THE OSCILLATING WING IN A SUBSONIC FLOW.

R 53, Int 1.

Computation Department Mathematical Centre.

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INTRODUCTION

This report R 53, Int 1 is the first of a number of interim reports giving information about computations carried out by the Computation Department of the Mathematical Centre on behalf of the National Aeronautical Research Institute in Amsterdam under contract R 53. The final report R 53 that will be made up eventually will not contain much else than the final results of the computations and, moreover, will be not available for general distribution. As however in the course of the computations a lot of information has to be compiled for internal use, and part of this information may be of some value to others, this compilation will be done in the form of interim reports, that will be made available for limited circulation.

Series expansions of the functions P_1, P_2 and R_n.

1. In the second part of the work a role is played by the functions

$$P_1(\beta^2\Omega, \eta_1, \eta_0) = \int_0^{\eta_0} e^{i\beta^2\Omega\cos\eta} \ln\frac{1-\cos(\eta+\eta_1)}{1-\cos(\eta-\eta_1)} \sin\eta . d\eta , \tag{1.1}$$

$$P_2(\beta^2\Omega, \eta_1, \eta_0) = \int_0^{\eta_0} e^{i\beta^2\Omega\cos\eta} \ln\frac{1-\cos(\eta+\eta_1)}{1-\cos(\eta-\eta_1)} \cos\eta . \sin\eta . d\eta , \tag{1.2}$$

$$R_n(\beta^2\Omega, \eta_1) = \int_0^{\eta_1} e^{i\beta^2\Omega\cos\eta} \cos n\eta \, d\eta . \tag{1.3}$$

We are more particularly interested in the cases $\eta_0 = \pi$ and $\eta_0 = \eta_1$ as these only occur in the final formulas.

Use is made of the two following Fourierexpansions

$$e^{i\beta^2\Omega\cos\eta} = J_0(\beta^2\Omega) + 2\sum_{\nu=1}^{\infty} i^\nu J_\nu(\beta^2\Omega)\cos\nu\eta =$$

$$\sum_{\nu=-\infty}^{\infty} i^\nu J_\nu(\beta^2\Omega)\cos\nu\eta , \tag{1.4}$$

$$\ln\frac{\cosh\xi-\cos(\eta+\eta_1)}{\cosh\xi-\cos(\eta-\eta_1)} = 4\sum_{n=1}^{\infty}\frac{1}{n}e^{-n\xi}\sin n\eta_1 \sin n\eta \quad \text{if } \xi > 0 . \tag{1.5}$$

We can take ξ equal to zero at a later stage of the computation.

2. In order to compute P_1 we first introduce

$$P_1^*(\beta^2\Omega, \eta_1, \eta_0, \xi) = \int_0^{\eta_0} e^{i\beta^2\Omega\cos\eta} \ln\frac{\cosh\xi-\cos(\eta+\eta_1)}{\cosh\xi-\cos(\eta-\eta_1)} \sin\eta \, d\eta . \tag{2.1}$$

Apparently

$$P_1(\beta^2\Omega, \eta_1, \eta_0) = P^*(\beta^2\Omega, \eta_1, \eta_0, 0). \tag{2.2}$$

Furthermore

$$P_1^*(\beta^2\Omega, \eta_1, \eta_0, \xi) = 4\int_0^{\eta_0}\left\{J_0(\beta^2\Omega) + 2\sum_{\nu=1}^{\infty} i^\nu J_\nu(\beta^2\Omega)\cos\nu\eta\right\} \sum_{n=1}^{\infty}\frac{e^{-n\xi}\sin n\eta_1}{n}\sin n\eta . \sin\eta \, d\eta .$$

Changing the order of integration and summations and having ξ vanish, we get

$$P_1(\beta^2\Omega, \eta_1, \eta_0) = 2 J_0(\beta^2\Omega) \sum_{n=1}^{\infty} \left\{ \frac{\sin n\eta_1 \sin(n-1)\eta_0}{n(n-1)} - \frac{\sin n\eta_1 \sin(n+1)\eta_0}{n(n+1)} \right\} + 2 \sum_{\nu=1}^{\infty} i^{\nu} J_\nu(\beta^2\Omega) \sum_{n=1}^{\infty}$$

$$\left\{ \frac{\sin n\eta_1 \sin(n+\nu-1)\eta_0}{n(n+\nu-1)} + \frac{\sin n\eta_1 \sin(n-\nu-1)}{n(n-\nu-1)} - \frac{\sin n\eta_1 \sin(n+\nu+1)\eta_0}{n(n+\nu+1)} - \frac{\sin n\eta_1 \sin(n-\nu+1)}{n(n-\nu+1)} \right\}. \quad (2.3)$$

where here, and in what follows, the indetermined expression $\frac{\sin \mu\eta}{\mu}$ for vanishing μ means simply η. We shall denote this case as the singular case.

