Application of Cagniard's method to the calculation of shock-waves in elastic and liquid media.

R 06, R 25, R 26.

Computation Department Mathematical Centre.

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Introduction.

This report contains a survey of the calculations performed by the Computation Department of the Mathematical Centre, Amsterdam for the Geological Department of the B.P.M., The Hague in connection with the orders R 6, R 25 and R 26.

In each of these orders the problem stated can be formulated as follows:

A three-dimensional space is filled with an elastic or liquid medium, this medium being divided into two semi-spaces by a horizontal layer of a medium of the same kind, but with higher velocity of wave propagation and of thickness h.

A point-sourse above this layer emits a wave starting at time t = 0 of a prescribed shape. Part of the energy generated by this primary is reflected into the upper medium again.

Due to the fact that the velocity of propogation of sound-waves is larger in the layer than in the remaining part of the space, there is a certain region of the upper medium in which the waves that can be regarded as due to direct reflection at the upper interface is not the first to arrive, but others will come before it.

It was the purpose of the computations to evaluate the displacements of the medium in a point lying at the same distance above the layer as the point-source and due to the longitudinal refracted and reflected waves arriving before the direct reflection.

In the case of order R 6 the thickness h was infinite, so that in fact we had two media filling a halfspace. Both media were supposed to be elastic with the same density; the velocities of propagation were:

for longitudinal waves in the upper medium $a = 1$
for transversal waves in the upper medium $b = \frac{1}{3}\sqrt{3}$
for longitudinal waves in the lower medium (layer) $a' = \sqrt{2}$
for transversal waves in the lower medium (layer) $b' = \frac{1}{3}\sqrt{6}$.

The primary wave is assumed to be spherically symmetrical in which case the displacements can be derived from a potential

$\Phi_o = F(t-\frac{R}{a})/R$ in which R = distance to point-source
t = time

Especially F(t) has been taken equal to the unit functions

$$F(t) = 0 \qquad t < 0$$
$$= 1 \qquad t > 0.$$

The height of the point-source above the layer is taken as unity. The displacements were to be calculated at points at horizontal distances 3, 4, 5, 6, 7, 8, 9 and 10 from the point-sour The horizontal as well as the vertical displacement were calculated.

In the case of order R 25 we had again elastic media. The thickness of the layer was h = 0,15 now. The other parameters were the same as in R 6, only now $\Phi_o = F(t-\frac{R}{a})/R$ should be the velocity-potential and only the vertical displacement was calculated in points lying at distances 5, 7 and 9 from the point-source.

Finally in R 26 the media were liquid, the layer had a thickness h = 0,15 and Φ_o was the velocity-potential again. There are only longitudinal waves possible now, b and b' being zero. The other parameters were the same as in R 25.

As R 25 gives the most general features of the problem and moreover part of R 6 and R 26 can be considered as a special case of it, we shall first consider this order.

A good deal of the theory on which the computations were based was developed in

a) L. Cagniard: Réflexion et Réfraction des Ondes Séismiques Progressives;

b) B.L.v.d. Waerden: Reflection and Refraction of Seismic Waves. GA Report No. 22363 of N.V. De Bataafse Petroleum Maatschappij;

c) P. Hazebroek: Two internal Reports of the B.P.M.

We shall therefore often refer to these authors.

Chapter I.

Reflection of shock-waves in an elastic medium divided into two parts by another elastic medium with higher velocities of propagation of sound-waves (R 25).

1.1. The notation for the parameters as described in the introduction will be used here. Further we may notice, that the problem

has a cylindrical symmetry around a vertical axis going through the point-source. Therefore we can reduce the problem to one which can be put in a form, introducing, besides the time coordinates, two special coordinates viz. the vertical coordinate z, measured along the axis of symmetry from the point-source and positive when the point in consideration lies below the source, and the horizontal coordinate r, giving the distance to the axis. Then of course $R = \sqrt{r^2 + z^2}$ is the distance to the point-source.

Taking the positive direction of the z-axis downwards has some advantages as regards signs and corresponds with the notations of Mr. Hazebroek. As, however, it is more convenient to take vertical displacements and velocities positive when they are upwards, we shall not derive the vertical velocity V_z from its potential as its negative but as its positive derivative with respect to z, and analogously for the vertical displacement .

We denote the height of the point-source above the layer by z_0. This should be 1 as stated in the introduction. Also we introduce $z_1 = 2z_0 - z$, this being a vertical coordinate with positive direction upwards and having its origin in the image of the point-source with respect to the upper interface. Then $R_1 = \sqrt{r^2 + z_1^2}$ gives the distance to this image.

Let ρ be the density of the medium and ρ' that of the layer. We will here take $\rho = \rho'$, and so $\sigma = \frac{\rho'}{\rho} = 1$.

Let λ and μ be Lamé's constants in the medium, λ' and μ' those in the layer.

We then have

$$a = \sqrt{\frac{\lambda + 2\mu}{\rho}} \qquad b = \sqrt{\frac{\mu}{\rho}}$$

$$a' = \sqrt{\frac{\lambda' + 2\mu'}{\rho'}} \qquad b' = \sqrt{\frac{\mu'}{\rho'}}$$

or

$$\lambda = \rho\,(a^2 - 2b^2) \qquad (1.1.1.)$$
$$\mu = \rho\, b^2 \qquad (1.1.2.)$$
$$\lambda' = \sigma\rho\,(a'^2 - 2b'^2) \qquad (1.1.1'.)$$
$$\mu = \sigma\rho\, b'^2 \qquad (1.1.2'.)$$

Now according to Hazebroek the Laplace-transform of the velocity-potential of the reflected waves can be written as

$$\varphi = \sum_{\substack{m,n=0 \\ m+n=\text{even}}}^{\infty} \varphi_{mn} \qquad (1.1.3.)$$

in which

$$\varphi_{mn} = \int_0^\infty a_{mn} \frac{P'}{P} e^{-\frac{p}{a}\{\alpha Z + h(m\alpha' + n\beta')\}} J_0\left(\frac{p}{a}\xi r\right) \frac{\xi\, d\xi}{\alpha} \quad (1.1.4)$$

p being the Laplace-variable

$$\begin{aligned}
-a_{00} &= 1 \\
-a_{20} &= q(q-q') \quad -a_{11} = 2(t'-t) \quad -a_{02} = r(r-r') \\
-a_{40} &= q^3(q-q') \quad -a_{31} = 2qt\{q(t'-t) + t(q'-q)\} \\
-a_{22} &= q^2 r(r-r') + q r^2 (q-q') + 4t^3(t-t') + s(s'-s) \quad (1.1.5) \\
-a_{13} &= 2rt\{r(t'-t) + t(r'-r)\} \\
-a_{04} &= r^3(r-r') \quad -a_{06} = q^5(q-q') \\
-a_{51} &= 2q^3 t\{q(t'-t) + 2t(q'-q)\} \\
-a_{42} &= q\{(3t^2 + q^2 r^2 + 2qrt^2)(q-q') + q^3 r(r-r') + 8qt^3(t-t') + qs(s'-s)
\end{aligned}$$

etc.

Generally the a_{mn} are the coefficients of the expansion

$$\frac{1 - qq'x^2 - rr'y^2 + ss'x^2y^2 + 2tt'xy}{1 - q^2x^2 - r^2y^2 + s^2x^2y^2 + 2t^2xy} = -\sum_{m,n} a_{mn} x^m y^n \quad (1.1.5')$$

$$\left.\begin{aligned}
q &= \frac{Q}{P} & q' &= \frac{Q'}{P'} \\
r &= \frac{R}{P} & r' &= \frac{R'}{P'} \\
s &= \frac{S}{P} & s' &= \frac{S'}{P'} \\
t &= \frac{T}{P} & t' &= \frac{T'}{P'}
\end{aligned}\right\} \quad (1.1.6)$$

$$\left.\begin{aligned}
P &= f_1 - f_2 - f_3 + g_1 - g_2 - g_3 \\
P' &= f_1 - f_2 - f_3 - g_1 + g_2 + g_3 \\
Q &= f_1 + f_2 + f_3 - g_1 - g_2 - g_3 \\
Q' &= f_1 + f_2 + f_3 + g_1 + g_2 + g_3 \\
R &= f_1 + f_2 - f_3 - g_1 - g_2 + g_3 \\
R' &= f_1 + f_2 - f_3 + g_1 + g_2 - g_3 \\
S &= f_1 - f_2 + f_3 + g_1 - g_2 + g_3 \\
S' &= f_1 - f_2 + f_3 - g_1 + g_2 - g_3 \\
T &= 2\sqrt{f_1 f_2} - 2\sqrt{g_1 g_2} \\
T' &= 2\sqrt{f_1 f_2} + 2\sqrt{g_1 g_2}
\end{aligned}\right\} \quad (1.1.7)$$

$$f_1 = \xi^2(\gamma'^2\mu' - \gamma^2\mu)^2 \qquad g_1 = \xi^2\alpha\beta\alpha'\beta'(\mu' - \mu)^2$$
$$f_2 = \alpha'\beta'(\xi^2\mu' - \gamma^2\mu)^2 \qquad g_2 = \alpha\beta(\gamma'^2\mu' - \xi^2\mu)^2 \quad (1.1.8)$$
$$f_3 = \alpha'\beta\mu\mu' a^4/4b^2b'^2 \qquad g_3 = \alpha\beta\mu\mu' a^4/4b^2b'^2$$

$$\alpha = \sqrt{\xi^2 + 1} \qquad \beta = \sqrt{\xi^2 + \frac{a^2}{b^2}} \qquad \gamma = \sqrt{\xi^2 + \frac{a^2}{2b^2}} \quad (1.1.9)$$
$$\alpha' = \sqrt{\xi^2 + \frac{a^2}{a'^2}} \qquad \beta' = \sqrt{\xi^2 + \frac{a^2}{b'^2}} \qquad \gamma' = \sqrt{\xi^2 + \frac{a^2}{2b'^2}}$$

(We use here R, r and t in another meaning than before; there is, however, no danger of confusion).

The μ and μ' could be calculated from (1.1.2) and (1.1.2) if only we knew the value of ϱ. As, however, the expressions (1.1.8) and therefore also (1.1.7) are all homogeneous of the second degree in μ and μ', the ϱ is eliminated by division in (1.1.6) and we can do without its value.

As we shall see afterwards we have generally to deal with complex quantities; therefore the expressions $\sqrt{f_1 f_2}$ and $\sqrt{g_1 g_2}$ which occur in T and T' are somewhat difficult to be calculated. Looking at the formulae (1.1.5), we see that t and t' can always be taken in combinations of the forms t^2 and $t\,t'$, and T and T' in combinations of the forms T^2 and TT'. For these we can derive the expressions

$$T^2 = 4\{(f_1 - g_2)(f_2 - g_1) + f_3 g_3\}$$
$$TT' = 4(f_1 f_2 - g_1 g_2) \quad (1.1.7')$$

1.2. From (1.1.3) and (1.1.4) we can derive the following formulae for the Laplace-transform V_z of the vertical velocity v_z by differentiating with respect to z.

$$V_z = \sum_{\substack{m,n=0 \\ m+n=\text{even}}}^{\infty} V_{z,mn} \quad (1.2.1)$$

$$V_{z,mn} = \frac{p}{a}\int_0^\infty a_{mn}\frac{p'}{p}e^{-\frac{p}{a}\{\alpha Z + h(m\alpha' + n\beta')\}} J_{(\frac{p}{a}\xi r)}\xi d\xi \quad (1.2.2)$$

The vertical displacement u_z is the time-integral of v_z, and we find therefore its Laplace-transform U_z by division by p.

$$U_z = \sum_{\substack{m,n=0 \\ m+n=\text{even}}}^{\infty} U_{z,mn} \quad (1.2.3)$$

$$u_{z,mn} = \frac{1}{d}\int_0^\infty a_{mn}\frac{P'}{P}e^{-\frac{p}{a}\{\alpha Z_1 + h(m\alpha' + n\beta')\}}\,J_0\left(\frac{p}{a}\xi r\right)\xi\,d\xi \qquad (1.2.4)$$

The inverse transformation is performed in the same way as Cagniard did; we shall expose it roughly. As to the proof of its validity we may refer to the considerations that Cagniard gives, and which can similarly be followed here.

Let us substitute for the Bessel-function J_0 its integral-representation

$$J_0\left(\frac{p}{a}\xi r\right) = \frac{1}{\pi}\int_0^\pi e^{-i\frac{p}{a}\xi r\cos\omega}\,d\omega \qquad (1.2.5)$$

$$u_{z,mn} = \frac{1}{\pi a}\int_0^\infty\int_0^\pi a_{mn}\frac{P'}{P}e^{-\frac{p}{a}\{\alpha Z_1 + h(m\alpha'+n\beta') + i\xi r\cos\omega\}}\,\xi\,d\xi\,d\omega \qquad (1.2.6)$$

and let us then replace the variable ξ by

$$K = \frac{1}{a}\left\{\alpha Z_1 + h(m\alpha'+n\beta') + i\xi r\cos\omega\right\} \qquad (1.2.7)$$

Then we get

$$u_{z,mn} = \frac{1}{\pi a}\int_{K_0}^\infty\int_0^\pi a_{mn}\frac{P'}{P}e^{-pK}\,\xi\,\frac{\partial\xi}{\partial K}\,dK\,d\omega \qquad (1.2.8)$$

with

$$K_0 = K(\xi = 0) = \frac{Z_1}{a} + h\left(\frac{m}{a'} + \frac{n}{b'}\right) \qquad (1.2.9)$$

and in which the path of integration may be taken along the real axis.

