THE OSCILLATING WING IN A SUBSONIC FLOW.

R 53, Int 4.

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1. Definition of integrals.

In this report the method of calculation is given of those integrals that play a role in the first part of R 53. So we shall treat successively:

$$I_n(\Omega)= \int_0^\infty e^{-i\Omega\cosh\xi} \sinh n\xi \, d\xi \, . \tag{1.1}$$

$$I_n'(\Omega)= \int_0^\infty e^{-i\Omega\cosh\xi} \, \xi \sinh n\xi \, d\xi \, . \tag{1.2}$$

$$\Lambda_n(\Omega)= \int_0^\infty e^{-i\Omega\cosh\xi} \, e^{-n\xi} \, d\xi \tag{1.3}$$

$$K_n(\beta,\Omega)= \Big(\sum_{m=1}^{\infty} m \, B_m^{(n)} \Big) \int_0^\infty e^{-i\Omega \cosh\xi} \Big\{ se_n'(0) \, \frac{Ne_n^{(2)}(\xi)}{Ne_n^{(2)\prime}(0)} + \sum_{m=1}^{\infty} B_m^{(n)} e^{-m\xi} \Big\} d\xi \, . \tag{1.4}$$

$$\Lambda_n(\beta,\Omega)=(-1)^n \, se_n'(0) \int_0^\infty e^{-i\Omega\cosh\xi} \, \frac{Ne_n^{(2)}(\xi)}{Ne_n^{(2)\prime}(0)} \, d\xi \, . \tag{1.5}$$

Furthermore we notice that

$$H_n^*(\Omega)= \frac{\pi}{2} \, i^{-n-1} \, H_n^{(2)}(\Omega)= \int_0^\infty e^{-i\Omega \cosh\xi} \cosh n\xi \, d\xi \, , \tag{1.6}$$

where $H_n^{(2)}(\Omega)$ means the Hankelfunction of the second type.

Most of the integrals are only defined for those values of Ω that have a negative imaginary part. But we may consider them to be defined on the real axis by analytic continuation.

Also, some of the formulas, used hereafter, are only justified again by analytic continuation.

2. Series expansion of the integral $I_n(\Omega)$.

We state first of all, that for $n \neq 0$ the integral only converges if the imaginary part of Ω is negative. For real Ω $I_n(\Omega)$ is defined by analytic continuation.

From formula (1,1) one sees easily, that the following recurrence-relation holds

$$I_{n+1}(\Omega) - I_{n-1}(\Omega)= - \frac{2ni}{\Omega} \, I_n(\Omega) - \frac{2i}{\Omega} \, e^{-i\Omega} \, . \tag{2.1}$$

When Ω is real, we find

$$I_0(\Omega) = 0 \qquad (2.2)$$
$$I_1(\Omega) = -\frac{i}{\Omega}e^{-i\Omega}.$$

By means of these relations (2.2) and (2.1) it is possible to compute $I_n(\Omega)$ for each n and each Ω. It is however our intention to compute separately the part of the integral that is singular for $\Omega = 0$ and the other part (the regular part). The reason for this lies in the fact that we have to form later on $i^{-n}\frac{\pi}{2}Y_n(\Omega) + I_n(\Omega)$, where $Y_n(\Omega)$ is the Besselfunction of the first kind and second type of order n. The singular parts of the two functions cancel eachother apart from a logarithmic term. Carrying out the recursion we would for higher values of n only obtain the singular part of $I_n(\Omega)$, unless we would perform the calculations to an excessive number of figures, and therefore the regular part could not be found with any accuracy. Therefore we put:

$$I_n(\Omega) = \sum_{k=0}^{\infty} c_{n,k}\,\Omega^{k-n} \qquad (2.3)$$

and we have the relations

$$c_{2,k} = \frac{2i^{-k}}{k!}(k-1)$$

$$c_{3,k} = \frac{i^{-k-1}}{k!}(3k-8)(k-1)$$

$$c_{4,k} = \frac{4i^{-k+2}}{k!}(k-1)(k-3)(k-4)$$

$$c_{5,k} = \frac{i^{-k+3}}{k!}(k-1)(k-3)(5k^2-50k+128)$$

$$c_{6,k} = \frac{2i^{-k}}{k!}(k-1)(k-3)(k-5)(3k^2-38k+128)$$

$$c_{7,k} = \frac{i^{-k+1}}{k!}(k-1)(k-3)(k-5)(7k^3-154k^2+1176k-3072)$$

$$c_{8,k} = \frac{8i^{-k+2}}{k!}(k-1)(k-3)(k-5)(k-7)(k^3-26k^2+240k-768)$$

$$c_{9,k} = \frac{i^{-k+3}}{k!}(k-1)(k-3)(k-5)(k-7)(9k^4-348k^3+5292k^2-36912k+98304)$$

$$c_{10,k} = \frac{2i^{-k}}{k!}(k-1)(k-3)(k-5)(k-7)(k-9)(5k^4-220k^3+3868k^2-31472k+98304).$$

We computed first for fixed n the coefficients $c_{n,k}$ $(k = 0,1,2\ldots)$, and then the singular part was formed as $\sum_{k=0}^{n-1} c_{n,k}\,\Omega^{k-n}$, and the regular part as $\sum_{k=n}^{\infty} c_{n,k}\,\Omega^{k-n}$.
All calculations were made in 8 figures.

3. Calculations of the functions $I_n'(\Omega)$.

We put
$$I_n(z,\Omega) = \int_z^\infty e^{-i\Omega\cosh\xi}\sinh n\xi\, d\xi \quad . \tag{3.1}$$

So
$$I_0(z,\Omega) = 0 \tag{3.2}$$
$$I_1(z,\Omega) = -\frac{1}{i\Omega}\, e^{-i\Omega\cosh z} \tag{3.3}$$

and we have again a recurrence-relation

$$I_{m+1}(z,\Omega) - I_{m-1}(z,\Omega) = \frac{2}{i\Omega}\, e^{-i\Omega\cosh z}\cosh m z + \frac{2m}{i\Omega}\, I_m(z,\Omega) \tag{3.4}$$

From these relations it is clear that $I_n(z,\Omega)$ can be written as a linear combination of terms $P_m(\Omega)\, e^{-i\Omega\cosh z}\cosh m z$, where $P_m(\Omega)$ is a polynomial in Ω^{-1} of degree $m \leq n$.

Further we know:

$$\begin{aligned}\int_0^\infty I_m(z,\Omega)\, dz &= -\int_0^\infty z\, d\int_z^\infty e^{-i\Omega\cosh\xi}\sinh m\xi\, d\xi = \\ &= \int_0^\infty e^{-i\Omega\cosh\xi}\,\xi\sinh m\xi\, d\xi = I_m'(\Omega) \quad .\end{aligned} \tag{3.5}$$

By means of (3.4) we get the recurrence-relation

$$I_{m+1}'(\Omega) - I_{m-1}'(\Omega) = \frac{2}{i\Omega}\, H_m^*(\Omega) + \frac{2m}{i\Omega}\, I_m'(\Omega) \tag{3.6}$$

and our results are

$$I_2'(\Omega) = -\frac{2}{\Omega^2}\, H_0^*(\Omega) + \frac{2}{i\Omega}\, H_1^*(\Omega) \quad .$$

$$I_3'(\Omega) = -i\left(\frac{2}{\Omega} - \frac{8}{\Omega^3}\right) H_0^*(\Omega) - \frac{12}{\Omega^2}\, H_1^*(\Omega) .$$

$$I_4'(\Omega) = \left(-\frac{23}{\Omega^2} + \frac{48}{\Omega^4}\right) H_0^*(\Omega) - i\left(\frac{4}{\Omega} - \frac{88}{\Omega^3}\right) H_1^*(\Omega) \quad .$$

$$I_5'(\Omega) = -i\left(\frac{5}{\Omega} - \frac{280}{\Omega^3} + \frac{384}{\Omega^5}\right) H_0^*(\Omega) + \left(-\frac{60}{\Omega^2} + \frac{800}{\Omega^4}\right) H_1^*(\Omega) \quad .$$