If we take $\eta_0 = \pi$ all the terms on the righthandside of (2.3) vanish but those that correspond to a singular case. Therefore:

$$P_1(\beta^2\Omega, \eta_1, \pi) = 2\pi J_0(\beta^2\Omega) \sin \eta_1 + 2\pi \sum_{\nu=1}^{\infty} i^{\nu} J_\nu(\beta^2\Omega) \frac{\sin(\nu+1)\eta_1}{\nu+1} - 2\pi \sum_{\nu=2}^{\infty} i^{\nu} J_\nu(\beta^2\Omega) \frac{\sin(\nu-1)\eta_1}{\nu-1}$$

$$= 2\pi \left\{ J_0(\beta^2\Omega) \sin \eta_1 + \sum_{\nu=2}^{\infty} i^{\nu-1} J_{\nu-1}(\beta^2\Omega) \frac{\sin \nu\eta_1}{\nu} + \sum_{\nu=1}^{\infty} i^{\nu-1} J_{\nu+1}(\beta^2\Omega) \frac{\sin \nu\eta_1}{\nu} \right\}$$

$$= 2\pi \sum_{\nu=1}^{\infty} i^{\nu-1} \left\{ J_{\nu-1}(\beta^2\Omega) + J_{\nu+1}(\beta^2\Omega) \right\} \frac{\sin \nu\eta_1}{\nu}$$

So:

$$P_1(\beta^2\Omega, \eta_1, \eta_0) = \frac{4\pi}{\beta^2\Omega} \sum_{\nu=1}^{\infty} i^{\nu-1} . J_\nu(\beta^2\Omega) . \sin \nu\eta_1 . \quad (2.4)$$

Next we take the case $\eta_0 = \eta_1$. Here we can again simplify considerably as now the sums in (2.3) cancel one another to a large extent. After some calculations we find the result:

$$P_1(\beta^2\Omega, \eta_1, \eta_1) = \frac{\eta_1}{\pi} P_1(\beta^2\Omega, \eta_1, \pi) + \frac{4}{\beta^2\Omega} \sum_{\nu=2}^{\infty} i^{\nu-1} . J_\nu(\beta^2\Omega) . \nu \sum_{n=1}^{\nu-1} \frac{\sin n\eta_1}{n} . \frac{\sin(\nu-n)\eta_1}{\nu-n} . \quad (2.5)$$

3. In order to compute P_2 we proceed in a similar way. So we find the result

$$P_2(\beta^2\Omega, \eta_1, \eta_o) = J_o(\beta^2\Omega) \sum_{n=1}^{\infty} \Big\{ \frac{\sin n\eta_1 \sin(n-2)\eta_o}{n(n-2)} - \frac{\sin n\eta_1 \sin(n+2)\eta_o}{n(n+2)} \Big\} + \sum_{\nu=1}^{\infty} i^{\nu} J_{\nu}(\beta^2\Omega) \sum_{n=1}^{\infty}$$

$$\Big\{ \frac{\sin n\eta_1 \sin(\nu+n-2)\eta_o}{n(\nu+n-2)} + \frac{\sin n\eta_1 \sin(\nu-n+2)\eta_o}{n(\ -n+2)} - \frac{\sin n\eta_1 \sin(\nu+n+2)\eta_o}{n(\nu+n+2)} - \frac{\sin n\eta_1 \sin(\nu-n-2)\eta_o}{n(\nu-n-2)} \Big\}. \quad (3.1)$$

First, we consider again the case $\eta_o = \pi$. Again all but the singular terms vanish. After some calculations we find

$$P_2(\beta^2\Omega, \eta_1, \pi) = \pi \sum_{\nu=1}^{\infty} i^{\nu} \frac{J_{\nu+2}(\beta^2\Omega) - J_{\nu-2}(\beta^2\Omega)}{\nu} \sin\nu\eta_1 . \quad (3.2)$$

In the second case, viz. $\eta_o = \eta_1$, again we can make drastic simplifications. After some more intricate calculations we find then

$$P_2(\beta^2\Omega, \eta_1, \eta_1) = \frac{\eta_1}{\pi} P_2(\beta^2\Omega, \eta_1, \pi) + \sum_{\nu=2}^{\infty} i^{\nu} .$$

$$\frac{J_{\nu+2}(\beta^2\Omega) - J_{\nu-2}(\beta^2\Omega)}{\nu} \nu \sum_{n=1}^{\nu-1} \frac{\sin n\eta_1}{n} \cdot \frac{\sin(\nu-n)\eta_1}{\nu-n} \quad (3.3)$$

4. At last we deal with the expansion of R_n. This is again performed along the same lines, and we give directly the result

$$R_n(\beta^2\Omega, \eta_1) = i^n \Big[\eta_1 J_n(\beta^2\Omega) + \sum_{\nu=1}^{\infty} i^{\nu} \big\{ J_{\nu+n}(\beta^2\Omega) + (-1)^n J_{\nu-n}(\beta^2\Omega) \big\} \frac{\sin \nu\eta_1}{\nu} \Big]. \quad (4.1)$$