This shows, that if we identify K with the time t, $u_{z,mn}$ is the Laplace-transform of a function

$$u_{z,mn} = \frac{1}{\pi a}\int_0^\pi a_{mn}\frac{P'}{P}\,\xi(\omega,t)\,\frac{\partial\xi}{\partial t}\,d\omega \quad \left(t > \frac{Z_1}{a} + h\left(\frac{m}{a'}+\frac{n}{b'}\right)\right) \qquad (1.2.10)$$

$$= 0 \quad \left(t < \frac{Z}{a} + h\left(\frac{m}{a'}+\frac{n}{b}\right)\right)$$

the ξ, ω, t connected by the relation

$$at = \alpha Z_1 + h(m\alpha'+n\beta') + i\xi r\cos\omega \qquad (1.2.11)$$

From this relation we transform ω to ξ back again, yielding

$$u_{z,mn} = \frac{1}{\pi a}\int_{\xi_0}^{\xi_\pi} a_{mn}\frac{P'}{P}\,\xi\,\frac{\partial\xi}{\partial t}\,\frac{\partial\omega}{\partial\xi}\,d\xi \quad \left(t > \frac{Z_1}{a} + h\left(\frac{m}{a'}+\frac{n}{b'}\right)\right) \qquad (1.2.12)$$

with ξ_0 and ξ_π resp. defined by the equations

$$at = \alpha Z + h(m\alpha' + n\beta') + i\xi r \qquad (1.2.13)$$

and

$$at = \alpha Z + h(m\alpha' + n\beta') - i\xi r \qquad (1.2.14)$$

Finally

$$\frac{\partial \xi}{\partial t} \cdot \frac{\partial \omega}{\partial \xi} = -\frac{\partial \omega}{\partial t} = \frac{-ia}{\xi r \sin\omega} = \frac{-ia}{\sqrt{\{at - \alpha Z - h(m\alpha' + n\beta')\}^2 + \xi^2 r^2}}$$

Thus

$$U_{Z,mn} = -\frac{i}{\pi} \int_{\xi_0}^{\xi_\pi} \frac{a_{mn} P'/P \; \xi \, d\xi}{\sqrt{\{at - \alpha Z - h(m\alpha' + n\beta')\}^2 + \xi^2 r^2}}$$

$$= -\frac{i}{\pi} \int_{\xi_0}^{\xi_\pi} a_{mn} \frac{P'/P \; \xi \, d\xi}{\sqrt{Q_{mn}}} \quad (+> \frac{Z}{a} + h\left(\frac{m}{\alpha'} + \frac{n}{\beta'}\right)) \qquad (1.2.15)$$

in which

$$Q_{mn} = \{at - \alpha Z - h(m\alpha' + n\beta')\}^2 + \xi^2 r^2 \qquad (1.2.16)$$

Note that ξ_0 and ξ_π are zeros of Q_{mn}. According to Cagniard they are to be chosen in the fourth and first quadrant respectively, which condition fixes them uniquely; only if they are lying on the imaginary axis there still might be two values for each. We then must choose the one with the smallest modulus.

1.3. First of all the case m = n = 0 will be considered.

As $a_{00} = -1$, we get from (1.2.15) and (1.2.16)

$$U_{Z,00} = \frac{i}{\pi} \int_{\xi_0}^{\xi_\pi} \frac{P'/P \; \xi \, d\xi}{\sqrt{Q_{00}}} \qquad (1.3.1.)$$

if $t > Z/a$

$$Q_{00} = \{at - \alpha Z\}^2 + \xi^2 r^2 \qquad (1.3.2.)$$

and ξ_0 and ξ_π defined by the equation

$$Q_{00} = 0 \qquad (1.3.3)$$

This term of the double-series for U_Z corresponds to the conical wave treated by Cagniard and v.d. Waerden.

Putting $t_{oo} = \frac{z}{Z}\sqrt{\frac{1}{a^2} - \frac{1}{a'^2}} + \frac{r}{a'}$ (1.3.4.)

there is a wave with conical front-ariving at the time in that region of the upper medium in which the condition $t_{oo} < t < \frac{R_1}{a}$ can be satisfied, that is t_{oo} must be smaller than R_1/a. (All this, assuming a' > a!)

Following the method as given by v.d. Waerden, we have

$$U_{z,oo} = Re\,\frac{2i}{\pi}\int_{i\,a/a'}^{\xi_\pi} \frac{P'/P\ \xi\, d\xi}{\sqrt{Q_{oo}}} \qquad (1.3.5)$$

Substituting α instead of ξ, we get

$$U_{z,oo} = Re\,-\frac{1i}{\pi}\int_{\alpha_\pi}^{\sqrt{1-a^2/a'^2}} \frac{P'/P\ \alpha\, d\alpha}{\sqrt{Q_{oo}}} \qquad (1.3.6)$$

in which now

$$Q_{oo} = \{dt - \alpha Z\}^2 + (\alpha^2 - 1)r^2 = R_1^2(\alpha - \alpha_\pi)(\alpha - \alpha_\pi^*) \qquad (1.3.7)$$

$$\left.\begin{aligned} \alpha_\pi &= \frac{1}{R_1^2}\left(adZt + r\sqrt{R_1^2 - a^2t^2}\right) \\ \alpha_\pi^* &= \frac{1}{R_1^2}\left(adZt - r\sqrt{R_1^2 - a^2t^2}\right) \end{aligned}\right\} \qquad (1.3.8)$$

Writing $c = \sqrt{1 - a^2/a'^2}$ (1.3.9)

and $Re(-iP'/p) = Im\ P'/p = H$ (1.3.10)

we get

$$U_{z,oo} = \frac{2}{\pi R_1}\int_{\alpha_0}^{c} \frac{H\ \alpha\, d\alpha}{\sqrt{(\alpha - \alpha_\pi)(\alpha - \alpha_\pi^*)}} \qquad (1.3.11)$$

As to the value of H, we notice that on the path of integration ξ has moduli which are smaller than 1, a/b and a/b', so that α, β and β' are real there. Then returning to the formulae (1.1.8) and (1.1.7) we see that we can write

$$P = \varepsilon + \delta\alpha' \ ;\ P' = \varepsilon' + \delta'\alpha' \qquad (1.3.12)$$

ε, δ, ε' and δ' being real, and so

$$P'/P = \frac{(\varepsilon' + \delta'\alpha')(\varepsilon - \delta\alpha')}{\varepsilon^2 - \delta^2\alpha'^2} = \frac{\varepsilon'\varepsilon - \delta\delta'\alpha'^2}{\varepsilon^2 - \delta^2\alpha'^2} + \frac{\varepsilon\delta' - \delta\varepsilon'}{\varepsilon^2 - \delta^2\alpha'^2}\alpha' \qquad (1.3.13)$$

$$H = \frac{\varepsilon\delta' - \delta\varepsilon'}{\varepsilon^2 - \delta^2\alpha'^2}\,\frac{\alpha'}{i} \qquad (1.3.14)$$

$$\text{But} \quad \alpha' = \sqrt{\xi^2 + \frac{\pi^2}{d'^2}} = \sqrt{\alpha^2 - c^2} \qquad (1.3.15)$$

therefore

$$H = \frac{\varepsilon\delta' - \delta\varepsilon'}{\varepsilon^2 + \delta^2(c^2 - \alpha^2)}\sqrt{c^2 - \alpha^2} \qquad (1.3.16)$$

The function

$$G = \frac{\varepsilon\delta' - \delta\varepsilon'}{\varepsilon^2 + \delta^2(c^2 - \alpha^2)}\sqrt{c + \alpha} \qquad (1.3.17)$$

which is independent of t, was tabulated with the desired accuracy between the limits of α where it was needed and with such an interval, that interpolation was easily performed.

The factor $\sqrt{c - \alpha}$ was taken apart because it causes a singularity in the point $\alpha = c$, one end of the integration-interval. At the other end $(\alpha - \alpha_\pi)^{-1/2}$ has a singularity, we can get rid of either of them by the substitution

$$\alpha = \frac{c + \alpha_\pi}{2} + \frac{c - \alpha_\pi}{2}\sin\frac{\pi}{2}\Psi \qquad (1.3.18)$$

giving

$$U_{z,\omega} = \frac{c - \alpha_\pi}{i R_1}\int_{-1}^{1}\frac{G\,\alpha}{\sqrt{\alpha - \alpha_\pi^*}}\left(1 - \sin\frac{\pi}{2}\Psi\right)d\Psi \qquad (1.3.19)$$

This integral could be calculated numerically.

1.4. Next we shall consider the case $m \neq 0$, n arbitrary. This corresponds to a wave that, after refraction, travels through the layer, being repeatedly reflected at the interfaces such that there are (m+n) paths between two consecutive reflections, and then goes into the upper medium again, m of the reflections being of the longitudinal type and n of them transversal ones. Because of the higher longitudinal velocity of propagation in the layer, these waves may arrive before the directly reflected one, provided the m and n are not too large. Now we have for

$$t > z/a + h(m/a' + n/b')$$

$$U_{z,mn} = -\frac{i}{\pi}\int_{\xi_0}^{\infty} a_{mn}\,P'/P\;\xi\;Q_{mn}^{-1/2}\,d\xi \qquad (1.4.1)$$

with

$$Q_{mn} = \left\{ \partial t - \alpha Z - h\left(m\alpha' + n\beta'\right) \right\}^2 + \xi^2 r^2 \qquad (1.4.2)$$

where ξ_o and ξ_π are the roots of the equation

$$Q_{mo} = 0 \qquad (1.4.3)$$

or $$\partial t - \alpha_o Z - h\left(m\alpha'_o + n\beta'_o\right) - i\,\xi_o r = 0 \qquad (1.4.4)$$

$$\partial t - \alpha_\pi Z - h\left(m\alpha'_\pi + n\beta'_\pi\right) + i\,\xi_\pi r = 0 \qquad (1.4.5)$$

For $t = Z/a + h\,(m/a' + n/b')$; $\xi_o = \xi_\pi = o$. Then if t grows ξ_o becomes negative imaginary and ξ_π positive imaginary, until t reaches the point where $\frac{\partial t}{\partial \xi_o} = o$ and $\frac{\partial t}{\partial \xi_\pi} = o$, corresponding to a branch-point of ξ_o and ξ_π as a function of t. (They have the same value of t as ξ_o and ξ_π are conjugate complex and so are $\frac{\partial t}{\partial \xi_o}$ and $\frac{\partial t}{\partial \xi_\pi}$ therefore.). In these branch-points the values of ξ_o and ξ_π have moduli which are smaller than those of any of the branch-points $\pm i\,a/a'$, $\pm i$, $\pm i\,a/a'$; $\pm i\,a/b$ of the integrand (compare Cagniard). Denoting the corresponding value of t by t_{mn} , we have

$$U_{z,mn} = o \qquad t < t_{mn}\,, \qquad (1.4.6)$$

as the integrand of $U_{z,mn}$ is an antimetric function of ξ now. When $t > t_{mn}$, ξ_o and ξ_π become complex.

Taking into account that we may take as path of integration the paths followed by ξ_o and ξ_π as t grows from $Z/a + h\,(m/a' + n/b')$, and that the integrand reaches conjugate complex values for two conjugate complex arguments, we find

$$U_{z,mn} = Re - \frac{2i}{\pi} \int_{\xi_\pi(t_{mn})}^{\xi_\pi(t)} a_{mn} \frac{P'/P\ \xi\ d\xi}{\sqrt{Q_{mn}}} \qquad (1.4.7)$$

Now we perform the transformation

$$a\tau = \alpha Z + h\left(m\alpha' + n\beta'\right) \qquad (1.4.8)$$

Comparing this with (1.4.5) we see, that this means, that τ is t that value of t for which the point ξ would be the ξ_n . This means, that the integration-variable is now a real one. We get

$$U_{z,mn} = Re - \frac{2i}{\pi} \int_{t_{mn}}^{t} a_{mn} \frac{P'/P \cdot \xi\ d\xi/d\tau \cdot d\tau}{\sqrt{Q_{mn}}} \quad (t > t_{mn}) \qquad (1.4.9)$$

in which according to (1.4.2) and (1.4.8)

$$Q_{mn} = a^2 (t - \tau)(t - \tau - 2i\xi_r) \qquad (1.4.10)$$

so that we again have two singularities at the ends t_{mn} and t of the path of integration. The first is cancelled by the substitution

$$\sigma = \sqrt{\tau - t_{mn}} \qquad (1.4.11)$$

and the other by applying the transformation

$$\sigma = \sqrt{t - t_{mn}} \sin \frac{\pi}{2} \psi \qquad (1.4.12)$$

The result is

$$U_{2,mn} = Re\, -2i \int_0^1 a_{mn} \frac{P'/P\, \xi \frac{d\xi}{d\tau} \sigma\, d\psi}{a (t - t_{mn} - \sigma^2 - 2i\xi_r)^{3/2}} \qquad (1.4.13)$$

Now if $$\frac{a_{mn}}{a} P'/P\, \xi \frac{d\xi}{d\tau} \sigma = L_r + i L_i \qquad (1.4.14)$$

and $$(t - t_{mn} - \sigma^2 - 2i\xi_r)^{-3/2} = K + iM \qquad (1.4.15)$$

then

$$U_{2,mn} = 2 \int_0^1 (K L_i + M L_r)\, d\psi \qquad (1.4.16)$$

which integral can be calculated nulerically if only we know L_r, L_i, K and M as functions of ψ. Now ξ, L_r and L_i are functions of σ independent of t, so that if we could make a table or draw a graph of them (for ξ as well of ξ_r as of ξ_i, being respect. its real and imaginary part) as a function of σ, and then, after also having calculated t_{mn} find the σ-value corresponding to particular values of t and ψ, also the L_i, L_r and ξ, and consequently from (1.4.15) the K and M for these ψ- and t-values could be found.

The easiest way, and the only one feasible without taking too long a time, appeared to be drawing graphs of ξ_r, ξ_i, L_r and L_i. The method by which the graphs of ξ_r and ξ_i were found will be treated in the next two sections separately for the cases n = 0 and n ≠ 0. For the same σ-values as for which ξ had been calculated, the formulae (1.1.5-9) and the additional formula

$$\frac{d\xi}{d\tau} = \frac{i\, d}{\frac{\xi}{\alpha} Z + h\left(\frac{m}{\alpha}\xi + \frac{n}{\beta}\xi\right) - i\tau} \qquad (1.4.17)$$

give us L_r and L_i, except for $\sigma = 0$ where $\frac{d\xi}{d\tau} = \infty$, but $\sigma \frac{d\xi}{d\tau}$ has a finite value. We shall consider this last case afterwards with the calculation of t_{mn} and the values of ξ for $\sigma = 0$ which will be developed in section (1.7).

The computation of K and M was performed as follows

$$\left(t - t_{mn} - \sigma^2 - 2i\xi\tau\right)^{-1/2} = \left(t - t_{mn} - \sigma^2 + 2\xi_i\tau - 2i\xi_r\tau\right)^{-1/2} \qquad (1.4.18)$$
$$= \Theta^{-1/2}\,(1 - ik)^{-1/2}$$

in which

$$\Theta = 2\tau\xi_i + t - t_{mn} - \sigma^2 \qquad (1.4.19)$$

and $$k = \frac{2\tau\xi_r}{\Theta}$$

Now a table of A and B from

$$(1 - ik)^{-1/2} = A + iB \qquad (1.4.20)$$

was made.

