$ and

$$-\tfrac{1}{15} \leqslant P_m(\alpha) \leqslant 0.1999.$$

(2) Consider the first order extrapolation formula (i.e. $q = 1$ in Table 2.1). The relations for γ_i with $i = 0, 1, 2, 3$ (see (A3)) may now be written as follows:

$$\gamma_0 = 1 + 3\, P_m(\alpha) + \tfrac{103}{25}\frac{1 - P_m(\alpha)}{1 - z}, \qquad \gamma_1 = 4 + 4\, P_m(\alpha) + \tfrac{52}{25}\frac{1 - P_m(\alpha)}{1 - z},$$

$$\gamma_2 = 4 + \frac{1 - P_m(\alpha)}{1 - z}(-\tfrac{4}{25} - 2z), \qquad \gamma_3 = \frac{1 - P_m(\alpha)}{1 - z}(\tfrac{24}{25} - 4z).$$

It can be easily shown that if $z \leqslant 0$ and $-\frac{1}{3} \leqslant P_m(\alpha) \leqslant 1$ then $\gamma_i \geqslant 0$ for all i. Substitution of γ_i in the nonlinear condition in (A2) gives

$$\frac{1 - P_m(\alpha)}{(1 - z)^3} R(z) \geqslant 0, \tag{A5}$$

where

$$R(z) = r_3 z^3 + r_2 z^2 + r_1 z + r_0,$$

$$r_3 = -64, \qquad r_2 = 92.16 + 97.28\, P_m(\alpha) - 30.72\, P_m^2(\alpha),$$

$$r_1 = -62.9456 - 70.8608\, P_m(\alpha) + 11.3664\, P_m^2(\alpha),$$

$$r_0 = 17.69472 + 13.76256\, P_m(\alpha) - 0.73728\, P_m^2(\alpha).$$

It is easily established that for $P_m(\alpha)$ in the interval $[-\frac{1}{3}, 1]$ $r_2 \geqslant 0$, $r_1 \leqslant 0$ and $r_0 \geqslant 0$. Thus, the inequality (A5) holds for all $z \leqslant 0$ and $P_m(\alpha)$ in the interval $[-\frac{1}{3}, 1]$. Again, application of Hurwitz's criterion reveals that the characteristic equation (A1) with the first order extrapolation formula has its roots on the unit disk if $z \leqslant 0$ and $-\frac{1}{3} \leqslant P_m(\alpha) \leqslant 1$.

In a completely analogous manner we can analyse the BDF4 with the other extrapolation formulas and derive for $P_m(\alpha)$ the bounds already given in Table 2.1. □

Remark 1. When in (A1) the variables z_1 and z_2 are largely negative, then the characteristic equation can be approximated by

$$\zeta^4 = P_m(\alpha) \sum_{l=1}^{4} \hat{a}_l \zeta^{4-l}. \tag{A6}$$

Applying Hurwitz's criterion to this equation we obtain a condition on $P_m(\alpha)$ for each extrapolation formula. For the zero order, first order, second order and third order extrapolation the bounds (D_1, D_2) for $P_m(\alpha)$ are in this case (1, 1), $(\frac{1}{3}, 1)$, $(\frac{1}{7}, \frac{1}{2})$ and $(\frac{1}{15}, \frac{1}{5})$, respectively.

A2. Stability tests

In this section the stability of SC(q, m, S^*) is tested. Therefore the SC(q, m, S^*) method is applied to the model problem I with a large number of integration steps and a relatively large step, say $\tau = \frac{1}{10}$.

The second order method SC(1, m, S^*) should not give difficulties, because theoretically it is unconditionally stable provided S^* is not too large (see Table A0). This is confirmed by the results in Table A1, where the model problem I is integrated from $t = 0$ until $t = t_e = 10$. Here the accuracy is measured in the points $t_j = \{j \mid j = 1, 2, \ldots, 10\}$ by sd $= -\log_{10}|$maximum absolute error in $t_j|$. Note that the maximal stable step $\tau_{\max}$ follows from Theorem 2.3.

The fourth order method SC(3, m, S^*) is only stable for relatively small values of τ. Taking again problem I we found the results listed in Table A2.

Here, an asterisk means instability (for $\tau_{\max}$ it indicates the S^* is larger than allowed by Table 2.2).