5. The formulae (2.4) and (2.5) are not very suitable for small $\beta^2\Omega$ as they take the indetermined form 0/0 for $\beta^2\Omega = 0$. This suggests the use of an expansion in a powerseries with respect to $\beta^2\Omega$. This is readily performed by expanding the Besselfunctions according to

$$J_{\nu}(\beta^2\Omega) = \sum_{j=0}^{\infty} (-1)^j \frac{2^{-\nu-2j}}{j!(\nu+j)!} (\beta^2\Omega)^{\nu+2j}, \quad (5.1)$$

and interchanging the order of summation with respect to n and j. We find without difficulty recurrence relations between the coeff ciets which give rise to the following expansion:

$$\frac{4}{z}\sum_{\nu=1}^{\infty} i^{\nu-1} s_\nu J_\nu(z) = 2s_1 - (\frac{1}{4}s_1 + \frac{1}{12}s_3)z^2 + (\frac{1}{96}s_1 + \frac{1}{192}s_3 + \frac{1}{960}s_5)z^4$$

$$- (\frac{1}{4608}s_1 + \frac{1}{7680}s_3 + \frac{1}{23040}s_5 + \frac{1}{161280}s_7)z^6$$

$$+ (\frac{1}{368640}s_1 + \frac{1}{552960}s_3 + \frac{1}{1290240}s_5 + \frac{1}{5160960}s_7 + \frac{1}{46448640}s_9)z^8$$

$$- (\frac{1}{44236800}s_1 + \frac{1}{61931520}s_3 + \frac{1}{123863040}s_5 + \frac{1}{371589120}s_7 + \frac{1}{1857945600}s_9 + \frac{1}{20437401600}s_{11})z^{10}$$

$$+ \dots\dots\dots$$

$$+ i\Big\{\frac{1}{2}s_2 z - (\frac{1}{24}s_2 + \frac{1}{96}s_4)z^3 + (\frac{1}{768}s_2 + \frac{1}{1920}s_4 + \frac{1}{11520}s_6)z^5$$

$$- (\frac{1}{46080}s_2 + \frac{1}{92160}s_4 + \frac{1}{322560}s_6 + \frac{1}{2580480}s_8)z^7$$

$$+ (\frac{1}{4423680}s_2 + \frac{1}{7741440}s_4 + \frac{1}{20643840}s_6 + \frac{1}{92897280}s_8 + \frac{1}{928972800}s_{10})z^9$$

$$- (\frac{1}{619315200}s_2 + \frac{1}{990904320}s_4 + \frac{1}{2229534720}s_6 + \frac{1}{7431782400}s_8 + \frac{1}{40874803200}s_{10} + \frac{1}{490497638400}s_{12})z^{11}$$

$$+ \dots\dots\dots\Big\} \qquad (5.2)$$

By inserting $\beta^2\Omega$ for z, and $\sin \nu\eta_1$ for s_ν we find the sum that occurs in (2.4), whereas insertion of $\sum_{n=1}^{\nu-1} \frac{\sin n\eta_1}{n} \cdot \frac{\sin(\nu-n)\eta_1}{\nu-n}$ for s_ν gives us the sum that occurs in (2.5).

Although not strictly necessary we also expand the sums in (3.2) and (3.3) with respect to $\beta^2\Omega$ so as to afford at least an independent check on the computations. The result can easily be expressed in the foregoing one. We write (5.2) in the form

$$\frac{4}{z}\sum_{\nu=1}^{\infty} i^{\nu-1} s_\nu J_\nu(z) = S_0 + S_2 z^2 + S_4 z^4 + \dots + i(S_1 z + S_3 z^3 + S_5 z^5 + \dots) \qquad (5.3)$$

Furthermore, we have

$$\sum_{\nu=1}^{\infty} i^{\nu} s_{\nu} \frac{J_{\nu+2}(z) - J_{\nu-2}(z)}{\nu} = \sum_{\nu=1}^{\infty} - i^{\nu} s_{\nu} \frac{\partial}{\partial z} \left\{ \frac{4}{z} J_{\nu}(z) \right\} =$$
$$-i \frac{\partial}{\partial z} \sum_{\nu=1}^{\infty} i^{\nu-1} s_{\nu} J_{\nu}(z). \qquad (5.4)$$

This relation is an obvions result from the directly from the definitions (1.1) and (1.2) following identity:

$$P_2 (\beta^2\Omega, \eta_1, \eta_0) = -i \frac{\partial}{\partial(\beta^2\Omega)} P_1 (\beta^2\Omega, \eta_1, \eta_0), \qquad (5.5)$$

which we have not used before, but which checks independently all the foregoing expansions of both P_1 and P_2. Moreover, we have now directly:

$$\sum_{\nu=1}^{\infty} i^{\nu} s_{\nu} \frac{J_{\nu+2}(z) - J_{\nu-2}(z)}{\nu} = (S_1 + 3 S_3 z^2 + 5 S_5 z^4 + \ldots)$$
$$+ i(2 S_2 z + 4 S_4 z^3 + \ldots). \qquad (5.6)$$

The coefficients of both series can therefore be calculated in one go, what makes the use also for P_2 attractive.

6. The way in which we have written the expansions (2.4), (2.5), (3.2), (3.3) and (4.1) suggests already the method of computation we have in mind.

First of all we prepare a 8 decimal table of $J_n(\beta^2\Omega)$ for all n for which the function does not vanish identically. This table is given on Datasheet 3. For the highest value of $\beta^2\Omega$, i.e. 11.2 n goes up to 26.