The result is $$K = A/\sqrt{\Theta} \;;\; M = B/\sqrt{\Theta} \qquad (1.4.21)$$

and so $$U_{z,mn} = 2\int_0^1 \frac{AL_i + BL_r}{\sqrt{\Theta}}\, d\psi$$

In the case of large values of m and n it was often sufficient to take

$$A = 1 \;;\; B = \frac{k}{2} \qquad (1.4.22)$$

1.5. Now the method of getting the graphs of $\xi(\sigma)$ in the case $m \neq 0$, $n = 0$ will be shown.

Let (ξ lies in the first quadrant)

$$i\xi = \sin(-I_{1r} + i I_{1i}) \qquad (1.5.1)$$

$$\left.\begin{array}{l} \text{with} \quad 0 \leq I_{1r} \leq \frac{\pi}{2} \\ I_{1i} \geq 0 \end{array}\right\} \qquad (1.5.2)$$

Then

$$\alpha = \cos(-I_{1r} + i I_{1i}) \qquad (1.5.3)$$

Further let be

$$\alpha' = \frac{a}{a'}\cos(-I_{2r} + i I_{2i}) \qquad (1.5.4)$$

$$\left.\begin{array}{l} 0 \leq I_{2r} \leq \frac{\pi}{2} \\ I_{2i} \geq 0 \end{array}\right\} \qquad (1.5.5)$$

Then from (1.1.9) follows

$$\sin(-I_{1r} + i I_{1i}) = \frac{a}{a'} \sin(-I_{2r} + i I_{2i}) \qquad (1.5.6)$$

We choose some values of I_{1r}, and try to find values of I_{1i}, I_{2r} and I_{2i} such, that with (1.5.1), (1.5.3), (1.5.4) and (1.4.8) we should get a real value for τ. Then (1.5.6) and the equation $\mathrm{Im}\,\tau = 0$ give us

$$\sin I_{1r} \cosh I_{1i} = \frac{a}{a'} \sin I_{2r} \cosh I_{2i} \qquad (1.5.7)$$

$$\cos I_{1r} \sinh I_{1i} = \frac{a}{a'} \cos I_{2r} \sinh I_{2i} \qquad (1.5.8)$$

$$Z_1 \sin I_{1r} \sinh I_{1i} + m h \frac{a}{a'} \sin I_{2r} \sinh I_{2i} = r \cos I_{1r} \sinh I_{1i} \qquad (1.5.9)$$

We now can calculate I_{1i}, I_{2r} and I_{2i} according to the following scheme

$$\left.\begin{aligned} \operatorname{tg} I_{2r} &= \frac{r - Z_1 \operatorname{tg} I_{1r}}{m h} \\ \mu &= \frac{\operatorname{tg} I_{2r}}{\operatorname{tg} I_{1r}} \\ \lambda &= \frac{a'}{a} \frac{\cos I_{1r}}{\cos I_{2r}} \\ \cosh^2 I_{2i} &= \frac{\lambda^2 - 1}{\mu^2 - 1} \\ \cosh I_{1i} &= \frac{\mu}{\lambda} \cosh I_{2i} \end{aligned}\right\} \qquad (1.5.10)$$

Then we have

$$\left.\begin{aligned} \xi_r &= \cos I_{1r} \sinh I_{1i} \\ \xi_i &= \sin I_{1r} \cosh I_{1i} \end{aligned}\right\} \qquad (1.5.11)$$

Moreover we can easily calculate α_r, α_i, α'_r and α'_i, these being the real and imaginary parts of α and α'. These values are necessary for the computation of L_r and L_i, and now that we find them in this comfortable way we avoid the square-rooting of complex numbers, which should have had to be performed if we had calculated them directly from ξ.

We have

$$\left.\begin{aligned} \alpha_r &= \cos I_{1r} \cosh I_{1i} \\ \alpha_i &= \sin I_{1r} \sinh I_{1i} \\ \alpha'_r &= \frac{a}{a'} \cos I_{2r} \cosh I_{2i} \\ \alpha'_i &= \frac{a}{a'} \sin I_{2r} \sinh I_{2i} \end{aligned}\right\} \qquad (1.5.12)$$

Finally

$$\vartheta\tau = \sum \alpha_r + m h \alpha'_r + r \xi_i \qquad (1.5.13)$$

and after the calculation of t_{mn}, which will be exposed in (1.7) and applying (1.4.11) we get corresponding values of σ and ξ .

1.6. The problem is much more complicated, if also $n \neq 0$. We now must also introduce

$$\beta' = \frac{a}{b'} \cos(-I_{3r} + i I_{3i}) \qquad (1.6.1)$$

and then the equations are

$$\begin{aligned} \sin I_{1r} \cosh I_{1i} &= \frac{a}{a'} \sin I_{2r} \cosh I_{2i} = \frac{a}{b'} \sin I_{3r} \cosh I_{3i} \\ \cos I_{1r} \sinh I_{1i} &= \frac{a}{a'} \cos I_{2r} \sinh I_{2i} = \frac{a}{b'} \cos I_{3r} \sinh I_{3i} \\ \sum \sin I_{1r} \sinh I_{1i} + hm \frac{a}{a'} \sin I_{2r} \sinh I_{2i} + hn \frac{a}{b'} \sin I_{3r} \sinh I_{3i} &= r \cos I_{1r} \sinh I_{1i} \end{aligned} \qquad (1.6.2)$$

From these we obtain the two equations

$$\sum t_1 + hm\, t_2 + hn\, t_3 - r = 0 \qquad (1.6.3)$$

$$\text{and} \left(\frac{a^2}{a'^2} s_2^2 - s_1^2\right)\left(t_1^2 - t_3^2\right) = \left(\frac{a^2}{b'^2} s_3^2 - s_1^2\right)\left(t_1^2 - t_2^2\right) \qquad (1.6.4)$$

$$\text{if} \quad t_k = \operatorname{tg} I_{kr}, \qquad s_k = \sin I_{kr} \qquad (1.6.5)$$

By $s_k^2 = \frac{t_k^2}{1 + t_k^2}$ we can write (1.6.4) as a quadratic equation in t_3^2 . The root that comes into consideration is

$$t_3^2 = \frac{-X + \sqrt{X^2 + Y}}{N} \qquad (1.6.6)$$

$$\left.\begin{aligned} X &= \alpha_1 t_2^4 - \alpha_2 t_2^2 - \alpha_3 \\ Y &= t_2^2 (\beta_1 t_2^4 + \beta_2 t_2^2 - \beta_3) \\ N &= \gamma_1 t^2 + \gamma_2 \end{aligned}\right\} \qquad (1.6.7)$$

$$\left.\begin{aligned} \alpha_1 &= 2 t_1^2 (a^2/b'^2 - 1) + 2 a^2/b'^2 \\ \alpha_2 &= 2 (t_1^4 - 1)(a^2/b'^2 - a^2/a'^2) \\ \alpha_3 &= 2 t_1^2 \{ t_1^2 a^2/b'^2 + (a^2/b'^2 - 1) \} \end{aligned}\right\} \qquad (1.6.8)$$

$$\left.\begin{aligned} \beta_1 &= 16\, t_1^2 \left\{ t_1^2 \left(1 - \frac{a^2}{a'^2}\right) - a^2/a'^2 \right\} \\ \beta_2 &= 16\, t_1^4 \left\{ 2 - 4 a^2/a'^2 + 4 a^4/a'^4 \right) - a^2/a'^2 \left(1 - a^2/a'^2\right) + t_1^2 (t_1^2 - 1)^2 \} \\ \beta_3 &= 16\, t_1^4 \left\{ a^2/a'^2 t_1^2 - \left(1 - a^2/a'^2\right) \right\} \end{aligned}\right\} \qquad (1.6.9)$$

$$\left.\begin{aligned} \gamma_1 &= 4\left\{t_1^2\left(1-a^2/a'^2\right)-a^2/a'^2\right\} \\ \gamma_2 &= 4t_1^2 \end{aligned}\right\} \qquad (1.6.10)$$

An extra factor 2 has been introduced regarding the factor that occurs in the expressions a^2/a'^2, a^2/b'^2 and a^2/b^2 for the values chosen for our parameters. Inserting these values and introducing an auxiliary quantity

$$\eta = \gamma_1 t_1^2 \qquad (1.6.11)$$

we get

$$\left.\begin{aligned} \gamma_1 &= 2\left(t_1^2-1\right) \\ \gamma_2 &= 4t_1^2 \end{aligned}\right\} \qquad (1.6.10')$$

$$\left.\begin{aligned} \alpha_1 &= t_1^2+3 \\ \alpha_2 &= \gamma_1\left(t_1^2+1\right) \\ \alpha_3 &= t_1^2\left(3t_1^2+1\right) \end{aligned}\right\} \qquad (1.6.8')$$

$$\left.\begin{aligned} \beta_1 &= 4\eta \\ \beta_2 &= \gamma_2^2-\gamma_1\eta \\ \beta_3 &= \gamma_2\eta \end{aligned}\right\} \qquad (1.6.9')$$

As I_{1r} is chosen, t_1 is known, and we can tabulate the α, β and γ for those values of t_1 that occurred most frequently in our computations, independent of t, m and n.

Now, assuming a value for t_3 we can solve (1.6.3) for t_2. Substituting this t_2 into (1.6.7) and (1.6.6.) gives a new value for t_3, that goes into (1.6.3) again, etc. This iteration-process proved to be fairly fast . A difficulty is, that t_2 is large and therefore $y \ll X^2$, so that if we work for instance with 8 significant digits for X and $\sqrt{X^2+y}$ hardly any digit is left in $C = -X+\sqrt{X^2+y}$. We could therefore apply a power-series in y/X^2, but that often converged too slowly.

The way we overcame this difficulty is as follows: Suppose the errors in X and y to be ΔX and Δy respectively, then the error in $C = -X+\sqrt{X^2+y}$ is

$$\Delta C = -\Delta X+\sqrt{(X+\Delta X)^2+(y+\Delta y)}-\sqrt{X^2+y} \qquad (1.6.12)$$

Now $X^2 \gg y$ and $(\Delta X)^2 \ll \Delta y$, so that

$$\Delta C = -\Delta X+\frac{2X\Delta X+\Delta y}{2X} = \frac{1}{2}\,\frac{\Delta y}{X} \qquad (1.6.12')$$

and $$C \approx y/2X \qquad (1.6.13)$$

thus

$$\frac{\Delta C}{C} \approx \frac{\Delta y}{y} \qquad (1.6.14)$$

From (1.6.14) it follows that the answer in C will be correct to the same relative accuracy as y if we handle X as if it were correct to the same number of decimal places as y.

A good start for t_3 is found by taking t_2 to be unfinite in (1.6.7) and (1.6.6). We then find

$$t_3^2 = \beta_1/2\alpha_1\gamma_2$$
$$\text{or} \quad t_3^2 = \frac{2t_1^2}{\alpha_1} \qquad (1.6.15)$$

Now that the I_{kr} have been calculated, we find I_{2i} from

$$th^2 I_{2i} = \frac{1-(2\cos^2 I_{1r} - \cos^2 I_{2r})}{2\cos^2 I_{1r} - \cos^2 I_{2r}} \cdot \frac{1}{t_1^2} \qquad (1.6.16)$$

and the others from

$$\frac{th\, I_{1i}}{t_1} = \frac{th\, I_{2i}}{t_2} = \frac{th\, I_{3i}}{t_3} \qquad (1.6.17)$$

Then everything is ready for the computation of ξ and L.

1.7. The waves begin at the time t_{mn}, which is defined by

$\frac{dl_+}{d\xi_\pi} = 0$, or

$$\frac{\xi_\pi}{\alpha_\pi} Z_1 + h\left(\frac{m\xi_\pi}{\alpha'_\pi} + \frac{n\xi_\pi}{\beta'_\pi}\right) - ir = 0 \qquad (1.7.1)$$

Now putting

$$\left.\begin{aligned} \xi_\pi &= i \sin I_2 \\ \alpha_\pi &= \cos I_1 \\ \alpha'_\pi &= \frac{d}{d'} \cos I_2 \\ \beta'_\pi &= \frac{d}{b'} \cos I_3 \end{aligned}\right\} \qquad (1.7.2)$$

and using (1.1.9) and (1.7.1.) we find

$$\sin I_1 = \frac{d}{d'} \sin I_2 = \frac{d}{b'} \sin I_3 \qquad (1.7.3)$$

and $$Z_1 tg\, I_1 + hm\, tg\, I_2 + hn\, tg\, I_3 - r = 0 \qquad (1.7.4)$$

These equations correspond to (1.6.2) and (1.6.3) with $I_{ki} = 0$ and $I_{kr} = I_k$.

When n = 0 we can omit I_3, and we have

$$\sin I_1 = \frac{d}{d'} \sin I_2 \qquad (1.7.3')$$

$$Z + g I_1 + hm + g I_2 - r = 0 \qquad (1.7.4')$$

Beginning with any given value of I_1 and then solving (1.7.4') for I_2, we insert the value found for I_2 in (1.7.3') and solve it for I_1 etc., and so find I_1 and I_2 by iteration.

Then we find t_{m0} from

$$t_{m0} = Z \cos I_1 + r \sin I_1 + mh \frac{a}{d'} \cos I_2 \qquad (1.7.5)$$

and the formulae (1.7.2) give us values of ξ_n, $\varkappa_n$, α_n' that are the values of ξ, α and α' for $\sigma = 0$.

When $n \neq 0$ we proceed in the following way: Take any given value of I_2, solve (1.7.3) for I_1 and I_3, substitute these values in (1.7.4) and solve this equation for I_2 etc.

Then, after having found I_1, I_2 and I_3 we have

$$t_{mn} = Z \cos I_1 + r \sin I_1 + mh \frac{a}{d'} \cos I_2 + nh \frac{a}{b} \cos I_3 \qquad (1.7.6)$$

and the formulae (1.7.2) give us values of ξ_n, α_n, α_n', β_n' that are the values of ξ, α, α' and β' for $\sigma = 0$.

Now we can continue in the ordinary way the computation of L_i and l_{ir} for $\sigma = 0$, but for the factor $\frac{d\xi}{d\tau}$ for σ.