Table A0
Maximal S^* values for the first order predictor ($q = 1$)

m	1	2	3	4	5	6
$S^* =$	2.96	33.2	157	486	1176	2425

Table A1
Results obtained by SC(1, m, S^*) with $\tau = 1/10$ when applied to problem I with $t_e = 10$

Method	h	$\tau_{\max}$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$	$t=7$	$t=8$	$t=9$	$t=10$
SC(1, 2, 10)	$\frac{1}{10}$	∞	3.0	3.4	3.9	4.3	4.7	5.2	5.6	6.0	6.5	6.9
SC(1, 4, 10)	$\frac{1}{20}$	∞	3.0	3.4	3.9	4.3	4.8	5.2	5.6	6.0	6.5	6.9

Table A2
Results obtained by SC(3, m, S^*) with $\tau = \frac{1}{10}$ when applied to problem I with $t_e = 10$

Method	h	$\tau_{\max}$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$	$t=7$	$t=8$	$t=9$	$t=10$
SC(3, 2, 10)	1/10	*	4.8	4.5	3.7	2.7	1.6	0.5	*	*	*	*
SC(3, 4, 10)	1/10	0.12	6.1	6.5	6.9	7.4	7.8	8.3	8.7	9.1	9.6	10.0
SC(3, 2, 10)	1/20	*	4.0	2.8	1.4	0.1	*	*	*	*	*	*
SC(3, 4, 10)	1/20	0.03	4.5	3.7	2.7	1.5	0.2	*	*	*	*	*
SC(3, 4, 40)	1/20	0.08	5.3	5.7	6.1	6.1	5.7	5.5	4.9	4.7	4.2	3.7
SC(3, 4, 52)	1/20	0.10	5.2	5.6	6.0	6.5	6.9	7.3	7.7	8.0	8.2	8.2
SC(3, 4, 80)	1/20	*	5.0	5.2	4.8	4.1	3.5	2.9	2.3	1.6	1.0	0.4

Table A3
Results obtained by SC(3, 4, 10) with $\tau = \frac{1}{34} = 0.0294\ldots$ when applied to problem I with $t_e = 10$

Method	h	$\tau_{\max}$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$	$t=7$	$t=8$	$t=9$	$t=10$
SC(3, 4, 10)	$\frac{1}{20}$	0.0295	8.0	8.4	8.8	9.3	9.7	10.2	10.6	11.0	11.5	11.9

According to the theory only the experiment SC(3, 4, 10) with $h = \frac{1}{10}$ is stable. Yet if τ is sufficiently close to τ_{max} the instabilities seem to vanish. By decreasing the step size τ such that $\tau \cong \tau_{max}$ we should get completely stable results. In Table A3 these results are listed confirming the theory.

A.2.1. Amplification factors

In order to see by what factor perturbations are amplified we have plotted the magnitude of the largest characteristic root of equation (A1) with the third order extrapolation ($q = 3$) as a function of α (see (2.18)) with $z_1 = z_2 = \frac{1}{2}z$. In Figs. A1(a), (b) the value of $|\zeta|_{max}$ is shown for SC(3, 2, 10) and SC(3, 4, 10), respectively. These plots show that amplifications by a factor as large as 1.4 occur if α is less than 0.1 and 0.05, respectively.

This happens if $\tau\sigma$ is larger than 300 in SC(3, 2, 10) and larger than 700 in SC(3, 4, 10). For convenience we list for both methods in Table A4 the values of ω defined by (2.5), b defined in (2.4), α_0 defined in (2.13), D defined by (2.13') and $\tilde{a}$ given by (2.21). The SC(3, 2, 10) method is unstable unless the integration step τ is so small that α lies in the interval $[\tilde{a}, 0.83]$, i.e. $\tau\sigma < 1.14$. It can be easily shown that if $z_1 = z_2 = \frac{1}{2}z$ and $z \leqslant 0$ then α (given by (2.18)) lies in the interval [0, 1]. Notice that the curves in Figs. A1(a) and (b) are obtained by calculating the values of $|\zeta|_{max}$ for a large number of values of α in the interval [0, 1]. Fig. A1(b) shows that SC(3, 4, 10) has a stability interval $[\tilde{a}, b]$, i.e. $\tau\sigma < 94.7$ (see Table A4 and Theorem 2.3); serious instabilities are to be expected if $\alpha \to 0.05$, i.e. $\tau\sigma$ becomes as large as 700.

A.2.2. Smoothed third order extrapolation

Finally, we consider the method SC(4, 4, S^*) with its extended stability intervals (see Section 3). Theoretically, the experiment in Table A5 should be stable which is confirmed by the sd-values obtained.

