THE OSCILLATING WING IN A SUBSONIC FLOW.

R 53, Int 7.

Computation Department Mathematical Centre.

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INTRODUCTION

This report R 53, Int 7 is the seventh of a number of interim reports giving information about computations carried out by the Computation Department of the Mathematical Centre on behalf of the National Aeronautical Research Institute in Amsterdam under contract R 53. The final report R 53 that will be made up eventually will not contain much else than the final results of the computations and, moreover, will be not available for general distribution. As however in the course of the computations a lot of information has to be compiled for internal use, and part of this information may be of some value to others, this compilation will be done in the form of interim reports, that will be made available for limited circulation.

Summary.

Section 2 of this report deals with the computation of the function

$$L_1(\Omega,\eta_1) = \sin\eta_1 \int_0^\infty e^{-i\Omega\cosh\xi} \frac{d\xi}{\cosh\xi + \cos\eta_1} . \qquad (1,1)$$

The other sections are devoted to the summation of the following series

$$\sum_{n=1}^{\infty} G_n(\beta,\Omega,\eta_1), \qquad (1,2)$$

$$\sum_{n=1}^{\infty} \left\{ se_n(\eta_1) \frac{Ne_n^{(2)}(0)}{Ne_n^{(2)\prime}(0)} + \sum_{m=1}^{\infty} B_m^{(n)} \frac{\sin m\eta_1}{m} \right\} S_n(\beta,\Omega,\eta_1), \qquad (1,3)$$

$$\sum_{n=1}^{\infty} \left\{ se_m(\eta_1) \frac{Ne_n^{(2)}(0)}{Ne_n^{(2)\prime}(0)} + \sum_{m=1}^{\infty} B_m^{(n)} \frac{\sin m\eta_1}{m} \right\} T_n(\beta,\Omega,\eta_1), \qquad (1,4)$$

$$\sum_{n=1}^{\infty} \frac{Ne_n^{(2)}(0)}{Ne_n^{(2)\prime}(0)} S_n(\beta,\Omega,\eta_1)\, S_n^*(\beta,\Omega,\eta_1), \qquad (1,5)$$

$$\sum_{n=1}^{\infty} \frac{Ne_n^{(2)}(0)}{Ne_n^{(2)\prime}(0)} S_n(\beta,\Omega,\eta_1)\, T_n^*(\beta,\Omega,\eta_1), \qquad (1,6)$$

$$\sum_{n=1}^{\infty} \frac{Ne_n^{(2)}(0)}{Ne_n^{(2)\prime}(0)} S_n^*(\beta,\Omega,\eta_1)\, T_n(\beta,\Omega,\eta_1), \qquad (1,7)$$

$$\sum_{n=1}^{\infty} \frac{Ne_n^{(2)}(0)}{Ne_n^{(2)\prime}(0)} T_n(\beta,\Omega,\eta_1)\, T_n^*(\beta,\Omega,\eta_1), \qquad (1,8)$$

$$\sum_{n=1}^{\infty} \Lambda_n(\beta,\Omega)\, S_n^*(\beta,\Omega,\eta_1) \qquad (1,9)$$

$$\sum_{n=1}^{\infty} \Lambda_n(\beta,\Omega)\, T_n^*(\beta,\Omega,\eta_1). \qquad (1,10)$$

The functionsymbols are explained in former reports, and the asterisk denotes the complex conjugated. Section 3 gives the method used, section 4 some expansions, and, finally, results are given in sections 5 through 12.

2. Computation of $L_1(\Omega,\eta_1)$.

From the definition (7,15) of the report F 54 of the National Aeronautical Research Institute, viz.

$$L_1(\Omega,\eta_1)= \sin\eta_1 \int_0^\infty e^{-i\Omega\cosh\xi} \frac{d\xi}{\cosh\xi+\cos\eta_1}, \qquad (2,1)$$

one sees that the integral converges for values of Ω with a non-positive imaginary part. If one chooses the imaginary part of Ω to be negative then it is easily proved by means of

$$\frac{\partial L_1(\Omega,\eta_1)}{\partial\Omega} = -i\sin\eta_1 \int_0^\infty e^{-i\Omega\cosh\xi} \frac{\cosh\xi\, d\xi}{\cosh\xi+\cos\eta_1}$$

that $L_1(\Omega,\eta_1)$ satisfies the differential equation of the first order:

$$\frac{d L_1(\Omega,\eta_1)}{d\Omega} - i\cos\eta_1 L_1(\Omega,\eta_1)= -\frac{\pi}{2}\sin\eta_1 H_0^{(2)}(\Omega). \qquad (2,2)$$