Now we have

$$\lim_{\sigma \to 0} \frac{1}{\sigma} \frac{d\tau}{d\xi} = \left\{ \frac{d^2\tau}{d\xi^2} \Big/ \frac{d\sigma}{d\xi} \right\}_{\sigma=0} \qquad (1.7.7)$$

and on the other hand

$$\lim_{\sigma \to 0} \frac{1}{\sigma} \frac{d\tau}{d\xi} = 2 \left\{ \frac{d\sigma}{d\xi} \right\}_{\sigma=0} \qquad (1.7.8)$$

So (1.7.7) and (1.7.8) together give

$$4 \left\{ \frac{d\sigma}{d\xi} \right\}^2_{\sigma=0} = 2 \left\{ \frac{d^2\tau}{d\xi^2} \right\}_{\sigma=0} = 2 \left\{ \frac{Z}{\alpha^3} + \left(\frac{a}{a'}\right)^2 \frac{mh}{\alpha'^3} + \left(\frac{a}{b'}\right)^2 \frac{nh}{\beta'^3} \right\}_{\sigma=0}$$

$$= 2 \left\{ \frac{Z}{\cos^3 I_1} + \frac{a'}{a} \frac{mh}{\cos^3 I_2} + \frac{b'}{a} \frac{nh}{\cos^3 I_3} \right\} \qquad (1.7.$$

and so from (1.7.8) again

$$\lim_{\sigma \to 0} \sigma \frac{d\tau}{d\xi} = \frac{1}{\sqrt{2\left\{\frac{2}{c_0^3 I_1} + \frac{2'}{d}\frac{mh}{\cos^3 I_2} + \frac{b'}{d}\frac{nh}{\cos^3 I_3}\right\}}} \qquad (1.7.10)$$

Now all values wanted for the calculation of L_r and L_i are known. Next we ask for the value of $U_{2,mn}$ at the time t_{mn}. Then in (1.4.13) $\sigma \equiv \omega$ independent of ψ, according to (1.4.12). Therefore L_r, L_i and ξ are constant having their values they adopt for $\sigma = 0$. So ξ is purely imaginary, $(+ - t_{mn} - \sigma^2 - 2i\xi\tau)^{-1/2}$ is real, M = 0, and $K = \frac{1}{\sqrt{2\xi_i r}}$. Also ξ^2, α, α', β, β', γ and γ' are real; from (1.1.8), (1.1.7), (1.1.6) and (1.1.5) we learn that $d_{mn} P''/P$ is real, and so $\left\{\frac{d\xi}{d\tau}\sigma\right\}_{\sigma=0}$. Therefore L is purely imaginary, $L_r = 0$. Thus we see that

$$\left[U_{2,mn}\right]_{t=t_{mn}} = \left[\sqrt{\frac{2}{\xi_i r}} L_i\right]_{\sigma=0} \qquad (1.7.11)$$

As $L_i \neq 0$ the waves begin with a discontinuity. This is not the case for m = n = 0. From (1.3.8) we obtain $\alpha_\pi = c$ for $t = t_{00}$ and thus $\left[U_{2,00}\right]_{t=t_{00}} = 0$ according to (1.3.19), as $c - \alpha_\pi = 0$.

1.8. As the velocity of propagation of transversal waves in the layer is smaller than that of the longitudinal waves in the main medium, waves with m = 0, $n \neq 0$ arrive later than the directly reflected one. Therefore we need not consider these waves.

Chapter II.

Reflection of stock-waves in an elastic medium filling a half-space and resting on a second elastic medium with higher velocities of propagation of sound-waves, filling the other half space (R 6).

2.1. As has been said in the introduction this problem corresponds to that of R 25 with $h = \infty$.

Now Φ_0, however, is the potential of the displacements, and the expressions (1.1.3) and (1.1.4) still hold, if we define φ as the Laplace-transform of the potential of the displacements of the reflected waves. Moreover, as $h = \infty$ we have to consider only the case m = n = 0.

Following the notations of (1.3) and quite similar considerations we get

$$\Phi = \frac{2a}{\pi R_1}\int_{\alpha_0}^{c}\frac{H\,d\alpha}{\sqrt{(\alpha-\alpha_\pi)(\alpha-\alpha_\pi^*)}}$$
$$= \frac{2a}{\pi R_1}\int_{\alpha_0}^{c}\frac{G_\tau}{\sqrt{\alpha-\alpha_\pi^*}}\cdot\sqrt{\frac{c-\alpha}{\alpha-\alpha_\pi^*}}\,d\alpha \qquad (2.1.1)$$

in which Φ is the potential of the displacements.

Then the displacements are

$$U_2 = \frac{\partial\Phi}{\partial z} = -\frac{\partial\Phi}{\partial Z} \qquad (2.1.2)$$

and

$$U_r = -\frac{\partial\Phi}{\partial r} \qquad (2.1.3)$$

We perform the differentiations by regarding Φ as a function of Z, $r, \alpha_\pi, \alpha_\pi^*$, so that α_π and α_π^* would be independent of Z and r, and then taking the relations between α_0, α_0^*, Z and r into account. The partial derivatives of Φ with respect to Z and r, which we get if α_π and α_π^* are considered as independent variables, will be denoted by $\frac{\partial\Phi}{\partial Z}$ and $\frac{\partial\Phi}{\partial r}$. Then we have

$$U_z = -\frac{\partial\Phi}{\partial Z} - \frac{\partial\Phi}{\partial\alpha_\pi}\cdot\frac{\partial\alpha_\pi}{\partial Z} - \frac{\partial\Phi}{\partial\alpha_\pi^*}\cdot\frac{\partial\alpha_\pi^*}{\partial Z} \qquad (2.1.4)$$

and

$$U_r = -\frac{\partial\Phi}{\partial r} - \frac{\partial\Phi}{\partial\alpha_\pi}\cdot\frac{\partial\alpha_\pi}{\partial r} - \frac{\partial\Phi}{\partial\alpha_\pi^*}\cdot\frac{\partial\alpha_\pi^*}{\partial r} \qquad (2.1.5)$$

Now we have

$$\frac{\partial\Phi}{\partial r} = -\frac{2ar}{\pi R_1^3}\int_{\alpha_\pi}^{c}\frac{G_\tau}{\sqrt{\alpha-\alpha_\pi^*}}\sqrt{\frac{c-\alpha}{\alpha-\alpha_\pi}}\,d\alpha \qquad (2.1.6)$$

$$\frac{\partial\Phi}{\partial Z} = -\frac{2aZ}{\pi R_1^3}\int_{\alpha_\pi}^{c}\frac{G_\tau}{\sqrt{\alpha-\alpha_\pi^*}}\sqrt{\frac{c-\alpha}{\alpha-\alpha_\pi}}\,d\alpha \qquad (2.1.7)$$

$$\frac{\partial\Phi}{\partial\alpha_\pi^*} = \frac{a}{\pi R_1}\int_{\alpha_\pi}^{c}\frac{G}{(\sqrt{\alpha-\alpha_0})^3}\sqrt{\frac{c-\alpha}{\alpha-\alpha_\pi}}\,d\alpha \qquad (2.1.8)$$

At last we get by integrating by parts

$$\Phi = -\frac{4a}{\pi R_1}\int_{\alpha_\pi}^{c}\sqrt{\alpha-\alpha_\pi}\,\frac{d}{d\alpha}\left\{G\sqrt{\frac{c-\alpha}{\alpha-\alpha_\pi^*}}\right\}d\alpha$$
$$\Phi = -\frac{4a}{\pi R_1}\int_{\alpha_\pi}^{c}\frac{dG}{d\alpha}\sqrt{(c-\alpha)(\alpha-\alpha_\pi)}\,d\alpha \qquad (2.1.9)$$
$$+\frac{2a}{\pi R_1}\int_{\alpha_\pi}^{c}G\sqrt{\frac{\alpha-\alpha_\pi}{(c-\alpha)(\alpha-\alpha_\pi^*)}}\,d\alpha$$
$$+\frac{2a}{\pi R_1}\int_{\alpha_\pi}^{c}\frac{G}{(\sqrt{\alpha-\alpha_0})^3}\sqrt{(c-\alpha)(\alpha-\alpha_\pi^*)}\,d\alpha$$

Differentiating (2.1.9) with respect to α_π yields

$$\frac{\partial \Phi}{\partial \alpha_\pi} = \frac{2a}{\pi R_1}\int_{\alpha_\pi}^{c} \frac{dG}{d\alpha}\sqrt{\frac{c-\alpha}{(\alpha-\alpha_\pi)}}\,\frac{d\alpha}{\sqrt{\alpha-\alpha_\pi}} - \frac{a}{\pi R_1}\int_{\alpha_\pi}^{c}\frac{G_\Gamma}{\sqrt{\alpha-\alpha_\pi}}\,\frac{d\alpha}{\sqrt{(c-\alpha)(\alpha-\alpha_\pi)}} - \frac{a}{\pi R_1}\int_{\alpha_\pi}^{c}\frac{G_\Gamma}{(\sqrt{\alpha-\alpha_\pi})^3}\sqrt{\frac{c-\alpha}{\alpha-\alpha_\pi}}\,d\alpha \tag{2.1.10}$$

We introduce

$$\left.\begin{aligned} J_1 &= \frac{1}{\pi}\int_{\alpha_\pi}^{c} G\sqrt{\frac{c-\alpha}{(\alpha-\alpha_\pi)(\alpha-\alpha_\pi^*)}}\,d\alpha \\ J_2 &= \frac{1}{\pi}\int_{\alpha_\pi}^{c} G\sqrt{\frac{c-\alpha}{(\alpha-\alpha_\pi)(\alpha-\alpha_\pi^*)^3}}\,d\alpha \\ J_3 &= \frac{1}{\pi}\int_{\alpha_\pi}^{c} G\,\frac{d\alpha}{\sqrt{(c-\alpha)(\alpha-\alpha_\pi)(\alpha-\alpha_\pi^*)}} \\ J_4 &= \frac{1}{\pi}\int_{\alpha_\pi}^{c}\frac{\partial G_\Gamma}{\partial\alpha}\sqrt{\frac{c-\alpha}{(\alpha-\alpha_\pi)(\alpha-\alpha_\pi^*)}}\,d\alpha \end{aligned}\right\} \tag{2.1.11}$$

and find from (2.1.6), (2.1.7), (2.1.8) and (2.1.10)

$$\frac{\partial \Phi}{\partial r} = -\frac{2ar}{R_1^3}J_1 \tag{2.1.12}$$

$$\frac{\partial \Phi}{\partial z} = -\frac{2az}{R_1^3}J_1 \tag{2.1.13}$$

$$\frac{\partial \Phi}{\partial \alpha_\pi^*} = \frac{a}{R_1}J_2 \tag{2.1.14}$$

$$\frac{\partial \Phi}{\partial \alpha_\pi} = \frac{a}{R_1}\{2J_4 - J_2 - J_3\} \tag{2.1.15}$$

In the J's we apply the transformation (1.3.18); by this we obtain

$$J_1 = \frac{c-\alpha_\pi}{4}\int_{-1}^{1}\frac{G_\Gamma}{\sqrt{\alpha-\alpha_\pi^*}}\left(1-\sin\frac{\pi}{2}\psi\right)d\psi \tag{2.1.16}$$

$$J_2 = \frac{c-\alpha_\pi}{4}\int_{-1}^{1}\frac{G_\Gamma}{(\sqrt{\alpha-\alpha_\pi})^3}\left(1-\sin\frac{\pi}{2}\psi\right)d\psi \tag{2.1.17}$$

$$J_3 = \frac{1}{2}\int_{-1}^{1}\frac{G_\Gamma}{\sqrt{\alpha-\alpha_\pi}}\,d\psi \tag{2.1.18}$$

$$J_4 = \frac{c-\alpha_\pi}{4}\int_{-1}^{1}\frac{dG}{d\alpha}\,\frac{\left(1-\sin\frac{\pi}{2}\psi\right)}{\sqrt{\alpha-\alpha_\pi}}\,d\psi \tag{2.1.19}$$

These integrals can be calculated numerically.

The function $\frac{\partial G_\Gamma}{\partial \alpha}$ was tabulated for the same values of α as G_Γ.

2.2. We further need expressions for the derivatives of α_π and α_π^*. They obey the equation $Q_{t_\infty} = 0$ or according to (1.3.7)

$$R_1^2\alpha_1^2 - 2aZt\alpha + a^2t^2 - r^2 = 0 \qquad (2.2.1)$$

and therefore

$$(R_1^2\alpha - aZt)d\alpha + (Z\alpha^2 - at\alpha)dZ + (r\alpha^2 - r)dr = 0$$

Thus

$$\frac{\partial\alpha}{\partial Z} = \frac{at\alpha - Z\alpha^2}{R_1^2\alpha - aZt} \qquad (2.2.2)$$

$$\frac{\partial\alpha}{\partial r} = \frac{1-\alpha^2}{R_1^2\alpha - aZt}\cdot r \qquad (2.2.3)$$

On the right handside we have to substitute α_π or α_π^* according to whether we want to find the derivatives of α_π or α_π^*.

From (2.2.1) follows

$$R_1^2(\alpha_\pi + \alpha_\pi^*) = 2aZt$$

or

$$R_1^2\alpha_\pi - aZt = -(R_1^2\alpha_\pi^* - aZt) \qquad (2.2.4)$$

so that, if $\nu = R_1^2\alpha_\pi - aZt = -(R_1^2\alpha_\pi^* - aZt)$ (2.2.5)

we get

$$\frac{\partial\alpha_\pi}{\partial Z} = \frac{at\alpha_\pi - Z\alpha_\pi^2}{\nu} \qquad \frac{\partial\alpha_\pi^*}{\partial Z} = -\frac{at\alpha_\pi^* - Z\alpha_\pi^{*2}}{\nu} \qquad (2.2.6)$$

$$\frac{\partial\alpha_\pi}{\partial r} = \frac{1-\alpha_\pi^2}{\nu}r \qquad \frac{\partial\alpha_\pi^*}{\partial r} = -\frac{1-\alpha_\pi^{*2}}{\nu}r \qquad (2.2.7)$$

2.3. For $t = t_\infty$ we have $\alpha_\pi = C$ as we saw in (1.7). Therefore $\Delta_1 = \Delta_2 = \Delta_4 = 0$, and we have only Δ_3 left. Then in the transformation (1.3.18) α is evidently independent of ψ and equal to C.