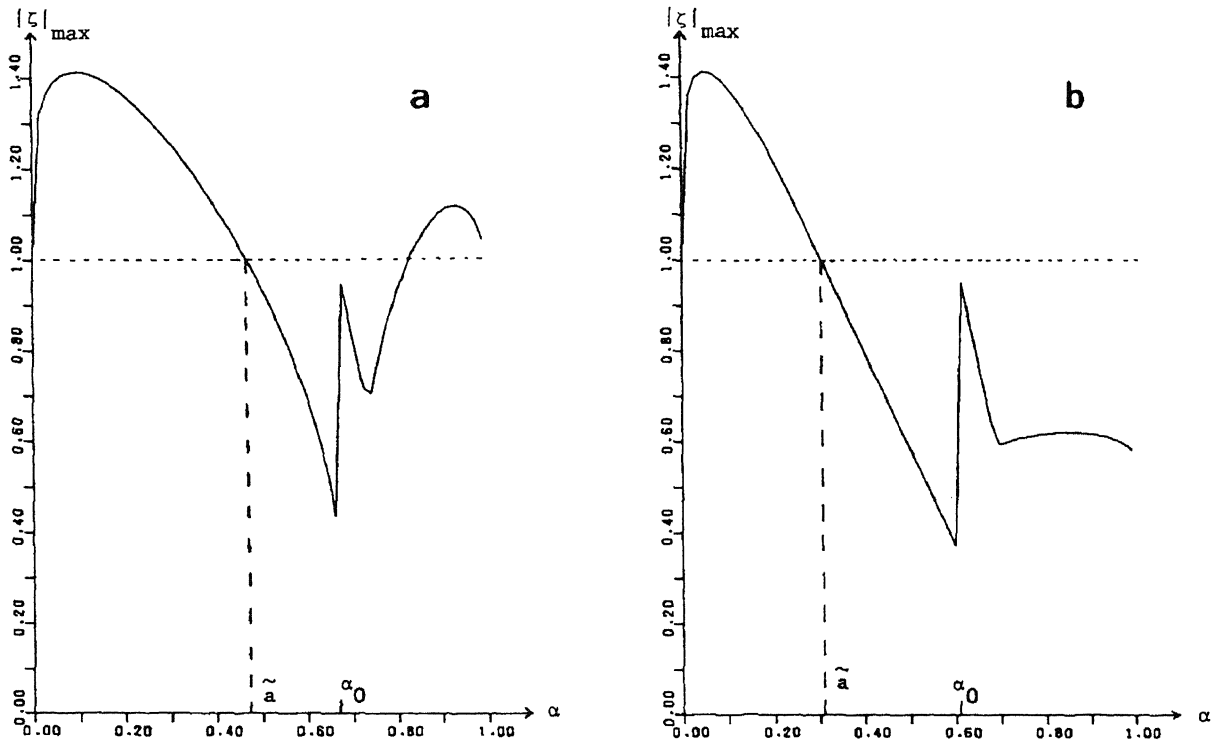

Fig. A1. The value of $|\zeta|_{max}$ of (A1) for (a) SC(3, 2, 10) and (b) SC(3, 4, 10) as a function of α with $z_1 = z_2 = \frac{1}{2}z$.

Table A4
The values of ω, b, α_0, D and $\bar{a}$ in the SC(3, 2, 10) and SC(3, 4, 10) method

Method	ω	b	α_0	D	$\bar{a}$
SC(3, 2, 10)	2.36	1.5763	0.6679	0.1492	0.4685
SC(3, 4, 10)	2.67	1.6255	0.6088	0.0087	0.3074

Table A5
Results obtained by SC(4, 4, S^*) with $\tau = \frac{1}{10}$ when applied to problem I with $t_e = 10$

Method	h	τ_{max}	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$	$t=7$	$t=8$	$t=9$	$t=10$
SC(4, 4, 10)	$\frac{1}{20}$	0.11	5.5	5.7	6.6	6.5	7.1	7.1	7.6	7.8	8.3	8.8
SC(4, 4, 40)	$\frac{1}{20}$	0.28	5.3	5.7	6.2	6.6	7.0	7.5	7.9	8.3	8.8	9.2
SC(4, 4, 52)	$\frac{1}{20}$	0.33	5.2	5.6	6.0	6.5	6.9	7.3	7.8	8.2	8.6	9.1

In order to suppress instabilities in the high frequency region we performed an adjusted Jacobi iteration on the third order extrapolation predictor which leads to the SC(4, m, S^*) method (see Section 3). It is well known that also Point Gauss–Seidel (PGS) iteration removes the high frequencies from the iteration error. Replacing the Jacobi iteration (3.1) by Point Gauss–Seidel iteration complicates the theoretical analysis considerably. Therefore, we give only some numerical results for the linear model problem I with a third order extrapolation predictor smoothed by PGS iteration.