$$I'_6(\Omega) = \left(-\frac{102}{\Omega^2} + \frac{3232}{\Omega^4} - \frac{3840}{\Omega^6}\right) H_0^*(\Omega) - i\left(\frac{6}{\Omega} - \frac{832}{\Omega^3} + \frac{8768}{\Omega^5}\right) H_1^*(\Omega)$$

$$I'_7(\Omega) = -i\left(\frac{7}{\Omega} - \frac{1792}{\Omega^3} + \frac{43008}{\Omega^5} - \frac{46080}{\Omega^7}\right) H_0^*(\Omega) + $$
$$+ \left(\frac{168}{\Omega^2} + \frac{12320}{\Omega^4} - \frac{112896}{\Omega^6}\right) H_1^*(\Omega)$$

$$I'_8(\Omega) = \left(-\frac{248}{\Omega^2} + \frac{32160}{\Omega^4} - \frac{652032}{\Omega^6} + \frac{645120}{\Omega^8}\right) H_0^*(\Omega) -$$
$$- i\left(\frac{8}{\Omega} - \frac{3760}{\Omega^3} + \frac{200448}{\Omega^5} - \frac{1672704}{\Omega^7}\right) H^*(\Omega)$$

$$I'(\Omega) = -i\left(\frac{9}{\Omega} - \frac{6720}{\Omega^3} + \frac{615168}{\Omega^5} - \frac{11123712}{\Omega^7} + \frac{10321920}{\Omega^9}\right) H_0^*(\Omega)$$
$$+ \left(-\frac{360}{\Omega^2} + \frac{82080}{\Omega^4} - \frac{3596544}{\Omega^6} + \frac{28053504}{\Omega^8}\right) H_1^*(\Omega)$$

$$I'_{10}(\Omega) = \left(-\frac{490}{\Omega^2} + \frac{172320}{\Omega^4} - \frac{12692736}{\Omega^6} + \frac{211193856}{\Omega^8} - \frac{185794560}{\Omega^{10}}\right) H_0^*(\Omega)$$
$$-i\left(\frac{10}{\Omega} - \frac{11840}{\Omega^3} + \frac{1850688}{\Omega^5} - \frac{70926336}{\Omega^7} + \frac{525606912}{\Omega^9}\right) H_1^*(\Omega)$$

$H_0^*(\Omega)$ and $H_1^*(\Omega)$ are found as

$$H_0^*(\Omega) = -\frac{\pi}{2} Y_0(\Omega) - \frac{\pi i}{2} J_0(\Omega)$$
$$H_1^*(\Omega) = -\frac{\pi}{2} J_1(\Omega) + \frac{\pi i}{2} Y_1(\Omega) \qquad (3.7)$$

It is also possible to use the relation (3.6) for the computation of the $I_n'(\Omega)$. The values (3.7) are taken from a table and when we need $Y_n(\Omega)$ with $n \geq 2$ we can use

$$Y_{n+1}(\Omega) + Y_{n-1}(\Omega) = \frac{2n}{\Omega} Y_n(\Omega) \qquad (3.8)$$

4. Computations of $\Lambda_n(\Omega)$.

In order to compute $\Lambda_n(\Omega)$ we proceed in the following way

$$\Lambda_n(\Omega) = \int_0^\infty e^{-i\Omega \cosh\xi} e^{-n\xi} d\xi =$$
$$= \int_0^\infty e^{-i\Omega \cosh\xi} \cosh n\xi d\xi - \int_0^\infty e^{-i\Omega \cosh\xi} \sinh n\xi d\xi$$
$$= H_n^*(\Omega) - I_n(\Omega). \qquad (4.1)$$

We know: $H_n^*(\Omega) = \frac{\pi}{2} i^{-n-1} \quad J_n(\Omega) - i\, Y_n(\Omega)$ (4.2)

where $Y_n(\Omega)$ has the same singular part for $\Omega = 0$ as $I_n(\Omega)$ except for a logarithmic term

$$Y_n(\Omega) = \frac{2}{\pi}\left\{\gamma + \log\frac{\Omega}{2}\right\} J_n(\Omega) - \frac{1}{\pi}\sum_{r=0}^{n-1}\frac{(n-r-1)!}{r!}\left(\frac{2}{\Omega}\right)^{n-2r}$$

$$- \frac{1}{\pi}\sum_{r=0}^{\infty}(-1)^r\frac{(\Omega/2)^{2r+n}}{r!\,(n+r)!}\left\{1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{r} + 1 + \frac{1}{2} + \dots + \frac{1}{n+r}\right\} \quad (4.3)$$