Further

$$\alpha_\pi = \frac{1}{R_1^2}\left\{CZ^2 + 2\frac{a}{a'}rZ - Cr^2\right\} \qquad (2.3.1)$$

and so as $\alpha = C$ in the integrand

$$\alpha - \alpha_\pi^* = \frac{2r}{R_1^2}\left(Cr - \frac{a}{a'}Z\right) \qquad (2.3.2)$$

Thus according to (2.1.18)

$$\Delta_3 = R_1\frac{\Gamma(c)}{\sqrt{2\pi\left(Cr - \frac{a}{a'}Z\right)}} \qquad (2.3.3)$$

and from (2.1.1)

$$\frac{\partial \Phi}{\partial \alpha_\pi} = - \frac{a\,\Gamma(c)}{\sqrt{2r(cr - \frac{a}{a'}z)}} \qquad (2.3.4)$$

Then from (2.2.5-7) we get

$$\frac{\partial \alpha_\pi}{\partial z} = \frac{\frac{a}{a'}c}{cr - \frac{a}{a'}z} \qquad (2.3.5)$$

and

$$\frac{\partial \alpha_\pi}{\partial r} = \frac{(\frac{a}{a'})^2}{cr - \frac{a}{a'}z} \qquad (2.3.6)$$

so

$$[u_z]_{t=t_{oo}} = \frac{\frac{a}{a'}c\ \Gamma(c)}{\sqrt{2r(cr - \frac{a}{a'}z)^3}} \qquad (2.3.7)$$

$$[u_r]_{t=t_{oo}} = \frac{a^3/a'^2\ \Gamma(c)}{\sqrt{2r(cr - \frac{a}{a'}z)^3}} \qquad (2.3.8)$$

Therefore the wave begins with a discontinuing. It starts with a displacement in the direction of the vector $(c, \frac{a}{a'})$ and as the conical wave-front is defined by the equation

$$cz + \frac{a}{a'}r = a\,t_{oo} \qquad (2.3.9)$$

this displacement is perpendicular to the wave-front.

Specially when $c = \frac{a}{a'} = \frac{1}{2}\sqrt{2}$ which is true for our parameter-values, $[u_z]_{t=t_{oo}}$ and $\frac{1}{2}[u_r]_{t=t_{oo}}$ are equal.

Chapter III.

Reflection of shock-waves in a liquid-medium divided into two parts by another liquid medium with higher velocity of propagation of sound-waves (R 26).

3.1. The whole problem is completely analogous to that of R 25. There are only two differences.

1°. Transversal waves do not exist and therefore we will always have n = 0.

2°. The expressions $a_{mn}\frac{\rho'}{\rho}$ have to be replaced by others, which we will call β_m (we drop the index n, as it should always be 0).

Hazebroek gives for them

$$\beta_0 = + \frac{\alpha - \sigma\alpha'}{\alpha + \sigma\alpha'} \qquad (3.1.1.)$$

and $$f_m = -\left(\frac{\alpha-\sigma\alpha'}{\alpha+\sigma\alpha'}\right)^{m-1} + \left(\frac{\alpha-\sigma\alpha'}{\alpha+\sigma\alpha'}\right)^{m+1} \qquad (3.1.2)$$

$m \geq 2$ and even.

3.2. Considering the case m = 0 and following the consideration of (1.3), we replace $a_{oo}\, P'/p = -P'/p$ by f_o and find

$$H = Re\, i f_o = -Im\, f_o = \frac{2\alpha_o\alpha'\sigma}{(1-\sigma^2\alpha'^2)}$$

$$= \frac{2\alpha\sqrt{c^2-\alpha^2}}{(1-\sigma^2)\alpha^2+c^2\sigma^2} \qquad (3.2.1)$$

So according to (1.3.11)

$$U_{2,o} = \frac{4}{\pi R_1}\int_{\alpha_\pi}^{c} \frac{\alpha^2}{(1-\sigma^2)\alpha^2+c^2\sigma^2}\sqrt{\frac{c^2-\alpha^2}{(\alpha-\alpha_\pi)(\alpha-\alpha_\pi^*)}}\,d\alpha \qquad (3.2.2)$$

Applying the transformations (1.3.18) again we get

$$U_{2,o} = \frac{c-\alpha_\pi}{R_1}\int_{\alpha_\pi}^{c} \frac{\alpha^2}{(1-\sigma^2)\alpha^2+c^2\sigma^2}\sqrt{\frac{c+\alpha}{\alpha-\alpha_\pi^*}}\,(1-\sin\tfrac{\pi}{2}\psi)\,d\psi \qquad (3.2.3)$$

The integral can be calculated numerically. Specially when $\sigma = 1$ we get the easier formula

$$U_{2,o} = \frac{c-\alpha_\pi}{c^2 R_1}\int_{\alpha_\pi}^{c} \alpha^2\sqrt{\frac{c+\alpha}{\alpha-\alpha_\pi}}\,(1-\sin\tfrac{\pi}{2}\psi)\,d\psi \qquad (3.2.3')$$

As in R 25 $U_{2,6}$ starts with zero.

3.3. In the case that $m \neq 0$ there is no need to calculate $\tau_{m,o}$, $\frac{d\xi}{d\tau}$ and the graph of $\xi(\sigma)$ again, as we can simply take them from the results of R_{15}. Then the computations of $\mathcal{L}_i$ and $\mathcal{L}_r$ can start, in the definitions of which $a_{mn}\, P'/p$ should be replaced by f_m. In the case of $\sigma = 1$ the computation is somewhat facilitated by using for f_m the formula

$$f_m = -4\,\frac{(\alpha-\alpha')^{2m}\alpha\alpha'}{c^{2m+2}} \qquad (3.3.1)$$

So we have

$$\mathcal{L}_r + i\mathcal{L}_i = -\frac{4(\alpha-\alpha')^{2m}\alpha\alpha'}{c^{2m+2}}\int \frac{d\xi}{d\tau}\sigma \qquad (3.3.2)$$

$(\alpha-\alpha')^{2m}$ was calculated by putting

$$\left.\begin{aligned} \alpha-\alpha' &= \zeta\, e^{i\chi} \\ \zeta &= |\alpha-\alpha'| \\ \chi &= \arg(\alpha-\alpha') \end{aligned}\right\} \qquad (3.3.3)$$

so that

$$(\alpha-\alpha')^{2m} = \zeta^{2m}\, e^{2im\chi} \qquad (3.3.4)$$
$$= \zeta^{2m}\cos 2m\chi + i\,\zeta^{m}\sin 2m\chi$$

3.4. As in R 25 $U_{2,m}$ starts with a discontinuity at $t = t_{m,n}$ if . We can give a simple formula for $[U_{2,m}]_{t=t_{m,0}}$ in terms of the I_1 and I_2 of (1.7) in the case of $\sigma = 1$. For $t = t_{m,0}$ we obtain from (3.3.2) and (1.7)

$$\zeta_n = 0$$

and

$$\zeta_i = -\frac{c^{\frac{4}{2m+2}}\cdot\frac{a}{a'}\left(\cos I_1 - \frac{a}{a'}\cos I_2\right)^{2m}\cos I_1\cos I_2\sin I_1}{\sqrt{2\left\{\frac{Z}{\cos^3 I_1} + \frac{a'}{a}\frac{mh}{\cos^3 I_2}\right\}}} \qquad (3.4.1)$$

This from (1.7.11) we get

$$[U_{2,m}]_{t=t_{m0}} = -\frac{4}{c^{2m+2}}\frac{a}{a'}\cos I_1\cos I_2\left(\cos I_1 - \frac{a}{a'}\cos I_2\right)^{2m}\sqrt{\frac{\sin I_1}{2\left\{\frac{Z}{\cos^3 I_1} + \frac{a'}{a}\frac{mh}{\cos^3 I_2}\right\}}} \qquad (3.4.2)$$

Chapter IV.

Discussion of the method of calculation. Results.

4.1. The method is in great lines that of Cagniard. This author, however, suggests to attack complex integrals of the type of (1.4.9) in another way. He deforms the path of integration to one, consisting of a large circle with centre in the origin and enclosing all the branch-points of the integrand, viz. ζ_0, ζ_n, $\pm i\frac{a}{a'}$, $\pm i$, $\pm i\frac{a}{a'}$ and $\pm\frac{a}{a}$, and a contour enclosing all the branch-points but ζ en ζ_n and that is contracted to the imaginary axis (see fig. 1). The integral along the circle can be expressed in a residue, the other one should be ccalculated numerically. The latter point, however, is a difficult one, as along this path of integration there are 8 singularities, which can be grouped into pairs, corresponding to their conjugate complex values. So 4 are

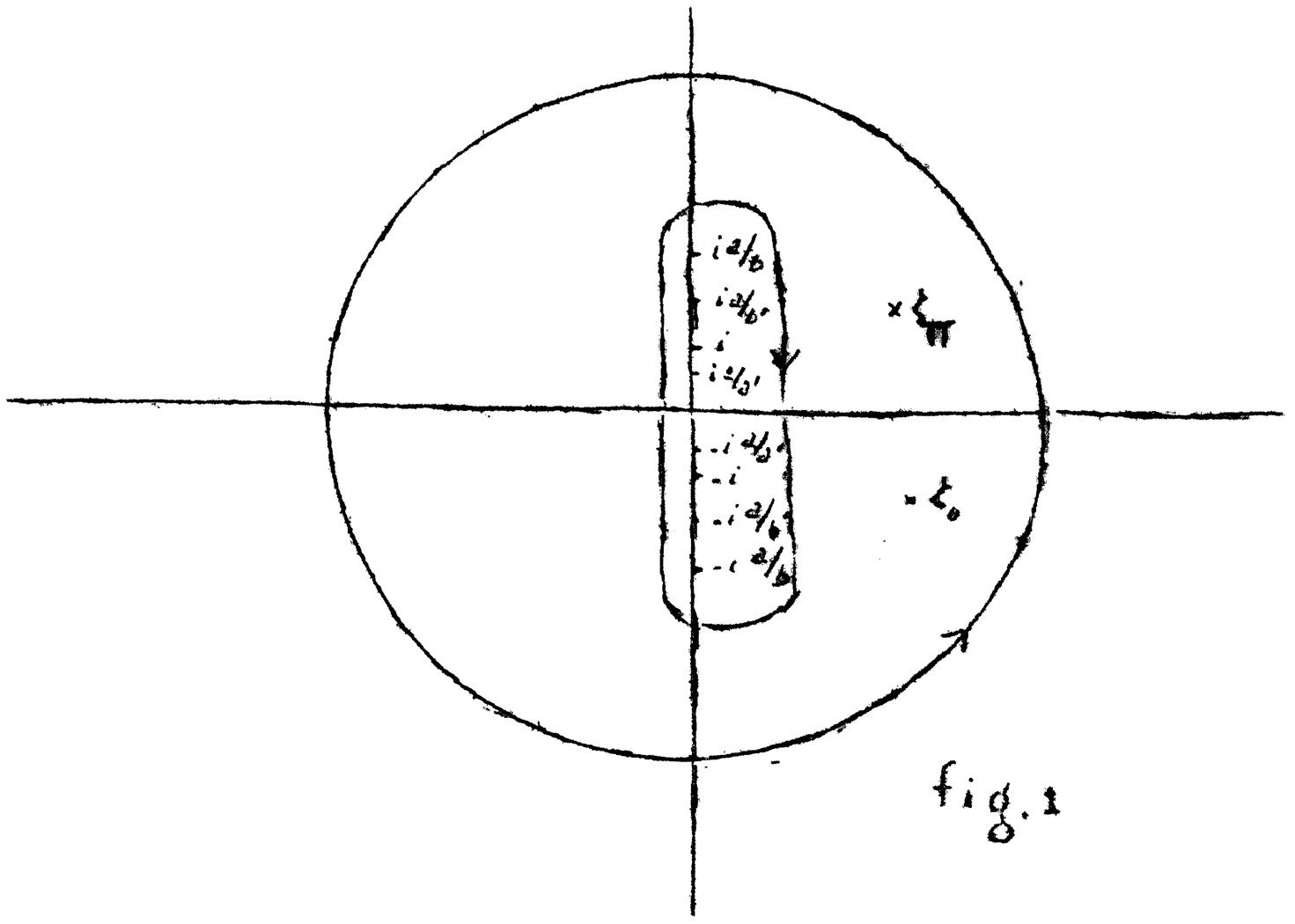


fig. 1

left, and we should have to split the path into three parts (the parts between $-i\frac{a}{a'}$ and $i\frac{a}{a'}$ gives zero again), applying a transformation of the type of (1.3.18) to each part. Moreover, when t does not differ too much from t_{min} and h is small ζ_0 lies quite near to the imaginary axis and this implies that the integrand is not so very smooth there. So numerical integration could not be performed without taking a very small interval. It is these reasons that we chose the other method.

4.2. The next pages give tables and graphs of the numerical results. The values of t, for which the displacements have been calculated, have been chosen in such a way, that they gave enough points for the drawing of the graphs.

The accuracy of the values is about 1 or ½ unit of the last digit for R 25 and R 26, 2 or 3 units of the last digit for R 6.

In the cases of R 25 and R 26 we calculated as many waves for each value of r as were necessary for the accuracy wanted (i.e. that accuracy with which the graphs could be drawn). Actually some more were calculated in order to secure that they were really negligible.

It may be noticed, that the U_z and U_r of R_6 can also be interpreted as the velocities V_{z0} and V_{r0} in the case that the $\Phi_0 = \frac{F(t - R/a)}{R}$ should be the velocity-potential of the original waves. Moreover the $U_{z\rho 0}$'s of R 25 are the integrals with respect to time of the U_z's of R 6. We could check them in that way.

The graphs of the displacements of the conical waves have asymptotes for the value of t corresponding to the arrival of the direct reflection.

Tables and graphs.

In all our calculations we put a = 1, $a'=\sqrt{2}$, $b=\frac{1}{3}\sqrt{3}$, $b'=\frac{1}{3}\sqrt{6}$, $\sigma=1$, $Z=2$ and $h=0,15$.

Table I gives the horizontal and vertical displacements u_x, and u_r corresponding with the calculations described in Chapter II for the values 3(1)10 of r and for different times t. For each r the first t-value gives the time of arrival of the conical wave, the last t-value, for which the displacements are infinite corresponds with the arrival of the direct reflection.

Table II gives the u_z following from the calculations described in Chapter III for r = 5,7,9. Not only the total displacement has been given, but also each of the components $u_{z\,m}$. Under the head "Starting-values" the time of arrival and the starting-value of the reflections of different order are given.

Table III gives the u_z following from the calculations described in Chapter I for r = 5,7,9; here also the different components are tabulated. As to the head "Starting-values" the same applies as to Table II.

After each Table the corresponding graphs have been drawn.