The PGS iteration leads to the following smoothed predictor:

$$\hat{y}^{(0)} = \Sigma + b_0 \tau F^*\big(t_{n+1}, \hat{y}^{(0)}, y^{(0)}\big), \tag{A7a}$$

$$y^{(0)} = 4y_n - 6y_{n-1} + 4y_{n-2} - y_{n-3}, \tag{A7b}$$

where $F^*(t, u, v)$ denotes a Gauss–Seidel type splitting function.

Let us define F^* by specifying its ith component F_i^* according to

$$F_i^*(t, u, v) = f_i(t, u_1, \ldots, u_{i-1}, u_i, v_{i+1}, \ldots, v_N), \tag{A8}$$

where $v_1, v_2, \ldots$ and $u_1, u_2, \ldots$ denote the components of the vectors v and u, respectively, and N is the number of vector components. Then the components $\hat{y}_i^{(0)}$, $i = 1, 2, \ldots, N$, of the smoothed predictor $\hat{y}^{(0)}$ (A7a) are determined by solving

$$y - b_0 \tau f_i\big(t_{n+1}, \hat{y}_1^{(0)}, \ldots, \hat{y}_{i-1}^{(0)}, y, y_{i+1}^{(0)}, \ldots, y_N^{(0)}\big) = \Sigma_i, \quad i = 1(1)N, \tag{A9}$$

for y and setting $\hat{y}_i^{(0)} = y$, $i = 1(1)N$. One (nonlinear) Point Gauss–Seidel iteration is now defined by (approximately) solving (A9) performing just one Newton iteration for each scalar equation. For the linear Problem I only linear scalar equations have to be solved. The iteration matrix in the PGS iteration, i.e. $\partial F^*/\partial u$ and $\partial F^*/\partial v$, does not have the same eigensystem for the class of model problems specified in Section 2.2.

The iterated BDF4 together with the predictor (A7) will be denoted by PGSSC(4, m, S^*). In Table A6 some results obtained by this method are listed for the linear Example I.

Table A6
Results obtained by PGSSC(4, 4, S^*) with $\tau = \frac{1}{10}$ when applied to Problem I with $t_e = 10$

Method	h	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$	$t=7$	$t=8$	$t=9$	$t=10$
PGSSC(4, 4, 10)	$\frac{1}{20}$	4.9	2.7	0.4	*	*	*	*	*	*	*
PGSSC(4, 4, 40)	$\frac{1}{20}$	5.3	5.7	5.7	5.3	4.6	4.1	4.2	4.9	6.2	7.6
PGSSC(4, 4, 52)	$\frac{1}{20}$	5.2	5.6	6.0	6.5	6.9	7.4	7.8	8.2	8.7	9.1

Only the PGSSC(4, 4, 52) method gives stable results and is competitive with the SC(4, 4, 52) method (see Table A5). Comparing the results listed in Tables A5 and A6 it seems that it is better to smooth the third order extrapolation predictor with Jacobi iteration (see Section 3) than with Point Gauss-Seidel iteration ((A7)–(A9)).

A3. The order of accuracy

From (4.2) it follows that a p-th order method satisfies the relation

$$\Delta\mathrm{sd} := \mathrm{sd}(\tau) - \mathrm{sd}(a\tau) = \log_{10} a^p \quad \text{as } \tau \to 0. \tag{A10}$$

Thus, on halving the step length we have $\Delta\mathrm{sd} \equiv 0.3\,p$; for the SC($q$, m, S^*) method we expect (cf. (2.14′))

$$\Delta\mathrm{sd} = 0.3 \min\{4, q+1\}, \quad S^* \text{ fixed and} \neq 0. \tag{A11}$$

If $S^* = 0$ the SC method reduces to the method discussed in [4] where it was shown that

$$\Delta\mathrm{sd} \equiv 0.3 \min\{4, q+2m\}, \quad S^* = 0. \tag{A12}$$

In the following tables of results the sd-values (cf. (4.2)) are listed for the model problem I obtained by the various methods. We also listed the maximal step $\tau_{\max}$ allowed by the stability conditions (see Theorem 2.3 and Section 2.3.1).