Next we prepare a 7 decimal table of $\sin n\eta_1$, for the six values of η_1, and the same range of n. This table is given on Datasheet 2. Furthermore we prepare a 7 decimal table of $\frac{\sin n\eta_1}{n}$ for the same range of η_1, and n. This table is given on Datasheet 4, and actually in two ways, viz. once with n = 1(1)27 and once with n = 27(-1)1. The reason for this arrangement is to perform easily the computation and checking of the coefficients $\sum_{n=1}^{\nu-1} \frac{\sin n\eta_1}{n} \cdot \frac{\sin(\nu-n)\eta_1}{\nu-n}$. By cutting the sheet in two, folding the lower part and shifting it alongside a column of the upper one, the two factors that have to be multiplied appear alongside one another. The results are given on Datasheet 5.

Next we compute $\frac{J_{\nu+2}(\beta^2\Omega) - J_{\nu-2}(\beta^2\Omega)}{\nu}$ and $J_{\nu+n}(\beta^2\Omega) + (-1)^n J_{\nu-n}(\beta^2\Omega)$.

We have then all the data, necessary for the computation of P_1, P_2 and R_n according to the formulae in 2, 3 and 4.

Furthermore we compute a table of $(\beta^2\Omega)^n$ for n = 1(1)11 and $\beta^2\Omega$ up to somewhat above 1 as far as they are relevant and of the coefficients of the powerseries after which P_1 and P_2 may be recalculated either as a check or in order to get the required accuracy of 7 decimals.

R 53. Gegevenblad 1.
Algemeen overzicht van parameters.

k	q	β = 0.35 Ω	ω	$\beta^2\Omega$	β = 0.50 Ω	ω	$\beta^2\Omega$	β = 0.60 Ω	ω	$\beta^2\Omega$	β = 0.70 Ω	ω	$\beta^2\Omega$	β = 0.80 Ω	ω	$\beta^2\Omega$
0.025	0.000625	0.14285714	0.12535714	0.0175												
0.050	0.002500	0.28571429	0.25071429	0.0350	0.2	0.150	0.050	0.16666667	0.10666667	0.06						
0.075	0.005625	0.42857143	0.37607143	0.0525							0.21428571	0.10928571	0.105			
0.100	0.010000	0.57142857	0.50142857	0.0700	0.4	0.300	0.100	0.33333333	0.21333333	0.12						
0.125	0.015625	0.71428571	0.62678571	0.0875										0.3125	0.1125	0.20
0.150	0.022500	0.85714286	0.75214286	0.1050	0.6	0.450	0.150	0.50000000	0.32000000	0.18	0.42857143	0.21857143	0.210			
0.200	0.040000	1.14285714	1.00285714	0.1400	0.8	0.600	0.200	0.66666667	0.42666667	0.24						
0.250	0.062500	1.42857143	1.25357143	0.1750	1.0	0.750	0.250	0.83333333	0.53333333	0.30	0.71428571	0.36428571	0.350	0.6250	0.2250	0.40
0.300	0.090000	1.71428571	1.50428571	0.2100	1.2	0.900	0.300	1.00000000	0.64000000	0.36						
0.375	0.140625	2.14285714	1.88035714	0.2625	1.5	1.125	0.375	1.25000000	0.80000000	0.45	1.07142857	0.54642857	0.525	0.9375	0.3375	0.60
0.450	0.202500	2.57142857	2.25642857	0.3150	1.8	1.350	0.450	1.50000000	0.96000000	0.54	1.28571429	0.65571429	0.630			
0.550	0.302500				2.2	1.650	0.550	1.83333333	1.17333333	0.66	1.57142857	0.80142857	0.770	1.3750	0.4950	0.88
0.650	0.422500	3.71428571	3.25928571	0.4550	2.6	1.950	0.650	2.16666667	1.38666667	0.78	1.85714286	0.94714286	0.910	1.6250	0.5850	1.04
0.800	0.640000							2.66666667	1.70666667	0.96	2.28571429	1.16571429	1.120	2.0000	0.7200	1.28
1.0	1.000000				4.0	3.000	1.000	3.33333333	2.13333333	1.20	2.85714286	1.45714286	1.400	2.5000	0.9000	1.60
1.2	1.440000	6.85714286	6.01714286	0.8400							3.42857143	1.74857143	1.680	3.0000	1.0800	1.92
1.5	2.250000							5.00000000	3.20000000	1.80	4.28571429	2.18571429	2.100	3.7500	1.3500	2.40
2.0	4.000000				8.0	6.000	2.000				5.71428571	2.91428571	2.800	5.0000	1.8000	3.20
3.0	9.000000							10.00000000	5.40000000	3.60				7.5000	2.7000	4.80
4.5	20.250000										12.85714286	6.55714286	6.300			
7.0	49.000000													17.5000	6.3000	11.20

τ =	0	0.10	0.15	0.20	0.25	0.30	0.35
η_1 =	0	0.64350111	0.79539883	0.92729522	1.04719755	1.15927948	1.26610368

N.B. Alle getallen, welke in minder dan 8 decimalen zijn opgegeven, zijn exact.