TABLE I

r = 3			r = 4		
t	U_z	U_r	t	U_z	U_r
3,5355	2,074	2,074	4,2426	0,635	0,635
3,54	2,061	2,104	4,25	0,619	0,629
3,55	2,041	2,187	4,27	0,576	0,616
3,56	2,061	2,329	4,29	0,534	0,601
3,57	2,148	2,573	4,31	0,496	0,592
3,58	2,387	3,050	4,33	0,463	0,587
3,59	3,071	4,207	4,35	0,435	0,591
3,60	6,629	9,724	4,37	0,415	0,607
3,6056	∞	∞	4,39	0,407	0,644
			4,41	0,420	0,724
			4,43	0,480	0,901
			4,45	0,720	1,448
			4,47	6,288	12,741
			4,4721	∞	∞

r = 5			r = 6		
t	U_z	U_r	t	U_z	U_r
4,9497	0,3092	0,3092	5,6569	0,1833	0,1833
4,96	0,2998	0,3047	5,66	0,1820	0,1827
5,00	0,2646	0,2874	5,72	0,1561	0,1688
5,04	0,2323	0,2714	5,78	0,1325	0,1556
5,08	0,2029	0,2573	5,84	0,1115	0,1438
5,12	0,1770	0,2459	5,90	0,0930	0,1335
5,16	0,1548	0,2388	5,96	0,0770	0,1256
5,20	0,1362	0,2367	6,02	0,0636	0,1198
5,24	0,1235	0,2452	6,08	0,0539	0,1185
5,28	0,1190	0,2726	6,14	0,0456	0,1239
5,32	0,1361	0,3548	6,20	0,0434	0,1442
5,36	0,9498	2,5138	6,26	0,0583	0,2191
5,3852	∞	∞	6,32	0,8141	2,5617
			6,3246	∞	∞

TABLE I continued.

r = 7			r = 8		
t	U_z	U_r	t	U_z	U_r
6,3640	0,1214	0,1214	7,0711	0,0864	0,0864
6,37	0,1201	0,1208	7,10	0,0821	0,0840
6,47	0,0964	0,1074	7,20	0,0678	0,0756
6,57	0,0760	0,0953	7,30	0,0552	0,0680
6,67	0,0586	0,0845	7,40	0,0443	0,0611
6,77	0,0443	0,0761	7,50	0,0351	0,0551
6,87	0,0328	0,0700	7,60	0,0272	0,0500
6,97	0,0242	0,0684	7,70	0,0206	0,0462
7,07	0,0182	0,0728	7,80	0,0151	0,0438
7,17	0,0190	0,1013	7,90	0,0108	0,0432
7,27	0,2121	0,8325	8,00	0,0076	0,0462
7,2801	∞	∞	8,10	0,0069	0,0592
			8,20	0,0224	0,1451
			8,2462	∞	∞

r = 9			r = 10		
t	U_z	U_r	t	U_z	U_r
7,7782	0,0647	0,0647	8,4853	0,0502	0,0502
7,80	0,0625	0,0634	8,50	0,0492	0,0497
7,95	0,0485	0,0551	8,70	0,0364	0,0419
8,10	0,0368	0,0477	8,90	0,0261	0,0352
8,25	0,0271	0,0413	9,10	0,0179	0,0296
8,40	0,0194	0,0360	9,30	0,0115	0,0251
8,55	0,0132	0,0322	9,50	0,0066	0,0220
8,70	0,0083	0,0298	9,70	0,0031	0,0210
8,85	0,0046	0,0295	9,90	0,0007	0,0232
9,00	0,0022	0,0347	10,10	0,0025	0,0527
9,15	0,0073	0,0873	10,1980	∞	∞
9,2195	∞	∞			

TABLE II

r = 5

t	m=0	2	4	6	Tot.
5,00	0,0089	-0,0039			-0,0007
5,04	0,0163	-0,0066	-0,0084		0,0013
5,08	0,0241	-0,0118	-0,0102	-0,0018	0,0003
5,12	0,0323	-0,0193	-0,0110	-0,0013	0,0007
5,16	0,0411	-0,0283	-0,0107	-0,0007	0,0014
5,20	0,0506	-0,0384	-0,0093	-0,0001	0,0028
5,24	0,0612	-0,0484	-0,0074	0,0004	0,0058
5,28	0,0735	-0,0572	-0,0050	0,0007	0,0120
5,32	0,0890	-0,0635	-0,0028	0,0008	0,0235
5,36	0,1426	-0,0669	-0,0007	0,0008	0,0758

Starting-values.

m	t_m	U_{zm} o
0	4,9497	0
2	4,9603	-0,0048
4	4,9915	-0,0057
6	5,0409	-0,0020
8	5,1072	-0,0003

TABLE II continued.

r = 7

t	m=0	2	4	6	8	Tot.
6,47	0,0073	0,0010	-0,0055			0,0028
6,57	0,0144	-0,0018	-0,0104	-0,0021	-0,0002	-0,0001
6,67	0,0217	-0,0096	-0,0125	-0,0002	0,0004	-0,0002
6,77	0,0293	-0,0210	-0,0107	0,0017	0,0004	-0,0003
6,87	0,0373	-0,0338	-0,0058	0,0022	0,0001	0,0000
6,97	0,0457	-0,0467	0,0002	0,0018	-0,0001	0,0009
7,07	0,0551	-0,0562	0,0050	0,0009	-0,0002	0,0046
7,17	0,0664	-0,0603	0,0069	-0,0000	-0,0001	0,0129
7,27	0,0921	-0,0560	0,0071	-0,0002	-0,0001	0,0429

Starting-values

m	t_m	$\left[U_{z,m}\right]_o$
0	6,3640	0
2	6,3703	-0,0013
4	6,3892	-0,0026
6	6,4204	-0,0018
8	6,4633	-0,0006

TABLE II continued

r = 9

t	m=0	2	4	6	8	Tot.
7,95	0,0062	0,0030	-0,0053	-0,0030	-0,0006	0,0003
8,10	0,0118	-0,0002	-0,0107	-0,0021	0,0004	-0,0008
8,25	0,0179	-0,0068	-0,0126	0,0011	0,0008	0,0004
8,40	0,0232	-0,0163	-0,0097	0,0032	0,0001	0,0005
8,55	0,0290	-0,0278	-0,0031	0,0029	-0,0004	0,0006
8,70	0,0349	-0,0389	0,0037	0,0010	-0,0005	0,0002
8,85	0,0410	-0,0474	0,0084	-0,0009	-0,0002	0,0009
9,00	0,0475	-0,0507	0,0090	-0,0014	0,0000	0,0044
9,15	0,0561	-0,0467	0,0062	-0,0011	0,0001	0,0146

Starting-values

m	t_m	$\left[U_{zm}\right]_o$
0	7,7782	0
2	7,7827	-0,0005
4	7,7965	-0,0013
6	7,8188	-0,0012
8	7,8501	-0,0007

TABLE III

r = 5

t	m,n=0,0	2,0	4,0	6,0	1,1	3,1	2,2	Tot.
5,00	0,0144	-0,0113						-0,0093
5,04	0,0243	-0,0215	-0,0018					0,0010
5,08	0,0330	-0,0304	0,0002	0,0001				0,0029
5,12	0,0406	-0,0373	0,0020	0,0001				0,0054
5,16	0,0473	-0,0414	0,0030	-0,0001	0,0000	-0,0001		0,0087
5,20	0,0529	-0,0426	0,0028	-0,0002	0,0001	-0,0002		0,0128
5,24	0,0580	-0,0412	0,0019	-0,0002	0,0007	-0,0003		0,0189
5,28	0,0629	-0,0379	0,0005	0,0000	0,0020	-0,0003	0,0000	0,0272
5,32	0,0682	-0,0331	-0,0011	0,0002	0,0040	-0,0003	0,0000	0,0379
5,36	0,0831	-0,0288	-0,0027	0,0004	0,0068	-0,0002	0,0002	0,0588

Starting-values

m,n	t_{mn}	$\left[U_{zmn}\right]_o$
0,0	4,9497	0
2,0	4,9603	-0,0058
4,0	4,8915	-0,0034
6,0	5,0409	-0,0001
1,1	5,1025	0,0000
3,1	5,1241	-0,0001
2,2	5,2611	-0,0001

TABLE III continued

r = 7

t	m,n=0,0	2,0	4,0	6,0	1,1	3,1	5,1
6,47	0,0115	-0,0090	-0,0033				
6,57	0,0202	-0,0216	0,0003	0,0003	0,0000	0,0000	0,0000
6,67	0,0268	-0,0321	0,0052	-0,0005	0,0000	0,0000	0,0000
6,77	0,0320	-0,0363	0,0055	-0,0005	0,0004	-0,0003	0,0000
6,87	0,0357	-0,0359	0,0023	0,0002	0,0012	-0,0008	0,0000
6,97	0,0388	-0,0321	-0,0018	0,0010	0,0026	-0,0010	0,0000
7,07	0,0408	-0,0268	-0,0059	0,0011	0,0046	-0,0006	0,0000
7,17	0,0423	-0,0205	-0,0088	0,0007	0,0071	0,0006	-0,0001
7,27	0,0471	-0,0145	-0,0101	-0,0002	0,0102	0,0029	-0,0001

t	m,n=2,2	Tot.
6,47		-0,0008
6,57		-0,0008
6,67		-0,0006
6,77	0,0001	0,0009
6,87	0,0004	0,0031
6,97	0,0008	0,0083
7,07	0,0011	0,0143
7,17	0,0008	0,0221
7,27	-0,0009	0,0344

Starting-values

m,n	t_{mn}	$[U_{zmn}]_o$
0,0	6,3640	0
2,0	6,3703	-0,0019
4,0	6,3892	-0,0018
6,0	6,4204	-0,0004
1,1	6,5156	0,0000
3,1	6,5285	0,0000
5,1	6,5542	0,0000
2,2	6,6706	0,0000

TABLE III continued

r = 9

t	m,n=0,0	2,0	4,0	6,0	1,1	3,1	5,1
7,95	0,0097	-0,0078	-0,0029	0,0005			
8,10	0,0160	-0,0187	0,0023	0,0000	-0,0016		
8,25	0,0208	-0,0268	0,0071	-0,0013	-0,0011	0,0000	
8,40	0,0243	-0,0299	0,0065	-0,0004	0,0002	-0,0005	
8,55	0,0268	-0,0286	0,0019	0,0016	0,0009	-0,0010	
8,70	0,0284	-0,0253	-0,0041	0,0023	0,0024	-0,0009	0,0003
8,85	0,0293	-0,0206	-0,0076	0,0019	0,0041	0,0001	0,0011
9,00	0,0297	-0,0158	-0,0105	0,0006	0,0062	0,0016	0,0020
9,15	0,0302	-0,0113	-0,0122	-0,0012	0,0084	0,0037	0,0026

t	m,n=2,2	4,2	Tot.
7,95			-0,0005
8,10			-0,0020
8,25			-0,0013
8,40	0,0004		0,0006
8,55	0,0009		0,0025
8,70	0,0011		0,0042
8,85	0,0008	-0,0001	0,0090
9,00	-0,0005	-0,0006	0,0127
9,15	-0,0030	-0,0010	0,0162

Starting-values

m,n	t_{mn}	$[U_{zmn}]_o$
0,0	7,7782	0
2,0	7,7827	-0,0008
4,0	7,7965	-0,0012
6,0	7,8188	-0,0005
1,1	7,9293	0,0000
3,1	7,9384	0,0001
5,1	7,9569	0,0000
2,2	8,083	0,0000
4,2	8,097	0,0000

Table 1

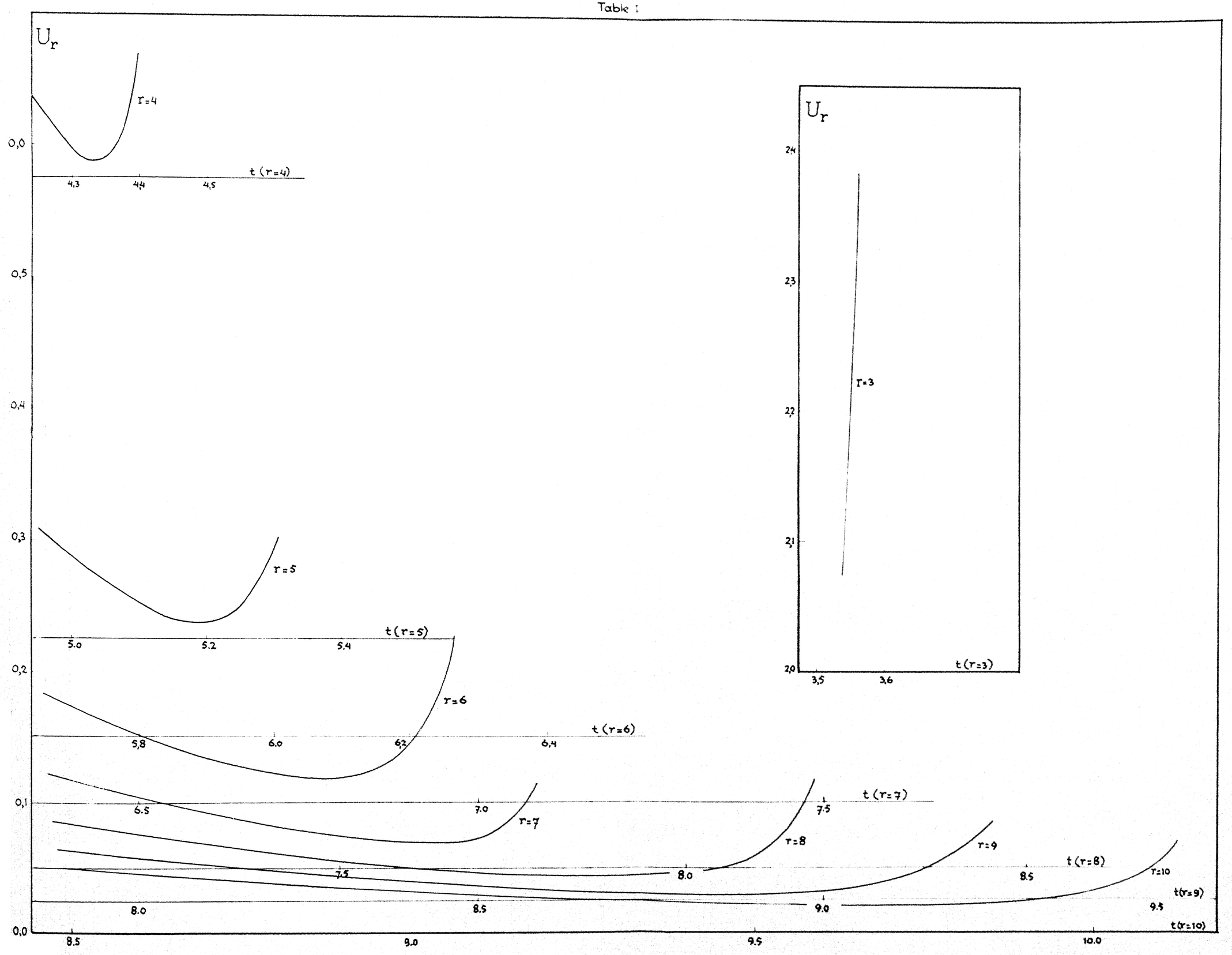

U_r
r=4
0,0
t (r=4)
4,3
4,4
4,5
0,5
0,4
0,3
r=5
t (r=5)
5.0
5.2
5.4
0,2
r=6
t (r=6)
5,8
6.0
6,2
6,4
0,1
t (r=7)
6.5
7.0
r=7
7.5
r=8
r=9
t (r=8)
7.5
8.0
8.5
r=10
t(r=9)
8.0
8.5
9.0
9.5
t(r=10)
0,0
8.5
9.0
9.5
10.0
U_r
24
23
22
21
20
r=3
t (r=3)
3,5
3,6