The results obtained for $S^* = 0$ should show a fourth order behaviour both for $q = 1, 3$ and $m = 2, 4$. Hence by virtue of (A12) we expect $\Delta\mathrm{sd} \equiv 1.2$ as $\tau \to 0$. For $q = 1$ this behaviour is roughly confirmed by Tables A7 and A8, but for $q = 3$ a much higher order of accuracy is shown in spite of the integration steps exceeding the maximal stable step $\tau_{\max}$. If $S^* > 0$ relation (A11) indicates that $\Delta\mathrm{sd} \equiv 0.6$ for $q = 1$ and $\Delta\mathrm{sd} \equiv 1.2$ for $q = 3, 4$, as $\tau \to 0$. This behaviour is more or less reflected in the tables of results. The most

Table A7
Results obtained by SC(q, 2, S^*) when applied to Problem I with $h = \frac{1}{10}$

τ	SC(q, 2, 0)		SC(q, 2, 4)			SC(1, 2, 20)	SC(1, 2, 30)
	$q=1$	$q=3$	$q=1$	$q=3$	$q=4$		
$\frac{1}{5}$	1.5	2.6	1.9	3.1	3.5	2.3	2.2
$\frac{1}{10}$	2.3	3.9	3.2	4.6	5.3	2.8	2.7
$\frac{1}{20}$	3.3	4.6	3.9	6.4	6.5	3.4	3.3
$\frac{1}{40}$	4.4	5.5	4.4	7.6	7.6	3.9	3.9
$\frac{1}{80}$	5.5	9.2	4.9	8.7	8.8	4.5	4.4
$\tau_{\max}$	∞	0.0105	∞	0.033	0.13	∞	∞

Table A8
Results obtained by SC(q, 4, S^*) when applied to Problem I with $h = \frac{1}{10}$

τ	SC(q, 4, 0)		SC(q, 4, 10)		SC(q, 4, 20)		SC(q, 4, 52)		
	$q=1$	$q=3$	$q=1$	$q=3$	$q=1$	$q=3$	$q=1$	$q=3$	$q=4$
$\frac{1}{5}$	1.8	2.9	2.9	4.1	3.2	4.4	2.8	4.0	4.1
$\frac{1}{10}$	2.8	4.4	4.3	6.1	3.7	5.8	3.2	5.2	5.2
$\frac{1}{20}$	3.9	5.9	4.5	7.3	4.1	6.7	3.7	6.3	6.3
$\frac{1}{40}$	5.2	8.3	5.0	8.3	4.6	7.8	.4.2	7.4	7.5
$\frac{1}{80}$	6.6	10.0	5.4	9.3	5.1	8.9	4.7	8.6	8.7
$\tau_{\max}$	∞	0.023	∞	0.12	∞	0.193	∞	0.39	1.3

Table A9
Results obtained by SC(4, m, S^*) when applied to Problem I with $h = \frac{1}{20}$

τ	SC(4, 2, 4)	SC(4, 4, 52)
$\frac{1}{5}$	2.9	4.0
$\frac{1}{10}$	4.1	5.2
$\frac{1}{20}$	6.1	6.3
$\frac{1}{40}$	7.6	7.4
$\frac{1}{80}$	8.7	8.6
τ_{max}	0.03	0.33

interesting method seems to be SC(4, m, S^*) because of its rather high accuracy, particularly for larger integration steps.

In order to increase the 'stiffness' of the problem we choose $h = \frac{1}{20}$. In Table A9 results are listed obtained by SC(4, m, S^*). The results again show the correct order behaviour. We also observe that comparing sd-values obtained for equal m/τ-values, that is requiring roughly the same computational effort, reveals that SC(4, 2, 4) is more efficient than SC(4, 4, 52), although SC(4, 2, 4) is stable only for $\tau = \frac{1}{40}$ and $\frac{1}{80}$. A similar conclusion can be drawn for SC(q, 2, 4) and SC(q, 4, 52) from Tables A7 and A8, where $q = 1$, 3 and 4.

Comparing the results listed in Tables A7, A8 and A9 obtained by SC(4, 2, 4) and SC(4, 4, 52) we observe that for large τ-values the SC(4, 2, 4) method is more sensitive to grid refinement than the SC(4, 4, 52) method.