Waarden van sin $n\eta_1$.

	$\tau = 0.10$	$\tau = 0.15$	$\tau = 0.20$	$\tau = 0.25$	$\tau = 0.30$	$\tau = 0.35$
	$\sin \eta_1 = 0.8$	$\sin \eta_1 = 0.7$	$\sin \eta_1 = 0.6$	$\sin \eta_1 = 0.5$	$\sin \eta_1 = 0.4$	$\sin \eta_1 = 0.3$
n	$\sin n\eta_1$	$\sin n\eta_1$	$\sin n\eta_1$	$\sin n\eta_1$	$\sin n\eta_1$	$\sin n\eta_1$
1	0.6000000	0.7141429	0.8000000	0.8660254	0.9165151	0.9539392
2	0.9600000	0.9998000	0.9600000	0.8660254	0.7332121	0.5723636
3	0.9360000	0.6855771	0.3520000	0	- 0.3299454	- 0.6105211
4	0.5376000	- 0.0399925	- 0.5376000	- 0.8660254	- 0.9971685	- 0.9386762
5	- 0.0758400	- 0.7415660	- 0.9971200	- 0.8660254	- 0.4677893	0.0473155
6	- 0.6589440	- 0.9982003	- 0.6589440	0	0.6229370	0.9670654
7	- 0.9784704	- 0.6559145	0.2063872	0.8660254	0.9661389	0.5329239
8	- 0.9066087	0.0799201	0.9066087	0.8660254	0.1499742	- 0.6473112
9	- 0.4721034	0.7678025	0.8815432	0	- 0.8461597	- 0.9213105
10	0.1512433	0.9950035	0.1512433	- 0.8660254	- 0.8269018	0.0945248
1	0.7140924	0.6252023	- 0.7000514	- 0.8660254	0.1846382	0.9780254
2	0.9913048	- 0.1197201	- 0.9913048	0	0.9746124	0.4922903
3	0.8719952	- 0.7928107	- 0.4895144	0.8660254	0.5950517	- 0.6826511
4	0.4038875	- 0.9902147	0.4038875	0.8660254	- 0.4985710	- 0.9018811
5	- 0.2257753	- 0.5934899	0.9741795	0	- 0.9939085	0.1415226
6	- 0.7651278	0.1593288	0.7651278	- 0.8660254	- 0.2965559	0.9867946
7	- 0.9984293	0.8165502	- 0.0560261	- 0.8660254	0.7566639	0.4505543
8	- 0.8323590	0.9838415	- 0.8323590	0	0.9018869	- 0.7164620
9	- 0.3333452	0.5608281	- 0.9428048	0.8660254	- 0.0351543	- 0.8804314
20	0.2990067	- 0.1986824	- 0.2990067	0.8660254	- 0.9300104	0.1882033
1	0.8117560	- 0.8389833	0.5839968	0	- 0.7088540	0.9933534
2	0.9998029	- 0.9758942	0.9998029	- 0.8660254	0.3629271	0.4078088
3	0.7879287	- 0.5272686	0.6157665	- 0.8660254	0.9991957	- 0.7486681
4	0.2608829	0.2377181	- 0.2608829	0	0.4354293	- 0.8570097
5	- 0.3705159	0.8600741	- 0.9288261	0.8660254	- 0.6500522	0.2344623
6	- 0.8537084	0.9663855	- 0.8537084	0.8660254	- 0.9564712	0.9976871
7	- 0.9954175	0.4928657	- 0.0956239	0	- 0.1151248	0.3641498

R 53. Datasheet 3.
Values of $J_n(\beta^2\Omega)$.