Solving this equation one has

$$L_1(\Omega,\eta_1)= e^{i\Omega\cos\eta_1}\left\{\eta_1 - \frac{\pi}{2}\sin\eta_1 \int_0^\Omega e^{-i\cos\eta_1 t} H_0^{(2)}(t)dt\right\}, \qquad (2,3)$$

where the following identity is used:

$$L_1(0,\eta_1)= \sin\eta_1 \int_0^\infty \frac{d\xi}{\cosh\xi+\cos\eta_1} = 2\int_{\frac{1+\cos\eta_1}{\sin\eta_1}}^\infty \frac{dt}{1+t^2}=\eta_1. \qquad (2,4)$$

Formula (2,3) can be justified also for real values of Ω by means of analytic continuation.

Putting $c = \cos\eta_1$, and using

$$A = \int_0^\Omega \cos ct\, J_0(t)dt - \int_0^\Omega \sin ct\, Y_0(t)dt, \qquad (2,5)$$

$$B = \int_0^\Omega \sin ct\, J_0(t)dt + \int_0^\Omega \cos ct\, Y_0(t)dt, \qquad (2,6)$$

yields

$$\mathrm{Re}\, L_1(\Omega,\eta_1)= \sin\eta_1\left\{\frac{\eta_1}{\sin\eta_1}\cos c\Omega - \frac{\pi}{2}A\cos c\Omega - \frac{\pi}{2}B\sin c\Omega\right\}, \qquad (2,7)$$

$$\mathrm{Im}\, L_1(\Omega,\eta_1)= \sin\eta_1\left\{\frac{\eta_1}{\sin\eta_1}\sin c\Omega - \frac{\pi}{2}A\sin c\Omega + \frac{\pi}{2}B\cos c\Omega\right\}, \qquad (2,8)$$

So the problem is now to compute A and B. Following L. Schwarz (Luftfahrtforschung 1944; 341-372) the following functions are defined:

$$
\begin{aligned}
Jc(\lambda,x) &= \int_0^x J_0(\lambda t)\cos t\, dt,\\
Js(\lambda,x) &= \int_0^x J_0(\lambda t)\sin t\, dt,\\
Yc(\lambda,x) &= \int_0^x Y_0(\lambda t)\cos t\, dt,\\
Ys(\lambda,x) &= \int_0^x Y_0(\lambda t)\sin t\, dt.
\end{aligned}
\qquad (2,9)
$$

Hence,

$$A = \left\{ Jc\left(\frac{1}{c},\frac{\Omega}{c}\right) - Ys\left(\frac{1}{c},\frac{\Omega}{c}\right)\right\},$$

$$B = \left\{ Js\left(\frac{1}{c},\frac{\Omega}{c}\right) + Yc\left(\frac{1}{c},\frac{\Omega}{c}\right)\right\}.$$

Schwarz tabulated the functions (2,9) for $\lambda = 0.1(0.1)1$. However, it is easily seen that here $\lambda = 1/c > 1$. Therefore the Computation Department tabulated the functions (2,9) also for values of $\lambda > 1$.

The computations are all made by numerical integration. Only, there rises a difficulty because of the singular part of $Y_0(t)$, viz. $\frac{2}{\pi} J_0(t)\log t$. When one integrates numerically the functions $Y_0(t)\sin ct$ and $Y_0(t)\cos ct$, one has to compute also the error caused by that singular term during the process of integration.

This error is found as follows:
Expand $J_0(t)\cos ct$ and $J_0(t)\sin ct$ into a power-series with respect to t. Now one first considers the error caused by numerical integration in $\int_0^\Omega t^n \log t\, d t$, $n = 0,1,\ldots,$ when exactly the same numerical process is used as is actually done in the numerical integration of the functions considered.

This error be $E_n(\Omega)$. $E_n(\Omega)$ is easily computed, because one can evaluate $\int_0^\Omega t^n \log t\, d t$ analytically.

The error caused by the singular term of $Y_0(t)$ in the integral $\int_0^\Omega \cos ct\ \ Y_0(t)dt$ resp. $\int_0^\Omega \sin ct\ \ Y_0(t)dt$ is given by

$$\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!n!2^{2n}}\sum_{k=0}^{\infty}\frac{(-1)^k c^{2k}}{(2k)!}\ E_{2k+2n}(\Omega),$$

resp.

$$\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!n!2^{2n}}\sum_{k=0}^{\infty}\frac{(-1)^k c^{2k+1}}{(2k+1)!}\ E_{2k+2n+1}(\Omega).$$

Both series are highly convergent and only three or four terms are needed in order to yield a result correct in 8 decimals.