From the results presented in this section we may draw the following conclusions:

(i) The asymptotic order of accuracy of the SC methods is roughly as predicted by the theory.

(ii) The instability is rather mild, which could be expected from the results presented in Section A2.

(iii) SC(3, m, S^*) is considerably more accurate than SC(1, m, S^*). SC(4, m, S^*) is more accurate than SC(3, m, S^*) for larger τ.

(iv) The accuracy increases as S^* decreases for $\tau \to 0$.

(v) SC(q, m_1, S^*_{max}) is more efficient than SC(q, m_2, S^*_{max}) if $m_1 < m_2$.

A4. The effect of grid refinement

It is well known that splitting methods loose accuracy if the mesh width h is decreased, particularly for large time steps. Therefore, we tested SC(q, 2, S^*) by performing calculations for a sequence of h values with the test problem I.

All experiments listed in Table A10 with $q = 3$ are theoretically unstable (an asterisk for the stability boundary β given in (2.20b) means that S^* exceeds its maximal value listed in Table 2.2) and with $q = 1$ the

Table A10
sd-values (4.2) obtained by SC(q, 2, S^*) with $\tau = \frac{1}{10}$ when applied to Problem I

h	$\tau\sigma$	SC(q, 2, 0)		SC(q, 2, 10)		SC(q, 2, 20)		SC(q, 2, 40)	
		$q=1$	$q=3$	$q=1$	$q=3$	$q=1$	$q=3$	$q=1$	$q=3$
$\frac{1}{5}$	20	3.2	4.8	3.0	4.9	2.8	4.0	2.7	3.1
$\frac{1}{10}$	80	2.3	3.9	3.0	4.8	2.8	3.8	2.7	2.7
$\frac{1}{20}$	320	1.4	3.2	2.5	4.0	2.8	3.8	2.7	2.7
$\frac{1}{40}$	1280	0.9	2.3	1.5	3.2	1.7	3.4	2.0	2.7
β		∞	8.4	∞	*	∞	*	*	*

Table A11
Results obtained by SC(1, m, S^*) with $\tau = \frac{1}{10}$ when applied to Problem I with $h = \frac{1}{20}$

m	$S^* = 0$	$S^* = 10$	$S^* = 20$	$S^* = 40$	$S^* = 80$
1	1.1	1.6*	1.7*	1.8*	1.9*
2	1.4	2.5	2.8	2.7*	2.3*
3	1.7	2.8	3.2	2.9	2.7
4	1.8	3.0	3.5	3.3	3.0
5	2.0	3.2	3.7	4.0	3.4
6	2.2	3.4	3.9	4.2	4.0

method is only unstable for $S^* = 40$ (see Table A0). Because of the small number of integration steps (only 7) the instabilities have not yet developed to an extent which seriously affects the numerical solution.

Therefore, these experiments suggest the following conclusions:

(vi) The accuracy of the SC(q, 2, S^*) methods is strongly sensitive to grid refinement.

(vii) The sensitivity decreases if S^* increases.

(viii) The accuracy increases if S^* increases as $h \to 0$.

A5. The effect of the parameters S^* and m

The method SC(1, m, S^*) is completely defined if the parameters m and S^* are specified. In particular, we are interested in the effect of S^*. In Table A10 this is investigated for the splitting method SC(q, m, S^*) for various values of h with m fixed ($m = 2$) and $q = 1$ and $q = 3$. In Table A11 sd-values (4.2) are given for h fixed ($h = \frac{1}{20}$) and various values of m and S^* (An asterisk indicates that S^* exceeds its maximal value listed in Table A0).

From the results listed in Table A10 and A11 we may draw the following conclusions.

(ix) For fixed h and m there is an optimal value for S^*.

(x) This optimal value increases if h decreases and is less sensitive to changes in m.

(xi) The rate of convergence slows down for $m > 2$.

The conclusions (ix)–(xi) roughly apply to SC(3, m, S^*) too.