$\beta^2\Omega$	$J_0(\beta^2\Omega)$	$J_1(\beta^2\Omega)$	$J_2(\beta^2\Omega)$	$J_3(\beta^2\Omega)$	$J_4(\beta^2\Omega)$	$J_5(\beta^2\Omega)$	$J_6(\beta^2\Omega)$	$J_7(\beta^2\Omega)$	$J_8(\beta^2\Omega)$
0.0175	0.99992344	0.00874967	0.00003828	0.00000011					
0.0350	0.99969377	0.01749732	0.00015311	0.00000089					
0.0500	0.99937510	0.02499219	0.00031243	0.00000260	0.00000002				
0.0525	0.99931106	0.02624096	0.00034445	0.00000301	0.00000002				
0.0600	0.99910020	0.02998650	0.00044987	0.00000450	0.00000003				
0.0700	0.99877538	0.03497857	0.00061225	0.00000714	0.00000006				
0.0875	0.99808685	0.04370814	0.00095642	0.00001395	0.00000015				
0.1000	0.99750156	0.04993753	0.00124896	0.00002082	0.00000026				
0.1050	0.99724565	0.05242768	0.00137686	0.00002410	0.00000032				
0.1200	0.99640324	0.05989206	0.00179784	0.00003579	0.00000054	0.00000001			
0.1400	0.99510600	0.06982864	0.00244600	0.00005710	0.00000100	0.00000001			
0.1500	0.99438291	0.07478926	0.00280723	0.00007021	0.00000132	0.00000002			
0.1750	0.99235839	0.08716547	0.00381836	0.00011144	0.00000244	0.00000004			
0.1800	0.99191639	0.08963599	0.00403908	0.00012125	0.00000273	0.00000005			
0.2000	0.99002497	0.09950083	0.00498335	0.00016625	0.00000416	0.00000008			
0.2100	0.98900535	0.10442225	0.00549227	0.00019241	0.00000505	0.00000011			
0.2400	0.98565176	0.11913807	0.00716550	0.00028696	0.00000862	0.00000021			
0.2500	0.98443593	0.12402598	0.00777189	0.00032425	0.00001014	0.00000025	0.00000001		
0.2625	0.98284748	0.13012275	0.00856393	0.00037521	0.00001232	0.00000032	0.00000001		
0.3000	0.97762625	0.14831882	0.01116586	0.00055934	0.00002100	0.00000063	0.00000002		
0.3150	0.97534716	0.15555457	0.01230088	0.00064714	0.00002551	0.00000080	0.00000002		
0.3500	0.96960868	0.17233396	0.01515678	0.00088641	0.00003884	0.00000136	0.00000004		
0.3600	0.96786150	0.17709970	0.01602575	0.00096415	0.00004346	0.00000157	0.00000005		
0.3750	0.96515154	0.18422336	0.01737303	0.00108901	0.00005114	0.00000192	0.00000006		
0.4000	0.96039823	0.19602658	0.01973466	0.00132005	0.00006613	0.00000265	0.00000009		
0.4500	0.95001213	0.21935254	0.02488805	0.00187453	0.00010571	0.00000477	0.00000018	0.00000001	
0.4550	0.94390959	0.22166329	0.02543455	0.00193716	0.00011046	0.00000503	0.00000019	0.00000001	
0.5250	0.93227172	0.25355932	0.03366856	0.00296307	0.00019513	0.00001027	0.00000045	0.00000002	
0.5400	0.92841789	0.26027735	0.03557230	0.00322115	0.00021822	0.00001181	0.00000053	0.00000002	
0.5500	0.92579283	0.26473180	0.03686828	0.00340111	0.00023472	0.00001294	0.00000059	0.00000002	
0.6000	0.91200486	0.28670099	0.04366510	0.00439966	0.00033147	0.00001995	0.00000100	0.00000004	
0.6300	0.90320943	0.29962838	0.04799179	0.00508136	0.00040216	0.00002542	0.00000134	0.00000006	
0.6500	0.89713164	0.30813545	0.05097744	0.00557186	0.00045513	0.00002969	0.00000161	0.00000008	
0.6600	0.89402917	0.31235468	0.05250018	0.00582829	0.00048347	0.00003203	0.00000177	0.00000008	
0.7700	0.85717803	0.35716294	0.07051790	0.00916384	0.00083864	0.00006877	0.00000443	0.00000024	0.00000001
0.7800	0.85358678	0.36108291	0.07226683	0.00951624	0.00093498	0.00007330	0.00000478	0.00000027	0.00000001
0.8400	0.83122844	0.38402922	0.08312686	0.01181297	0.00125146	0.00010575	0.00000743	0.00000045	0.00000002
0.8800	0.81557110	0.39876034	0.09070241	0.01352334	0.00150220	0.00013306	0.00000980	0.00000062	0.00000003
0.9100	0.80344653	0.40949915	0.09655160	0.01490348	0.00171313	0.00015698	0.00001196	0.00000078	0.00000004
0.9600	0.78253615	0.42678706	0.10660357	0.01739447	0.00211185	0.00020432	0.00001644	0.00000113	0.00000007
1.0000	0.76519769	0.44005059	0.11490348	0.01956335	0.00247664	0.00024976	0.00002094	0.00000150	0.00000009
1.0400	0.74733904	0.45279393	0.12341852	0.02189268	0.00288542	0.00030283	0.00002642	0.00000197	0.00000013
1.1200	0.71014613	0.47666336	0.14103843	0.02704534	0.00384732	0.00043549	0.00004095	0.00000329	0.00000023
1.2000	0.67113274	0.49828906	0.15934902	0.03287434	0.00502267	0.00061010	0.00006154	0.00000531	0.00000040
1.2800	0.63048224	0.51757660	0.17823119	0.03939588	0.00643701	0.00083546	0.00009000	0.00000829	0.00000067
1.4000	0.56685512	0.54194771	0.20735590	0.05049771	0.00906287	0.00129013	0.00015231	0.00001537	0.00000135
1.6000	0.45540217	0.56989594	0.25696775	0.07252344	0.01499516	0.00245236	0.00033210	0.00003840	0.00000387
1.6800	0.40952799	0.57652038	0.27680580	0.08254104	0.01798363	0.00309531	0.00044085	0.00005358	0.00000568
1.8000	0.33998641	0.58151695	0.30614354	0.09880202	0.02319652	0.00429361	0.00065690	0.00008571	0.00000975
1.9200	0.27020077	0.58059458	0.33458524	0.11645802	0.02934606	0.00581723	0.00095200	0.00013277	0.00001614
2.0000	0.22389078	0.57672481	0.35283403	0.12894325	0.03399572	0.00703963	0.00120243	0.00017494	0.00002218
2.1000	0.16660698	0.56829214	0.37462363	0.14527667	0.04045259	0.00882842	0.00158750	0.00024298	0.00003239
2.4000	0.00250768	0.52018527	0.43098004	0.19811480	0.06430696	0.01624172	0.00336689	0.00059274	0.00009076
2.8000	-0.18503603	0.40970925	0.47768550	0.27269860	0.10666866	0.03206898	0.00786343	0.00163142	0.00029367
3.2000	-0.32018817	0.26134325	0.48352770	0.34306638	0.15972176	0.05623801	0.01602203	0.00384461	0.00079815
3.6000	-0.39176898	0.09546555	0.44480540	0.39876267	0.21979906	0.08967968	0.02951115	0.00802417	0.00189395
4.8000	-0.24042533	-0.29849986	0.11605039	0.39520851	0.37796026	0.23472525	0.11105067	0.04290144	0.01407852
6.3000	0.22381201	-0.20808694	-0.28987135	0.02404164	0.31276815	0.37312427	0.27949259	0.15924257	0.07437980
11.2000	-0.13299194	-0.20385315	0.09658959	0.23834943	0.03109760	-0.21613685	-0.22407694	-0.02394558	0.19414496