This process, by the way, is valid for all linear operations in numerical analysis as e.g. interpolation, integration and differentiation.

A check can be made in another way, for

$$L_1(\Omega,\eta_1) = \sin\eta_1 \int_0^\infty e^{-i\Omega\cosh\xi} \frac{d\xi}{\cosh\xi + \cos\eta_1} =$$

$$= 2\sin\eta_1 \int_1^\infty \frac{e^{-i\frac{\Omega}{2}(x+\frac{1}{x})}\,dx}{1+2cx+x^2}\,, \qquad (2,10)$$

where $c = \cos\eta_1$,

$x = e^\xi$, $\frac{\Omega}{2} = \alpha$.

Be γ and γ^* the roots of the equation:

$x^2 + 2cx + 1 = 0$.

Hence, γ and γ^* are on the unit-circle. Indeed,

$$\gamma = -\cos\eta_1 + \sqrt{\cos^2\eta_1 - 1} = -\cos\eta_1 + i\sin\eta_1 = -e^{-i\eta_1},$$

$$\gamma^* = -\cos\eta_1 - i\sin\eta_1 = -e^{i\eta_1}\,.$$

So one has :

$$\frac{\gamma^n - \gamma^{*n}}{\gamma - \gamma^*} = (-1)^{n+1}\frac{e^{in\eta_1} - e^{-in\eta_1}}{e^{i\eta_1} - e^{-i\eta_1}} = (-1)^{n+1}\frac{\sin n\eta_1}{\sin\eta_1}\,.$$

Now one proceeds to compute the integral (2,10) as follows

$$\int_1^\infty e^{i\alpha x}\frac{e^{i\frac{\alpha}{x}}}{(x-\gamma)(x-\gamma^*)}\,dx = \int_1^\infty e^{i\alpha x}\frac{\sum_{h=0}^{\infty}\frac{(i\alpha)^h}{x^h h!}}{(x-\gamma)(x-\gamma^*)}\,dx$$

$$= \int_1^\infty e^{i\alpha x}\frac{1}{\gamma-\gamma^*}\left\{\sum_{n=0}^{\infty}\frac{\gamma^n}{x^{n+1}} - \sum_{n=0}^{\infty}\frac{\gamma^{*n}}{x^{n+1}}\right\}\sum_{h=0}^{\infty}\frac{(i\alpha)^h}{x^h\,h!}\,dx$$

$$= \int_1^\infty e^{i\alpha x}\sum_{j=0}^{\infty} x^{-j-1}\,dx\,,$$

with

$$A_j = \sum_{n=1}^{j}(-1)^{n+1}\frac{\sin n\eta_1}{\sin\eta_1}\frac{(i\alpha)^{j-n}}{(j-n)!}\,. \qquad (2,11)$$

Hence,

$$\int_1^{\infty} \frac{e^{i\alpha x}}{x^j}\, dx = \frac{e^{i\alpha}}{i\alpha} \sum_{h=1}^{j-1} \frac{(j-h-1)!}{(j-1)!} (i\alpha)^h + \frac{(i\alpha)^{j-1}}{(j-1)!} \int_1^{\infty} \frac{e^{i\alpha x}}{x}\, dx. \tag{2,12}$$

This result is obtained by means of partial integration. Also:

$$L_1(\Omega,\eta_1) = \sin\eta_1 \; 2e^{i\alpha} \left\{ \sum_{k=1}^{\infty} \frac{1}{(i\alpha)^k} \left[\sum_{j=k}^{\infty} A_j \frac{(i\alpha)^j}{j!} \right] (k-1)! \right\} + $$
$$+ 2\sin\eta_1 \sum_{j=1}^{\infty} A_j \frac{(i\alpha)^j}{j!} \int_1^{\infty} \frac{e^{i\alpha x}}{x}\, dx. \tag{2,13}$$

The terms of the series can be computed in the following way:

$$\frac{e^{i\alpha} n!}{(i\alpha)^{n+1}} \sum_{j=n+1}^{\infty} A_j \frac{(i\alpha)^j}{j!} = \frac{e^{i\alpha} n!}{(i\alpha)^{n+1}} \left[- \sum_{k=1}^{\infty} \frac{\sin k\eta_1}{\sin\eta_1} i^k J_k(\Omega) - \right.$$
$$\left. - \sum_{k=1}^{n} \sum_{\ell=0}^{n-k} \frac{\sin k\eta_1}{\sin\eta_1} (-1)^{n+1} \frac{(i\alpha)^{2\ell+k}}{\ell!(\ell+k)!} \right] \tag{2,14}$$

$$\sum_{j=1}^{\infty} A_j \frac{(i\alpha)^j}{j-1} = - \sum_{n=1}^{\infty} i^n \frac{\sin n\eta_1}{\sin\eta_1} J_n(\Omega). \tag{2,15}$$

By means (2,13), (2,14) and (2,15) the value of $L_1(\Omega,\eta_1)$ was checked for one value of Ω and found to be correct in 7 decimals. This sewed as an independent check against systematic errors.