Appendix B. Smoothed second order extrapolation

Consider the following smoothed predictor

$$\tilde{y}^{(0)} = \Sigma + b_0 \tau \tilde{F}\big(t_{n+1}, \tilde{y}^{(0)}, y^{(0)}\big), \tag{B1a}$$

$$y^{(0)} = 3y_n - 3y_{n-1} + y_{n-2}, \tag{B1b}$$

where Σ and b_0 are defined by (2.23) and $\tilde{F}(t, u, v)$ denotes a Jacobi type splitting function as defined in

Table B1
Maximal stability boundaries for BDF4 combined with the third order predictor (B1)

	$m = 1$	$m = 2$	$m = 3$	$m = 4$
$S^* \approx$	0.98	9.4	43	131
$\beta(m) \approx$	120	755	3090	9040
$\beta/m^4 \approx$	120	47.1	38.1	35.3

Table B2
Results obtained by SC(2, 2, 4) and SC_2(3, 2, 4) when applied to Problem I with h = $\frac{1}{10}$

τ	SC(2, 2, 4)	SC_2(3, 2, 4)
$\frac{1}{5}$	2.5	3.0
$\frac{1}{10}$	4.1	4.3
$\frac{1}{20}$	5.2	5.2
$\frac{1}{40}$	6.0	6.0
$\frac{1}{80}$	6.8	6.9

Table B3
sd/ce-values for problem I with $h = \frac{1}{24}$ obtained by SC_2

Method	$\tau = \frac{1}{5}$	$\tau = \frac{1}{10}$	$\tau = \frac{1}{20}$	$\tau = \frac{1}{40}$
SC_2	3.0/35	4.0/50	4.8/100	5.3/120

Section 3. The order of the predictor $\tilde{y}^{(0)}$ (B1) is 3 (cf. [4]). Then the order of the method (2.1)–(2.5) with the predictor (B1) is 4 (cf. (2.14')).

The parameter θ occurring in $\tilde{F}(t, u, v)$ ((3.2)–(3.4)) and the stability boundary β for this method can be derived in a similar way as for the SC method (see Section 3). Theorem 3.1 implies that the resulting method is stable if (see Table 2.1)

$$-\tfrac{1}{7} \leqslant \tilde{P}_m(z_1, z_2) \leqslant 0.4951, \tag{B2}$$

where $\tilde{P}_m(z_1, z_2)$ is defined by (3.7). The inequality $\tilde{P}_m(z_1, z_2) \geqslant -\frac{1}{7}$ is satisfied by choosing $P_m(\alpha) \geqslant -\frac{1}{7}$, $\theta = \frac{7}{8}$, $\tilde{\sigma} \geqslant \sigma - \frac{1}{7}b_0\tau$, where $\tilde{\sigma}$ is an estimate of the spectral radius of $\partial f/\partial y$ (see Section 3). The inequality $\tilde{P}_m(z_1, z_2) \leqslant 0.4951$ leads to the stability boundary β of the method. In table B1 these values for β are listed for $m = 1$ until 4 and S^* as large as allowed by (2.20a).

The method (2.1)–(2.5) with the predictor (B1b) and the predictor (B1) are denoted by SC(2, m, S^*) and $SC_2(3, m, S^*)$, respectively. In Table B2 the sd-values are listed for the modelproblem I obtained by SC(2, 2, 4) and $SC_2(3, 2, 4)$.

If $S^* > 0$ relation (A11) indicates that $\Delta\text{sd} \equiv 0.9$ for $q = 2$ and $\Delta\text{sd} \equiv 1.2$ for $q = 3$, as $\tau \to 0$. For SC(2, 2, 4) the third order behaviour is more or less reflected in Table B2. The fourth order $SC_2(3, 2, 4)$ method shows only a third order behaviour. It appears that the smoothed predictor (B1) shows its third order behaviour only for relatively small integration steps just as the method of successive corrections [4].

The iterated BDF4 together with the predictor (B1) generate a fourth order accurate splitting method which is stable for the S^*-values listed in Table B1 and $\tau \leqslant \beta(m)/\tilde{\sigma}$, where $\beta(m)$ is also listed in Table B1 (see also Section 3). This method will be denoted by SC_2 method.

In Table B3 the sd/ce-values are listed for problem I with $h = \frac{1}{24}$ obtained by SC_2.

The order behaviour of the SC_2 method is not reflected in Table B3. A comparison of the sd/ce-values for problem I with $h = \frac{1}{24}$ obtained by SC and SC_2 as listed in Tables 4.2 and B3 shows that the SC method is to be preferred. However, for a given value of m the maximal boundaries for the smoothed second order extrapolation (B1) as listed in Table B1 are much larger than the maximal boundaries for the smoothed third order extrapolation as listed in Table 3.1.

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