It is clear that we can omit at once the singular part of $I_n(\Omega)$. Furthermore it is necessary to omit this part, when we do not want to compute with twenty or more figures. When "Reg" means regular part and "Re and Im" real and imaginary part, and further γ means the constant of Euler, we have for odd n:

$$\operatorname{Im}\Lambda_n(\Omega) = (-1)^{\frac{n-1}{2}}\,\frac{\pi}{2}\operatorname{Reg} Y_n(\Omega) - \operatorname{Im}\operatorname{Reg} I_n(\Omega) + (-1)^{\frac{n-1}{2}}\left(\gamma + \log\frac{\Omega}{2}\right) J_n(\Omega)$$

$$\operatorname{Re}\Lambda_n(\Omega) = (-1)^{\frac{n+1}{2}}\,\frac{\pi}{2} J_n(\Omega) - \operatorname{Re} I_n(\Omega) \quad (4.4)$$

and for even n:

$$\operatorname{Im}\Lambda_n(\Omega) = (-1)^{\frac{n}{2}-1}\,\frac{\pi}{2} J_n(\Omega) - \operatorname{Im} I_n(\Omega)$$

$$\operatorname{Re}\Lambda_n(\Omega) = (-1)^{\frac{n}{2}+1}\,\frac{\pi}{2}\operatorname{Reg} Y_n(\Omega) - \operatorname{Re}\operatorname{Reg} I_n(\Omega) + (-1)^{\frac{n}{2}+1}\left(\gamma + \log\frac{\Omega}{2}\right) J_n(\Omega) \quad (4.5)$$

All computations were made again in 8 figures and checked by means of the recurrence relation:

$$\Omega\left[\Lambda_{n-1}(\Omega) - \Lambda_{n+1}(\Omega)\right] + 2\, i\, e^{-i\Omega} - 2\, n\, i\Lambda_n(\Omega) = 0 \quad (4.6)$$

This recurrence formula (4.6), however, could not be used for the computation of $\Lambda_n(\Omega)$, because we should lose significant digits at each step.

5. Calculations of $K_n(\beta,\Omega)$ and $\Lambda_n(\beta,\Omega)$.

First we express the function $Ne_n^{(2)}(\xi)$ defined by (11.2) of R 53, Int 2.1 into a modified Fourier-expansion of the form:

$$Ne_n^{(2)}(\xi) = b\left\{ -\xi\, Se_n(\xi) + \sum_{m=0}^{\infty} g_m^{(n)} \cosh m\xi \right\} + a\, Se_n(\xi) \tag{5.1}$$

Using the relation (13.10) of R 53, Int 2.1

$$\frac{a}{b} = -\left(\vartheta + \log\frac{k}{2} + \frac{\pi i}{2}\right), \tag{5.2}$$

and

$$Ne_n^{(2)\prime}(0) = a \sum_{m=1}^{\infty} m\, B_m^{(n)}, \tag{5.3}$$

we find

$$\frac{Ne_n^{(2)}(\xi)}{Ne_n^{(2)\prime}(0)} = \frac{1}{(\vartheta+\log\frac{k}{2}+\frac{\pi i}{2})\sum_{m=1}^{\infty} mB_m^{(n)}}\left\{ \sum_{m=1}^{\infty} B_m^{(n)}\xi \sinh m\xi - \sum_{m=0}^{\infty} T_m^{(n)} \cosh m\xi \right.$$

$$\left. + \left(\log\frac{k}{2} + \frac{\pi i}{2}\right) \sum_{m=0}^{\infty} B_m^{(n)} \cosh m\xi \right\} - \frac{1}{\sum_{m=1}^{\infty} m\, B_m^{(n)}} \sum_{m=1}^{\infty} B_m^{(n)} e^{-m\xi}, \tag{5.4}$$

following (9.3) of R 53, Int 2.13.