3. The method of summation.

All series mentioned in section 1 are convergent but the convergence is very poor. In order to overcome this difficulty, the following method was used.

1) An expansion is found for the terms of the series such that e few terms of this expansion give enough accuracy for a fixed value of n. So, e.g. the terms of the series (1,3) and (1,4) are expanded with respect to η_1, and the first two terms are sufficient to get the terms correct in 5 decimal places.

2) This expansion is summed analytically from that value of n till infinity.

3) Numerically the first n terms of the series are computed and also the result of 2). The sum of both gives a result correct in 5.

Now, it may be observed that all required expansions can be split up into partial fractions, and so can be expressed by means of the following auxiliary functions

$$F(\zeta,\alpha) = \sum_{0}^{\infty} \frac{e^{i\zeta(n+\frac{\alpha}{2})}}{2n+\alpha} , \qquad (3,1)$$

$$G(\zeta,\alpha) = \sum_{0}^{\infty} (-1)^n \frac{e^{i\zeta(n+\alpha)}}{n+\alpha} , \qquad (3,2)$$

$$C_2^{(2k)}(2,\zeta) = \sum_{k+1}^{\infty} \frac{\cos 2n\zeta}{(2n)^2} = \frac{1}{4}\zeta^2 - \frac{\pi}{4}\zeta + \frac{\pi^2}{24} - \frac{\cos 2\zeta}{4} - \frac{\cos 4\zeta}{16} \dots \frac{\cos 2k\zeta}{(2k)^2} \quad (0 \leq \zeta \leq \pi), \qquad (3,3)$$

$$S_2^{(2k)}(2,\zeta) = \sum_{k+1}^{\infty} \frac{\sin 2n\zeta}{(2n)^2} = -\frac{\zeta}{2}\ln(2\sin\zeta) + \frac{1}{2}\int_0^{\zeta} x \operatorname{cotg} x\,dx - \frac{\sin 2\zeta}{4} - \dots - \frac{\sin 2k\zeta}{(2k)^2}, \quad (0 \leq \zeta \leq \pi), \qquad (3,4)$$

$$C_3^{(2k)}(2,\zeta) = \sum_{k+1}^{\infty} \frac{\cos 2n\zeta}{(2n)^2} = \frac{\zeta^2}{4} \ln(2\sin\zeta) - \frac{\zeta}{2}\int_0^{\zeta} x \operatorname{cotg} x\,dx + \frac{1}{4}\int_0^{\zeta} x^2 \operatorname{cotg} x\,dx + \frac{1}{8}\zeta(3) - \frac{\cos 2\zeta}{2^3} - \dots - \frac{\cos 2k\zeta}{(2k)^3}, \quad (0 \leq \zeta \leq \pi), \qquad (3,5)$$

$$S_3^{(2k)}(2,\zeta) = \sum_{k+1}^{\infty} \frac{\sin 2n\zeta}{(2n)^3} = \frac{\zeta^3}{12} - \frac{\pi\zeta^2}{8} + \frac{\pi^2}{24} - \frac{\sin 2\zeta}{2^3} - \dots - \frac{\sin 2k\zeta}{(2k)^3}, \quad (0 \leq \zeta \leq \pi), \qquad (3,6)$$

$$\overline{C}_2^{(k)}(\zeta) = \sum_{k+1}^{\infty} (-1)^n \frac{\cos n\zeta}{n^2} = \frac{\zeta^2}{4} - \frac{\pi^2}{12} + \frac{\cos\zeta}{1} - \frac{\cos 2\zeta}{2^2} \dots + (-1)^{k+1}\frac{\cos k\zeta}{k^2}, \quad (0 \leq \zeta \leq \pi), \qquad (3,7)$$