Values of $J_n(\beta^2\Omega)$.

$\beta^2\Omega$	$J_9(\beta^2\Omega)$	$J_{10}(\beta^2\Omega)$	$J_{11}(\beta^2\Omega)$	$J_{12}(\beta^2\Omega)$	$J_{13}(\beta^2\Omega)$	$J_{14}(\beta^2\Omega)$	$J_{15}(\beta^2\Omega)$	$J_{16}(\beta^2\Omega)$	$J_{17}(\beta^2\Omega)$
1.0000	0.00000001								
1.0400	0.00000001								
1.1200	0.00000001								
1.2000	0.00000003								
1.2800	0.00000005								
1.4000	0.00000011	0.00000001							
1.6000	0.00000035	0.00000003							
1.6800	0.00000053	0.00000005							
1.8000	0.00000098	0.00000009	0.00000001						
1.9200	0.00000174	0.00000017	0.00000001						
2.0000	0.00000249	0.00000025	0.00000002						
2.1000	0.00000383	0.00000041	0.00000004						
2.4000	0.00001230	0.00000150	0.00000016	0.00000002					
2.8000	0.00004672	0.00000666	0.00000086	0.00000010	0.00000001				
3.2000	0.00014615	0.00002395	0.00000355	0.00000048	0.00000006	0.00000001			
3.6000	0.00039339	0.00007302	0.00001226	0.00000155	0.00000026	0.00000003			
4.8000	0.00402695	0.00102256	0.00023371	0.00004860	0.00000927	0.00000164	0.00000027	0.00000004	0.00000001
6.3000	0.02965849	0.01035876	0.00322646	0.00090823	0.00023346	0.00005526	0.00001212	0.00000248	0.00000048
11.2000	0.30129553	0.29007999	0.21670446	0.13558948	0.07384442	0.03583508	0.01574327	0.00633440	0.00235501

$\beta^2\Omega$	$J_{18}(\beta^2\Omega)$	$J_{19}(\beta^2\Omega)$	$J_{20}(\beta^2\Omega)$	$J_{21}(\beta^2\Omega)$	$J_{22}(\beta^2\Omega)$	$J_{23}(\beta^2\Omega)$	$J_{24}(\beta^2\Omega)$	$J_{25}(\beta^2\Omega)$	$J_{26}(\beta^2\Omega)$
6.3000	0.00000009	0.00000001							
11.2000	0.00081475	0.00026381	0.00008034	0.00002310	0.00000630	0.00000163	0.00000040	0.00000010	0.00000002

Values of $\frac{\sin n\eta_1}{n}$.