We have also:

$$K_n(\beta,\Omega) = \left\{\sum_{m=1}^{\infty} m\, B_m^{(n)}\right\} \int_0^{\infty} e^{-i\Omega\cosh\xi} \left\{ se_n'(0) \frac{Ne_n^{(2)}(\xi)}{Ne_n^{(2)\prime}(0)} + \sum_{m=1}^{\infty} B_m^{(n)} e^{-m\xi} \right\} d\xi =$$

$$= \frac{\sum_{m=1}^{\infty} m\, B_m^{(n)}}{\vartheta+\log\frac{k}{2}+\frac{\pi i}{2}} \left\{ \sum_{m=1}^{\infty} B_m^{(n)} I_m'(\Omega) + \sum_{m=0}^{\infty} \left[B_m^{(n)}\left(\log\frac{k}{2}+\frac{\pi i}{2}\right) - T_m^{(n)} \right] H^{*}_{r} \right. \tag{5.5}$$

and

$$\Lambda_n(\beta,\Omega) = (-1)^n\, se_n'(0) \int_0^{\infty} e^{-i\Omega\cosh\xi} \frac{Ne_n^{(2)}(\xi)}{Ne_n^{(2)\prime}(0)}\, d\xi =$$

$$= (-1)^n \left\{ \frac{K_n(\beta,\Omega)}{se_n'(0)} - \sum_{m=1}^{\infty} B_m^{(n)} \Lambda_m(\Omega) \right\}. \tag{5.6}$$

It must be noticed that although the $B_m^{(n)}$ are decreasing rapidly with increasing n for small values of q, the series

$$\sum_{m=1}^{\infty} B_m^{(n)} I_m'(\Omega) \text{ and } \sum_{m=0}^{\infty} \left[B_m^{(n)} (\log \frac{k}{2} + \frac{\pi i}{2}) - T_m^{(n)} \right] H_m^*(\Omega) = K_n^*(\Omega)$$

are difficult to compute. The terms of the last series appear to approach those of a geometrical series for large m.

Following R 53, Int 2.5 (3.1') we may write

$$B_{2r}^{(n)} = \frac{(-\tau)^r}{(r-p)!(r+p)!} \gamma_{0,p}^{(n)} \qquad (5.7)$$

$$B_{2r+1}^{(n)} = \frac{(-\tau)^r}{(r+p+1)!(r-p)!} \gamma_{0,p}^{(n)} \qquad (5.8)$$

when q is small and r large.

We shall use frequently the abbreviations

$$\tau = \frac{q}{4} \quad ; \quad k = q^2 \quad ; \quad k = \frac{\beta\Omega}{2} \quad . \qquad (5.9)$$

Following R 53, Int 2.14 (10.3), we have

$$T_{2r}^{(n)} = \frac{(-\tau)^r}{(r-p)!(r+p)!} \gamma_{0,p}^{(n)} \left\{ \frac{\psi(r-p)+\psi(r+p+1)}{2} \right\} \qquad (5.10)$$

$$T_{2r}^{(n)} = \frac{(-\tau)^r}{(r+p+1)!(r-p)!} \gamma_{0,p}^{(n)} \left\{ \frac{\psi(r-p)+\psi(r+p+1)}{2} \right\} \qquad (5.11)$$

$$H_m(\Omega) \sim -(i)^{-m} \frac{\pi}{2} Y_m(\Omega) \sim i^{-m} \frac{(m-1)!}{2} (\frac{2}{\Omega})^m \qquad (5.12)$$

when Ω is small enough.

Taking p = 0 we easily get the asymptotic approximation for odd n:

$$K_n^*(\Omega) = \frac{-i\,\gamma_{0,0}^{(n)}}{\Omega} \sum_{r=1}^{\infty} (\frac{\beta}{2})^{2r} \frac{(2r)!}{(r+1)!\,r!} \left\{ \log \frac{k}{2} + \frac{\pi i}{2} - \frac{\psi(r)+\psi(r+1)}{2} \right\} , \qquad (5.1)$$

and for even n

$$K_n^*(\Omega) = \frac{\gamma_{0,0}^{(n)}}{2} \sum_{r=1}^{\infty} (\frac{\beta}{2})^{2r} \frac{(2r-1)!}{r!\,r!} \left\{ \log \frac{k}{2} + \frac{\pi i}{2} - \psi(r) \right\} . \qquad (5.14)$$

These approximations, however, are not so suitable as we should prefer. The parameter Ω is namely too large to make the asymptotic representation (5.12) effective. On the other side, the relation (5.14) gives a valuable hint for the computations. When m is still increasing the quotient of (two) successive terms must converge to β^2.