Table I

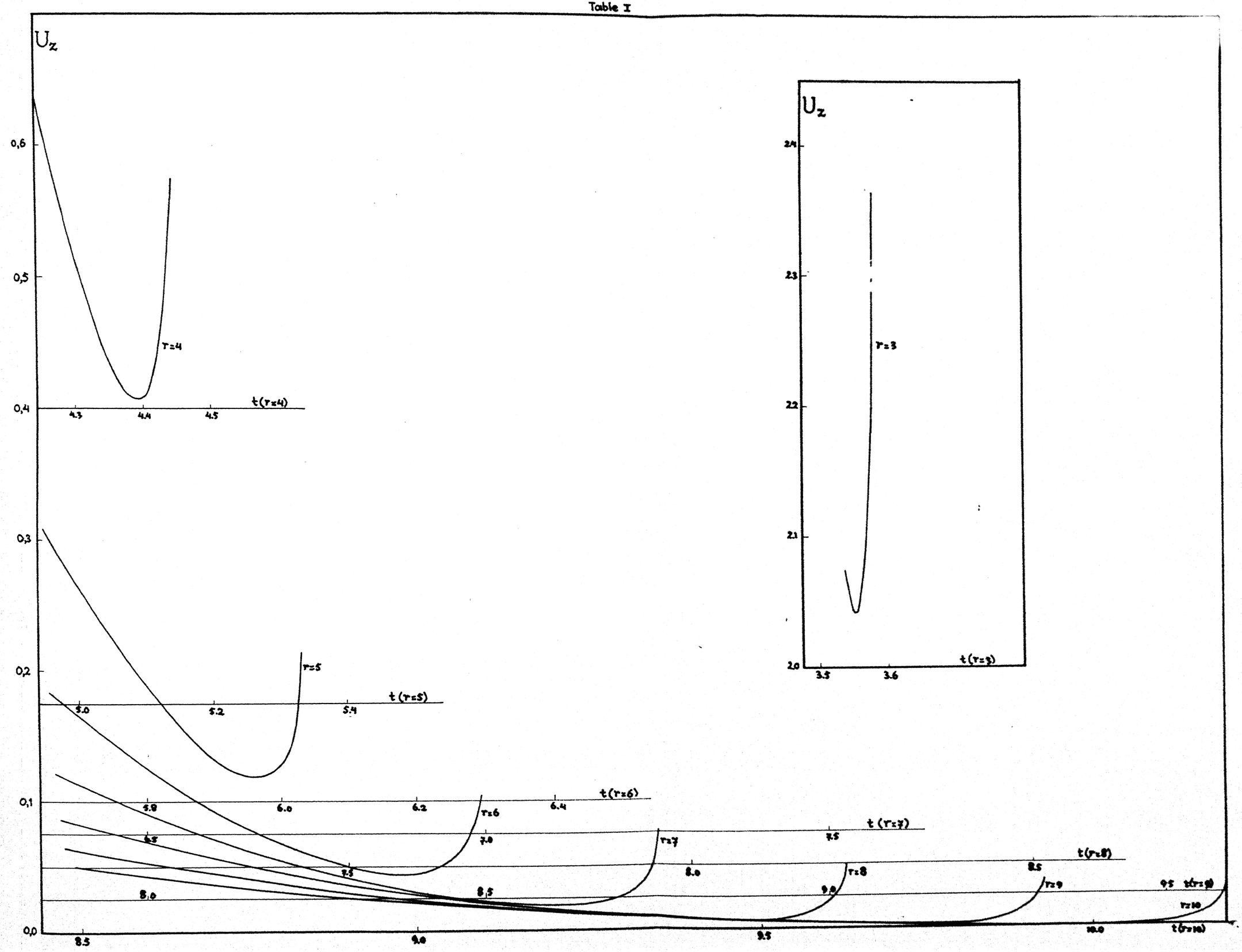
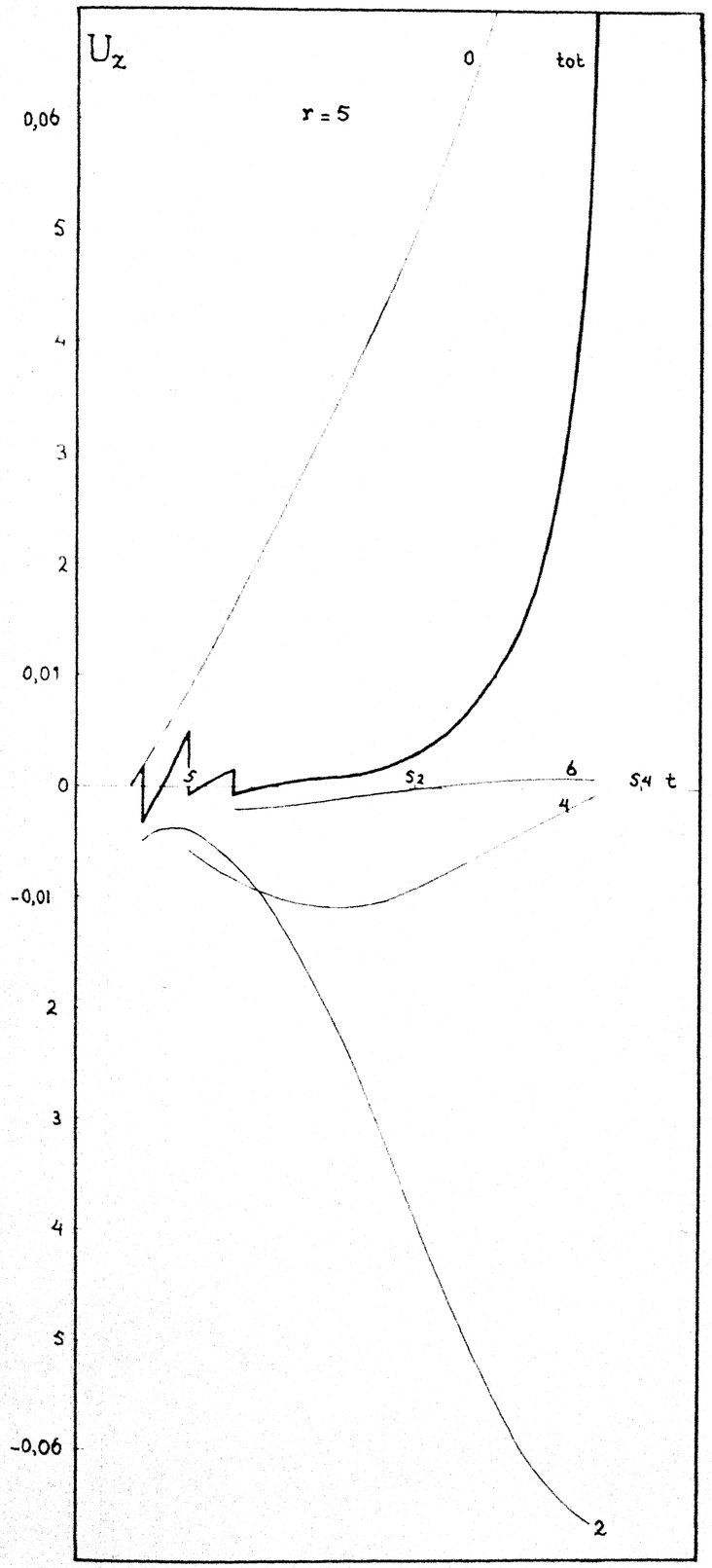

U_z
0,6
0,5
0,4
0,3
0,2
0,1
0,0
r=4
t(r=4)
4.3
4.4
4.5
r=5
t(r=5)
5.0
5.2
5.4
r=6
t(r=6)
5.8
6.0
6.2
6.4
r=7
t(r=7)
6.5
7.0
7.5
r=8
t(r=8)
7.5
8.0
8.5
r=9
t(r=9)
8.0
8.5
9.0
9.5
r=10
t(r=10)
8.5
9.0
9.5
10.0
U_z
2.4
2.3
2.2
2.1
2.0
r=3
t(r=3)
3.5
3.6