n	$\tau = 0.10$ $\frac{\sin n\eta_1}{n}$	$\tau = 0.15$ $\frac{\sin n\eta_1}{n}$	$\tau = 0.20$ $\frac{\sin n\eta_1}{n}$	$\tau = 0.25$ $\frac{\sin n\eta_1}{n}$	$\tau = 0.30$ $\frac{\sin n\eta_1}{n}$	$\tau = 0.35$ $\frac{\sin n\eta_1}{n}$
1	0.6000000	0.7141429	0.8000000	0.8660254	0.9165151	0.9539392
2	0.4800000	0.4999000	0.4800000	0.4330127	0.3666060	0.2861818
3	0.3120000	0.2285257	- 0.1173333	0.0000000	- 0.1099818	- 0.2035070
4	0.1344000	- 0.0099981	- 0.1344000	- 0.2165064	- 0.2492921	- 0.2346690
5	- 0.0151680	- 0.1483132	- 0.1994240	- 0.1732051	- 0.0935579	0.0094631
6	- 0.1098240	- 0.1663667	- 0.1098240	0.0000000	0.1038228	0.1611776
7	- 0.1397815	- 0.0937021	0.0294839	0.1237179	0.1380198	0.0761320
8	- 0.1133261	0.0099900	0.1133261	0.1082532	0.0187468	- 0.0809139
9	- 0.0524559	0.0853114	0.0979492	0.0000000	- 0.0940177	- 0.1023678
10	0.0151243	0.0995004	0.0151243	- 0.0866025	- 0.0826902	0.0094525
1	0.0649175	0.0568366	- 0.0636410	- 0.0787296	0.0167853	0.0889114
2	0.0826087	- 0.0099767	- 0.0826087	0.0000000	0.0812177	0.0410242
3	0.0670766	- 0.0609854	- 0.0376550	0.0666173	0.0457732	- 0.0525116
4	0.0288491	- 0.0707296	0.0288491	0.0618590	- 0.0356122	- 0.0644201
5	- 0.0150517	- 0.0395660	0.0549453	0.0000000	- 0.0662606	0.0094348
6	- 0.0478205	0.0099580	0.0478205	- 0.0541266	- 0.0185347	0.0616747
7	- 0.0587311	0.0480324	- 0.0032957	- 0.0509427	0.0445096	0.0265032
8	- 0.0462422	0.0546579	- 0.0462422	0.0000000	0.0501048	- 0.0398034
9	- 0.0175445	0.0295173	- 0.0496213	0.0455803	- 0.0018502	- 0.0463385
20	0.0149503	- 0.0099341	- 0.0149503	0.0433013	- 0.0465005	0.0094102
1	0.0386550	- 0.0399516	0.0278094	0.0000000	- 0.0337550	0.0473025
2	0.0454456	- 0.0443588	0.0454456	- 0.0393648	0.0164967	0.0185368
3	0.0342578	- 0.0229247	0.0267725	- 0.0376533	0.0434433	- 0.0325508
4	0.0108701	0.0099049	- 0.0108701	0.0000000	0.0181846	- 0.0357087
5	- 0.0148206	0.0344030	- 0.0371530	0.0346410	- 0.0260021	0.0093785
6	- 0.0328349	0.0371687	- 0.0328349	0.0333087	- 0.0367874	0.0383726
7	- 0.0368673	0.0182543	- 0.0035416	0.0000000	- 0.0042639	0.0134870
27	- 0.0368673	0.0182543	- 0.0035416	0.0000000	- 0.0042639	0.0134870
6	- 0.0328349	0.0371687	- 0.0328349	0.0333087	- 0.0367874	0.0383726
5	- 0.0148206	0.0344030	- 0.0371530	0.0346410	- 0.0266021	0.0093785
4	0.0108701	0.0099049	- 0.0108701	0.0000000	0.0181846	- 0.0357087
3	0.0342578	- 0.0229247	0.0267725	- 0.0376533	0.0434433	- 0.0325508
2	0.0454456	- 0.0443588	0.0454456	- 0.0393648	0.0164967	0.0185368
1	0.0386550	- 0.0399516	0.0278094	0.0000000	- 0.0337550	0.0473025
20	0.0149503	- 0.0099341	- 0.0149503	0.0433013	- 0.0465005	0.0094102
9	- 0.0175445	0.0295173	- 0.0496213	0.0455803	- 0.0018502	- 0.0463385
8	- 0.0462422	0.0546579	- 0.0462422	0.0000000	0.0501048	- 0.0398034
7	- 0.0587311	0.0480324	- 0.0032957	- 0.0509427	0.0445096	0.0265032
6	- 0.0478205	0.0099580	0.0478205	- 0.0541266	- 0.0185347	0.0616747
5	- 0.0150517	- 0.0395660	0.0649453	0.0000000	- 0.0662606	0.0094348
4	0.0288491	- 0.0707296	0.0288491	0.0618590	- 0.0356122	- 0.0644201
3	0.0670766	- 0.0609854	- 0.0376550	0.0666173	0.0457732	- 0.0525116
2	0.0826087	- 0.0099767	- 0.0826087	0.0000000	0.0812177	0.0410242
1	0.0649175	0.0568366	- 0.0636410	- 0.0787296	0.0167853	0.0889114
10	0.0151243	0.0995004	0.0151243	- 0.0866025	- 0.0826902	0.0094525
9	- 0.0524559	0.0853114	0.0979492	0.0000000	- 0.0940177	- 0.1023678
8	- 0.1133261	0.0099900	0.1133261	0.1082532	0.0187468	- 0.0809139
7	- 0.1397815	- 0.0937021	0.0294839	0.1237179	0.1380198	0.0761320
6	- 0.1098240	- 0.1663667	- 0.1098240	0.0000000	0.1038228	0.1611776
5	- 0.0151680	- 0.1483132	- 0.1994240	- 0.1732051	- 0.0935579	0.0094631
4	- 0.1344000	- 0.0099981	- 0.1344000	- 0.2165064	- 0.2492921	- 0.2346690
3	0.3120000	0.2285257	- 0.1173333	0.0000000	- 0.1099818	- 0.2035070
2	0.4800000	0.4999000	0.4800000	0.4330127	0.3666060	0.2861818
1	0.6000000	0.7141429	0.8000000	0.8660254	0.9165151	0.9539392