So we did put, when the actual quotiënt of the m^{th} term by the next one is named Q_m,

$$Q_m = \beta^2 + \frac{\alpha_1}{m} + \frac{\alpha_2}{m(m+1)} + \dots \tag{5.15}$$

First, this was used as a check upon original computations, but, afterwards, we could calculated the α_1, α_2, etc. from the $Q_m (m = 9,\dots 15)$ and then compute the other terms of our series $K_n^*(\Omega)$ as a product of Q_m and the $(m-1)^{th}$ term.

All calculations of $K_n(\beta,\Omega)$ and $Q_n(\beta,\Omega)$ were made with a absolute accuracy of 5 digits.

6. Computation of $\frac{Ne_n^{(2)}(0)}{Ne_n^{(2)\prime}(0)}$.

Of course, it is easy by means of Chapter III of R 53, Int 2 to calculate

$$\frac{Ne_n^{(2)}(0)}{Ne_n^{(2)\prime}(0)} = \frac{b\sum_{m=0}^{\infty} g_m^{(n)}}{a\sum_{m=1}^{\infty} m\, B_m^{(m)}} \tag{6.1}$$

This is, however, the right place for a check. Using the expansion of the Mathieu-functions in Bessel-function-product-series (cf. e.g. Mac Lachlan, Theory and Application of Mathieu-functions, p. 251):

We have:

$$Ne_{2n}^{(2)}(\xi) = \sum_{r=1}^{\infty} (-1)^r B_{2r}^{(2n)} \left\{ J_{r-1}(ke^{-\xi}) H_{r+1}^{(2)}(ke^{\xi}) - J_{r+1}(ke^{-\xi}) H_{r-1}^{(2)}(ke^{\xi}) \right\} \tag{6.2}$$

$$Ne_{2n+1}^{(2)}(\xi) = \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{(2n+1)} \left\{ J_r(ke^{-\xi}) H_{r+1}^{(2)}(ke^{\xi}) - J_{r+1}(ke^{-\xi}) H_{r-1}^{(2)}(ke^{\xi}) \right\} \tag{6.3}$$

$$Ne_{2n}^{(2)\prime}(\xi) = k \sum_{r=0}^{\infty} (-1)^r B_{2r}^{(2n)} \Big\{ -e^{-\xi} J'_{r-1}(ke^{-\xi}) H_{r+1}^{(2)}(ke^{\xi}) + e^{-\xi} J'_{r+1}(ke^{-\xi}) .$$
$$. H_{r-1}^{(2)}(ke^{\xi}) + e^{\xi} J_{r-1}(ke^{-\xi}) H_{r+1}^{(2)\prime}(ke^{\xi}) - e^{\xi} J_{r+1}(ke^{-\xi}) H_{r-1}^{(2)\prime}(ke^{\xi}) \Big\} \tag{6.4}$$

$$Ne_{2n+1}^{(2)\prime}(\xi) = k \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{(2n+1)} \Big\{ -e^{-\xi} J'_r(ke^{-\xi}) H_{r+1}^{(2)}(ke^{\xi}) + \tag{6.5}$$

$$+ e^{-\xi} J'_{r+1}(ke^{-\xi}) H_r^{(2)}(ke^{\xi}) + e^{\xi} J_r(ke^{-\xi}) H_{r+1}^{(2)\prime}(ke^{\xi}) - e^{+\xi} J_{r+1}(ke^{-\xi}) H_r^{(2)\prime}($$