Table II


U_z
r = 5
0
tot
0,06
5
4
3
2
0,01
0
-0,01
2
3
4
5
-0,06
5
S_2
6
4
S_4
t
2

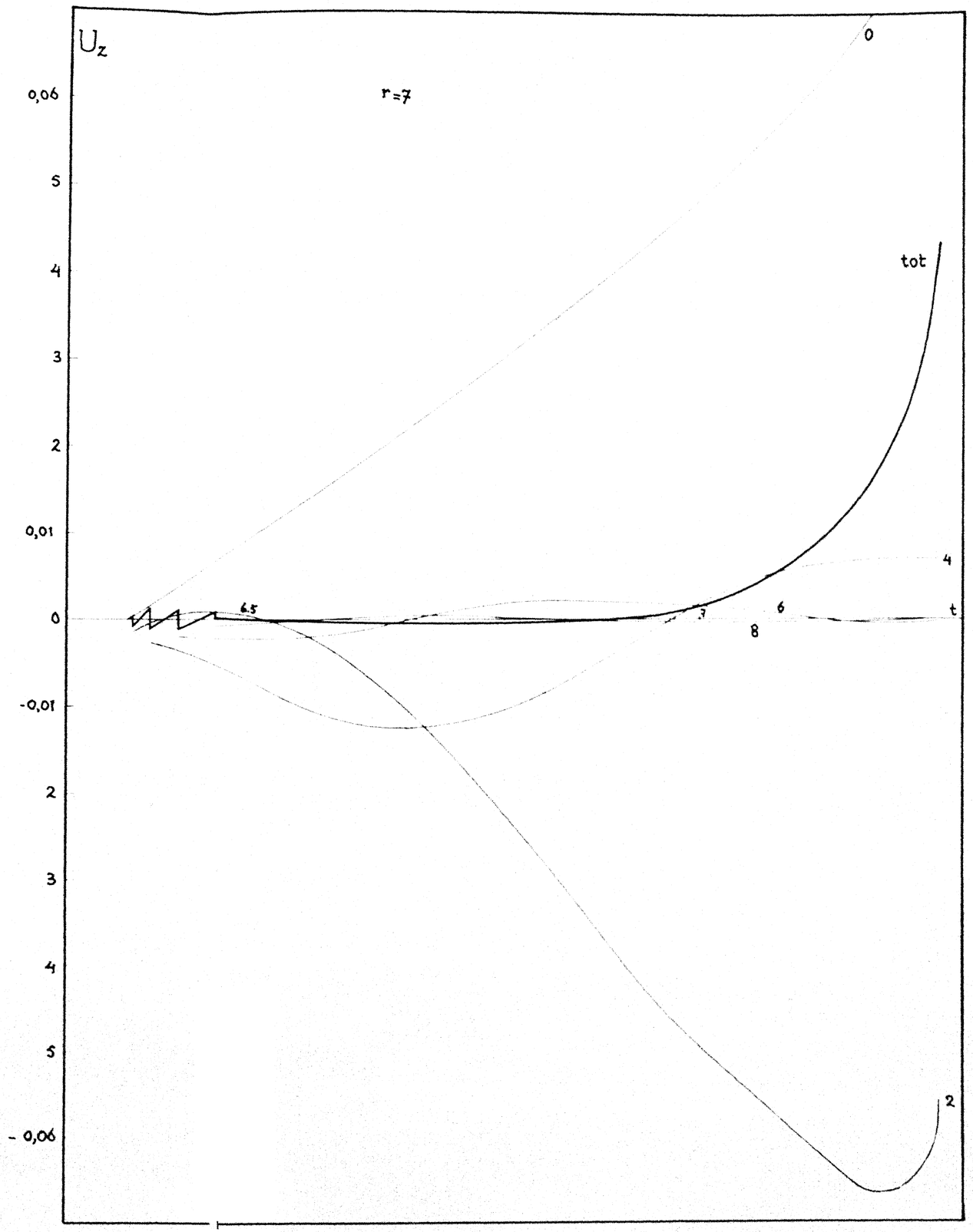

U_z
r=7
0
tot
0,06
5
4
3
2
0,01
0
-0,01
2
3
4
5
- 0,06
6,5
4
3
6
8
t
2

Table II

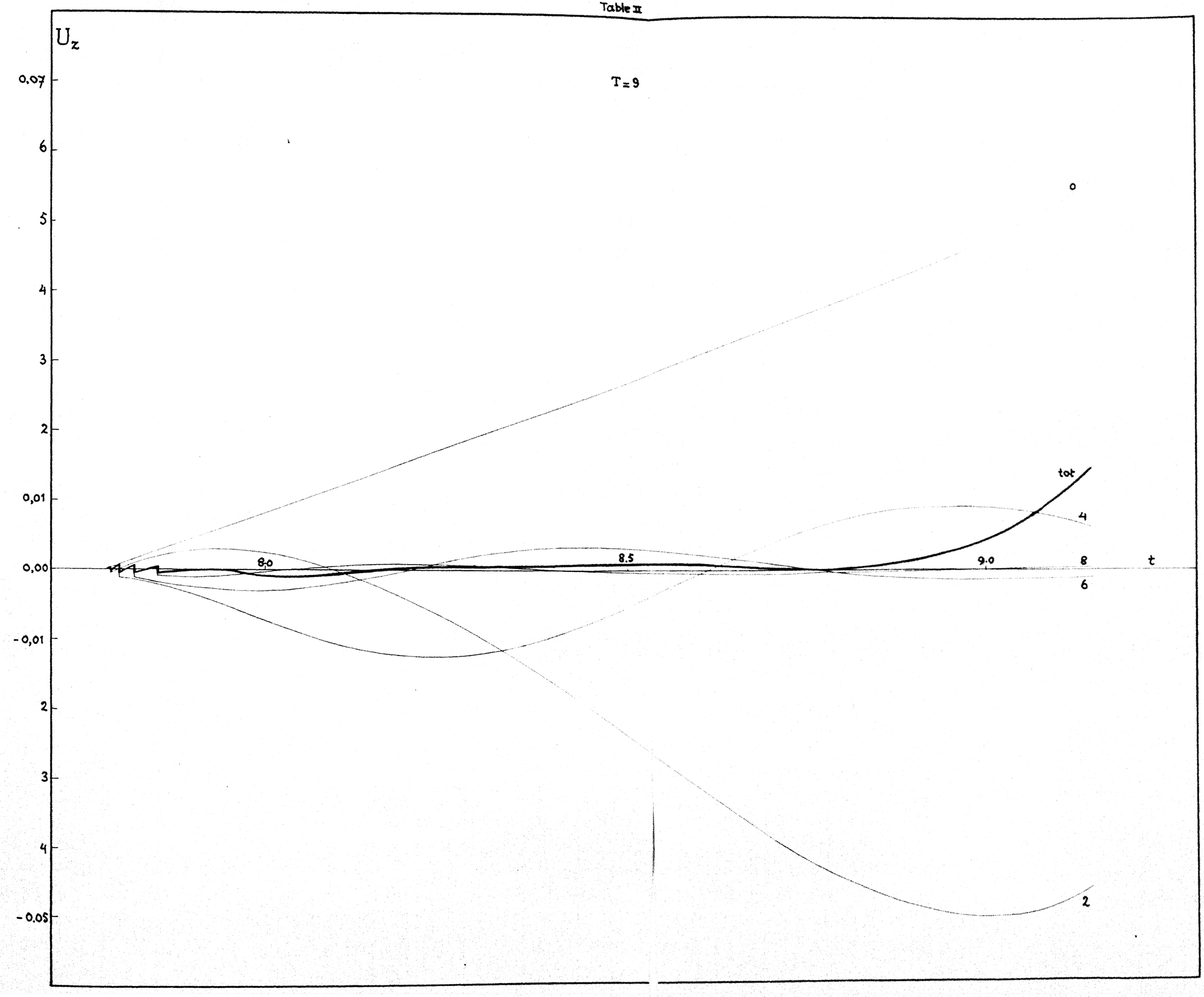
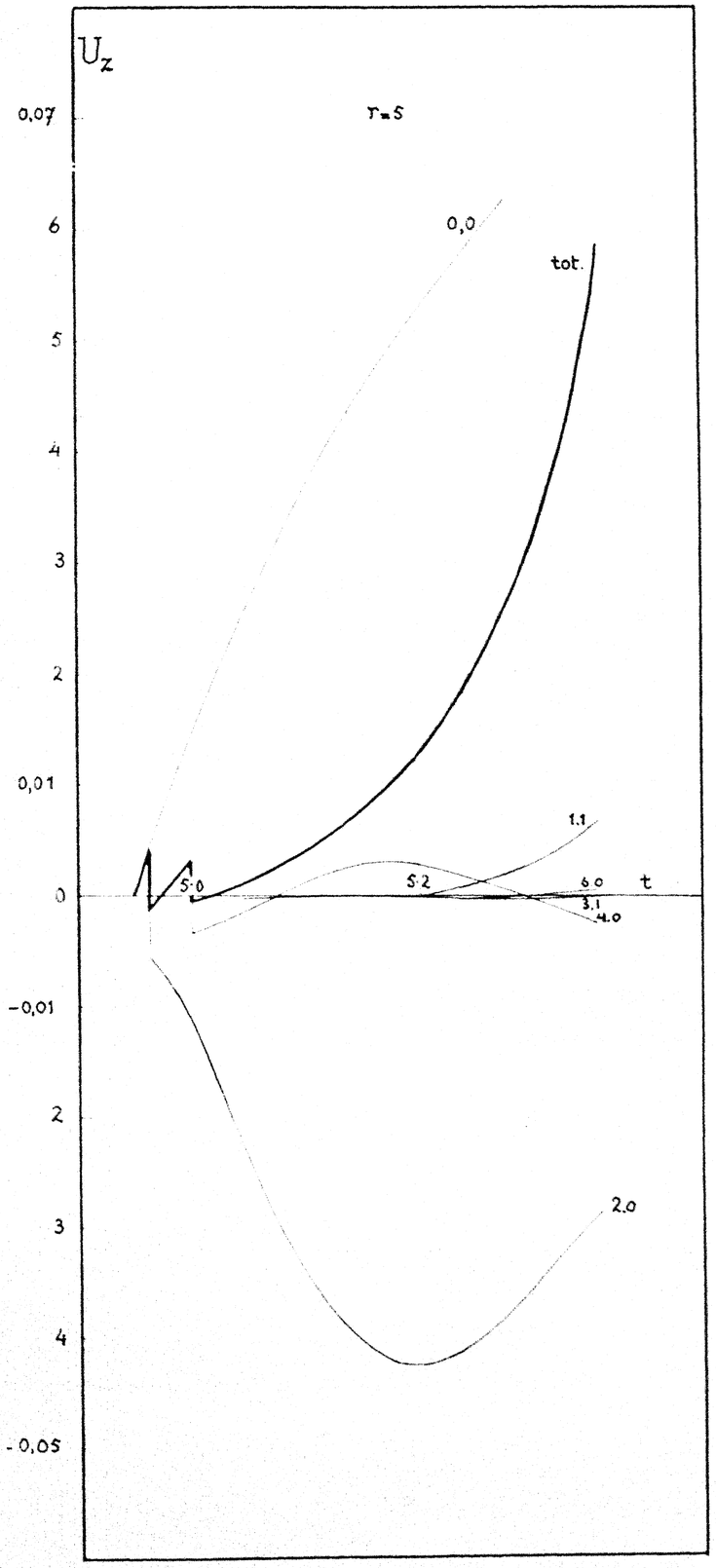


Table III


U_z
r=5
0,07
6
5
4
3
2
0,01
0
-0,01
2
3
4
-0,05
0,0
tot.
1.1
5·0
5·2
6·0
t
3.1
4.0
2.0

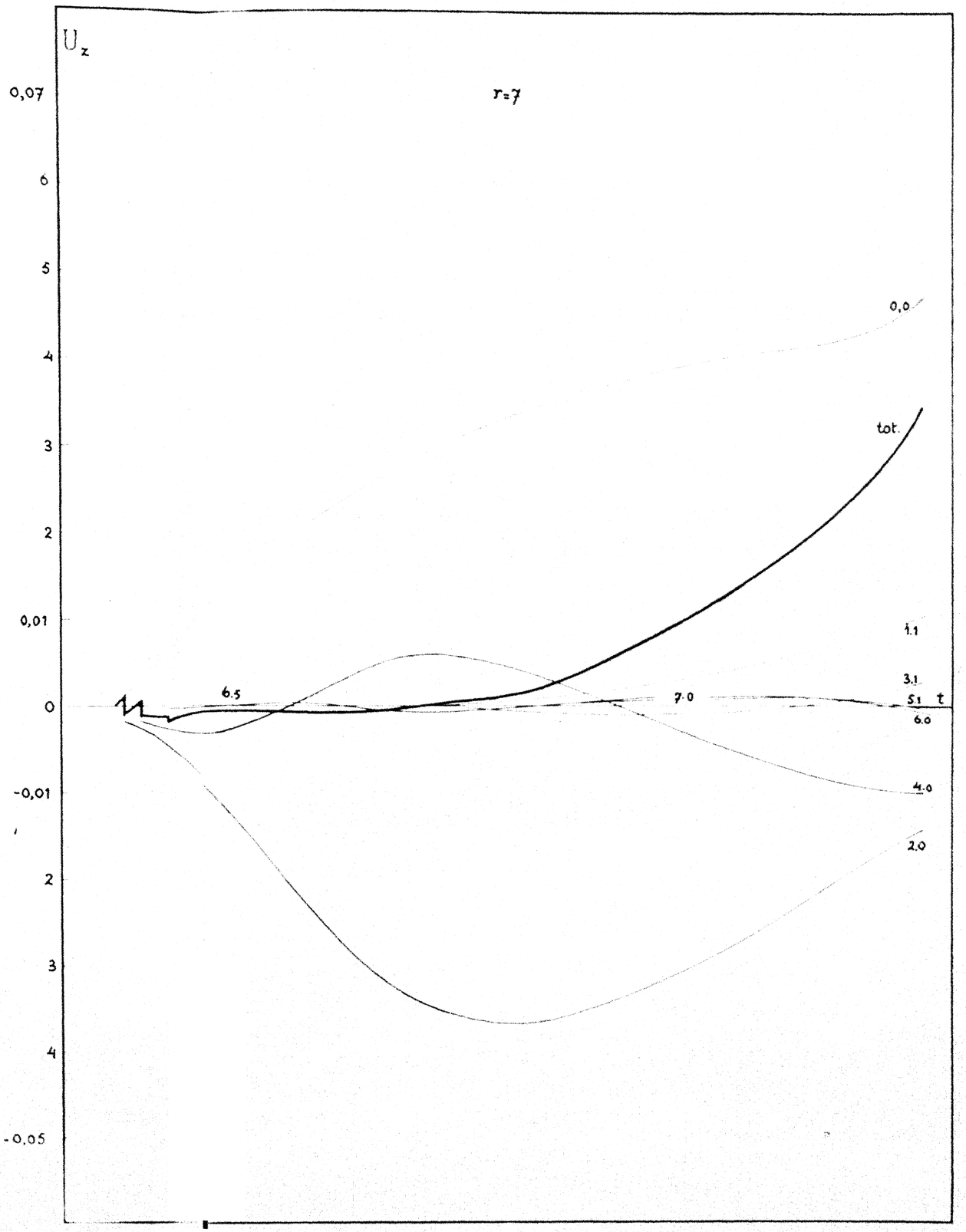
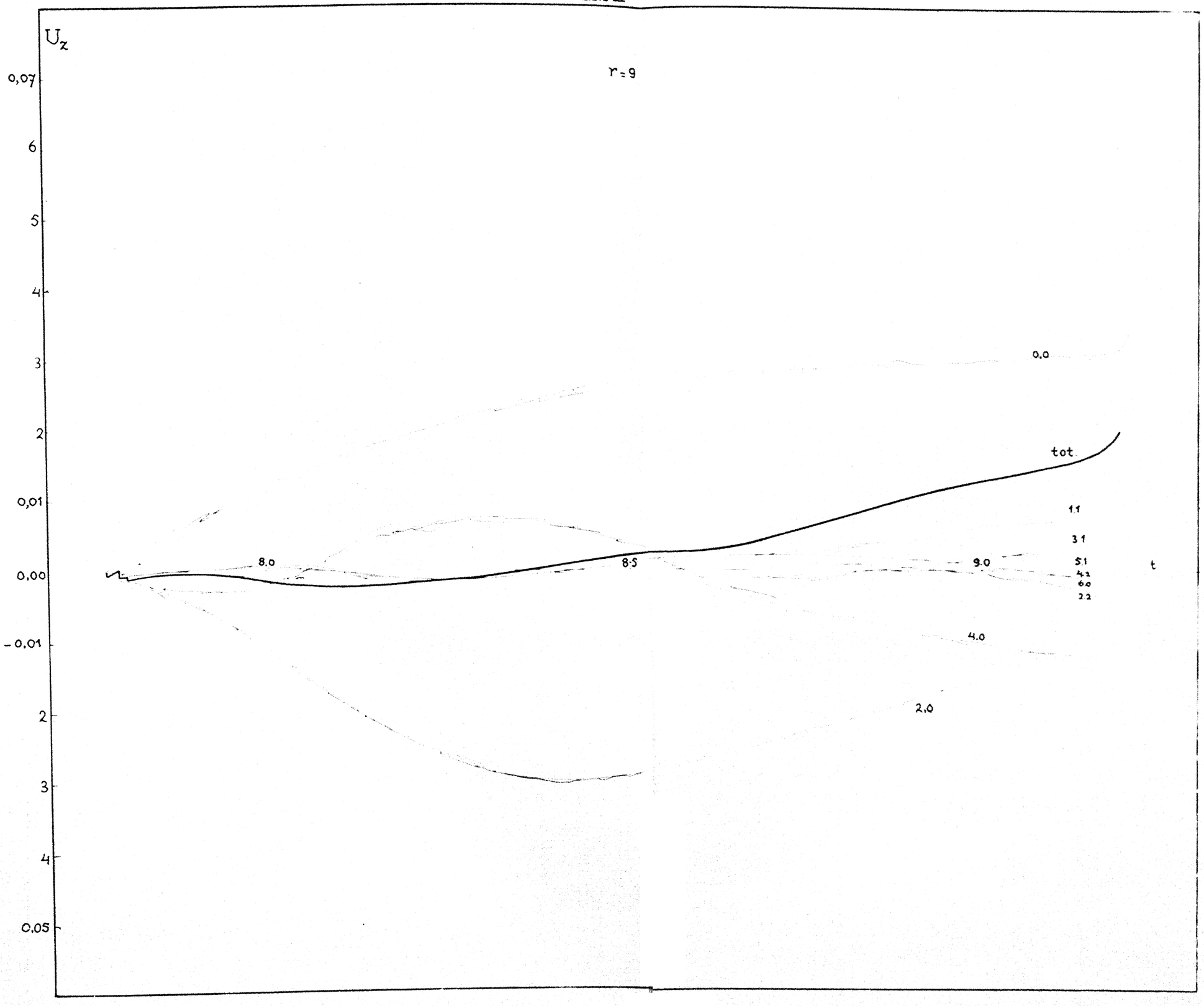

U_z
r=7
0,07
6
5
4
3
2
0,01
0
-0,01
2
3
4
-0,05
0,0
tot.
1.1
3.1
6.5
7·0
5.1
t
6.0
4.0
2.0

Table III


U_z
r = 9
0,07
6
5
4
3
2
0,01
0,00
- 0,01
2
3
4
0,05
0,0
tot.
1.1
3.1
8.0
8.5
9.0
5.1
4.2
6.0
2.2
4.0
2,0
t