We substitute the values of $J_r'(ke^{-\xi})$, $H_r^{(2)}(ke^{\xi})$ and $H_r^{(2)'}(ke^{\xi})$ and after a few reductions we get

$$Ne_{2n}^{(2)'}(0)=\sum_{r=1}^{\infty}(-1)^r B_{2r}^{(2n)}\Big[4r\Big\{J_r^2(k)-J_{r-1}(k)J_{r+1}(k)\Big\}+$$
$$+\, i\, 2r\Big\{J_{r-1}(k)Y_{r+1}(k)-2\,J_r(k)Y_r(k)+J_{r+1}(k)Y_{r-1}(k)\Big\}\Big] \tag{6.6}$$

$$Ne_{2n}^{(2)}(0)=\frac{2i}{\pi k^2}\sum_{r=1}^{\infty}(-1)^r\, 2r\, B_{2r}^{(2n)}=\frac{2i}{\pi k^2}\, se_{2n}'(\tfrac{\pi}{2}), \tag{6.7}$$

$$Ne_{2n+1}^{(2)'}(0)=\sum_{r=0}^{\infty}(-1)^r B_{2r+1}^{(2n+1)}\Big[2\Big\{k\Big\{J_r^2(k)-J_{r-1}(k)J_{r+1}(k)\Big\}\Big\}+$$
$$+\, i\Big\{k\Big\{J_{r-1}(k)Y_{r+1}(k)-2\,J_r(k)Y_r(k)+J_{r+1}(k)Y_{r-1}(k)+J_{r+1}(k)Y_r(k)+$$
$$J_r(k)\,Y_{r+1}(k)\Big\}\Big\}\Big], \tag{6.8}$$

$$Ne_{2n+1}^{(2)}(0)=\frac{2i}{\pi k}\sum_{r=0}^{\infty}(-1)^r B_{2r+1}^{(2n+1)}=\frac{2i}{\pi k}\, se_{2n+1}(\tfrac{\pi}{2}). \tag{6.9}$$

It may be noticed that the computations made by means of (6.1) and those made by means of (6.6) till (6.9) gives the same result in 8 digits.

7. The big catalogue.

Now we are able to compute several functions we need. All computations were made as far as necessary. So we have:

$$s_n=\sum_{m=1}^{\infty}B_m^{(n)} \tag{7.1}$$

$$se_n(\vartheta)=\sum_{m=1}^{\infty}B_m^{(n)}\sin m\vartheta \tag{7.2}$$

$$se_n'(\vartheta)=\sum_{m=1}^{\infty}m\,B_m^{(n)}\cos m\vartheta \tag{7.3}$$

$$se_n'(o)=\sum_{m=1}^{\infty}m\,B_m^{(n)} \tag{7.4}$$

$$se_n(\tfrac{\pi}{2})=\sum_{m=1}^{\infty}B_m^{(n)}\sin m\,\tfrac{\pi}{2} \qquad (n\text{ odd}) \tag{7.5}$$

$$se'_n\left(\frac{\pi}{2}\right)= \sum_{m=1}^{\infty} m\, B_m^{(n)} \cos m \frac{\pi}{2} \qquad (n \text{ even}) \tag{7.6}$$

$$\mu_n = \frac{B_1^{(n)}}{2\sqrt{1-\beta^2}\, se_n\left(\frac{\pi}{2}\right)} \qquad (n \text{ odd}) \tag{7.7}$$

$$\mu_n = -\frac{i\beta\Omega B_2^{(n)}}{4\sqrt{1-\beta^2}\, se'_n\left(\frac{\pi}{2}\right)} \qquad (n \text{ even}) \tag{7.8}$$

$$Q_n/A = -\frac{2\omega^2}{\sqrt{1-\beta^2}}\, \mu_n^*\, se_n(\vartheta) \tag{7.9}$$

The asterisk denotes the conjugate complex function.

$$Q_n/B = 2\, i\, \mu_n^* \left\{ \frac{se_n(\vartheta)}{\sqrt{1-\beta^2}} + \frac{se'_n(\vartheta)}{\beta} \right\} \tag{7.10}$$

Then

$$Z_1 = J_0(\beta^2\Omega) - i\, J_1(\beta^2\Omega) + 2 \sum_{n=1}^{\infty} (-1)^n \left\{ se'_n(0) \frac{Ne_n^{(2)}(0)}{Ne_n^{(2)\prime}(0)} + \sum_{m=1}^{\infty} B_m^{(n)} \right\} \mu_n\, se_n(\vartheta) \tag{7.11}$$

$$Z_2 = \sum_{n=1}^{\infty} Q_n/A \frac{Ne_n^{(2)}(0)}{Ne_n^{(2)\prime}(0)}\, \mu_n\, se_n(\vartheta) \tag{7.12}$$

$$Z'_2 = \sum_{n=1}^{\infty} Q_n/B \frac{Ne_n^{(2)}(0)}{Ne_n^{(2)\prime}(0)}\, \mu_n\, se_n(\vartheta) \tag{7.13}$$

$$Z_3 = -\frac{1}{2} J_0(\beta^2\Omega) + i\, J_1(\beta^2\Omega) + \frac{1}{2} J_2(\beta^2\Omega) - \frac{2i}{\Omega\sqrt{1-\beta^2}} \sum_{n=1}^{\infty} (-1)^n \cdot \left\{ se'_n(0) \frac{Ne_n^{(2)}(0)}{Ne_n^{(2)\prime}(0)} + \sum_{m=1}^{\infty} B_m^{(n)} \right\} x\, \mu_n \left\{ \frac{se_n(\vartheta)}{\sqrt{1-\beta^2}} - \frac{se'_n(\vartheta)}{\beta} \right\} \tag{7.14}$$

$$Z_4 = -\frac{i}{\Omega\sqrt{1-\beta^2}} \sum_{n=1}^{\infty} Q_n/A \frac{Ne_n^{(2)}(0)}{Ne_n^{(2)\prime}(0)}\, \mu_n \left\{ \frac{se_n(\vartheta)}{\sqrt{1-\beta^2}} - \frac{se'_n(\vartheta)}{\beta} \right\} \tag{7.15}$$

$$Z_4' = \frac{-i}{\Omega\sqrt{1-\beta^2}} \sum_{n=1}^{\infty} Q_n/B \frac{Ne_n^{(2)}(0)}{Ne_n^{(2)\prime}(0)} \mu_n \left\{ \frac{se_n(\vartheta)}{\sqrt{1-\beta^2}} - \frac{se_n'(\vartheta)}{\beta} \right\} \tag{7.16}$$

$$Z_5 = \frac{\pi}{2}\Omega \left\{ H_0^{(2)}(\Omega) - i\, H_1^{(2)}(\Omega) \right\} \tag{7.17}$$

$$Z_6 = \frac{-i\omega}{\sqrt{1-\beta^2}} \quad e^{-i\omega} \tag{7.18}$$

$$Z_7 = -\frac{(1-i\omega)}{\sqrt{1-\beta^2}} \quad e^{-i\omega} \tag{7.19}$$

$$a_0^{(1)} = \frac{\frac{2\omega^2}{\sqrt{1-\beta^2}} \sum_{n=1}^{\infty} \Lambda_n(\beta,\Omega)\mu_n^* \; se_n(\vartheta) + Z_6}{Z_5 + 2 \sum_{n=1}^{\infty} K_n(\beta,\Omega)} \tag{7.20}$$

$$a_0^{(2)} = \frac{-\sum_{n=1}^{\infty} \Lambda_n(\beta,\Omega) Q_n/B + Z_7}{Z_5 + 2 \sum_{n=1}^{\infty} K_n(\beta,\Omega)} \tag{7.21}$$

and then the coefficients used in (9.22) page 31 of the report F 54 of the National Aeronautical Research Institute are found by

$$k_a = 2\, a_0^{(1)} \; Z_1 + 2\, Z_2 \tag{7.22}$$

$$k_b = 2\, a_0^{(2)} \; Z_1 + 2\, Z_2' \tag{7.23}$$

$$m_a = 2\, a_0^{(1)} \; Z_3 + 2\, Z_4 \tag{7.24}$$

$$m_b = 2\, a_0^{(2)} \; Z_3 + 2\, Z_4' \tag{7.25}$$

All computations of these coefficients were made in 5 figures. The inaccuracy may be 3 or 4 units of the last decimal except for the highest value and the smallest values of Ω. In the last case the inaccuracy will be much better, i.e. 1 or 2 units, in the other the inaccuracy may exceed perhaps 4 units.