$$\overline{S}_2^{(k)}(\zeta) = \sum_{k+1}^{\infty} (-1)^n \frac{\sin n\zeta}{n^2} = -\zeta \ln(2\cos\zeta) + \int_{\zeta}^{\zeta} x \operatorname{cotg} x\,dx - 2\int_0^{\zeta/2} x \operatorname{cotg} x\,dx + \frac{\sin\zeta}{1^2} - \frac{\sin 2\zeta}{2^2} + \dots + (-1)^{k+1}\frac{\sin k\zeta}{k^2}, \quad (0 \leq \zeta \leq \pi), \qquad (3,8)$$

$$\overline{S}_3^{(k)}(\zeta) = \sum_{k+1}^{\infty} (-1) \frac{\sin n\zeta}{n^3} = \frac{\zeta^3}{12} - \frac{\pi^2\zeta}{12} + \frac{\sin\zeta}{1} - \frac{\sin 2\zeta}{2^3} + \dots + (-1)^{k+1}\frac{\sin k\zeta}{k^3}, \quad (0 \leq \zeta \leq \pi). \qquad (3,9)$$

The function (3,1) and (3,2) can be dealt with as follows:

$$F(\zeta,\alpha)= \sum_{5}^{\infty} {}^{n} \frac{e^{i\zeta(n+\frac{\alpha}{2})}}{2n+\alpha} \quad , \qquad \alpha \geqslant -9,$$

$$F_\zeta(\zeta,\alpha)= \frac{i}{2} \sum_{5}^{\infty} {}^{n} e^{i\zeta(n+\frac{\alpha}{2})} = \frac{ie^{i\zeta(5+\frac{\alpha}{2})}}{2(1-e^{i\zeta})} \quad .$$

Also,

$$F(\zeta,\alpha)= \int_{\pi}^{\zeta} \frac{ie^{iz(5+\frac{\alpha}{2})}}{2(1-e^{iz})} \, dz + F(\pi,\alpha) \; . \qquad (3,10)$$

In the same way,

$$G(\zeta,\alpha)= \int_{0}^{\zeta} \frac{ie^{i(\alpha+8)z}}{1+e^{iz}} \, dz + G(0,\alpha). \qquad (3,11)$$

So it is easily proved that for l being a positive integer

$$F(\zeta,\alpha)= -\frac{1}{2}\left[\ln\left(2 \sin \frac{\zeta}{2}\right)+ i \left(\frac{\zeta}{2} - \frac{\pi}{2}\right)\right] - \frac{1}{2} \sum_{k=1}^{l} \frac{1}{k} e^{ki\zeta},$$

$$\text{when } \alpha = -8 + 2l \qquad (3,12)$$

$$F(\zeta,\alpha)= \frac{1}{2} \ln \left(\operatorname{cotg} \frac{\zeta}{4}\right)+ \frac{\pi i}{4} - \sum_{k=1}^{l} \frac{e^{(2k+1)i z/2}}{2k+1} \; ,$$

$$\text{when } \alpha = -8 + 2l + 1, \qquad (3,13)$$

$$G(\zeta,\alpha)= (-1)^{l}\left\{ \left[\ln\left(2 \cos \frac{\zeta}{2}\right)+ \frac{i\zeta}{2}\right] + \sum_{k=1}^{l} (-1)^{k} \frac{e^{kiz}}{k} \right\},$$

$$\text{when } \alpha = -7 + l. \qquad (3,14)$$

These auxiliary functions were computed in 8 figures.

4. Some expansions.

First it is necessary to give expansions for

$S_n(\beta,\Omega,\eta_1)$ and $T_n(\beta,\Omega,\eta_1)$ (See for the definitions R 53. Int 1).

Putting

$$A = e^{i\beta^2\Omega\cos\eta_1}, \qquad (4,1)$$

one sees easily:

$$R_n(\beta^2,\Omega,\eta_1) = \int_{0}^{\eta_1} e^{i\beta^2\Omega\cos\eta} \cos n\eta \, d\eta =$$

$$= \frac{A \sin n\eta_1}{n} - \frac{i\beta^2\Omega}{2n} \int_0^{\eta_1} e^{i\beta^2\Omega\cos\eta}\left[\cos(n+1)\eta - \cos(n-1)\eta\right] d\eta =$$

$$= \frac{A \sin n\eta_1}{n} - \frac{i\beta^2\Omega}{2n}\left[R_{n+1}(\beta^2\Omega,\eta_1) - R_{n-1}(\beta^2\Omega,\eta_1)\right]. \qquad (4,2)$$

Calculating $R_{n+1}(\beta^2,\Omega,\eta_1)$ and $R_{n-1}(\beta^2,\Omega,\eta_1)$ by means of (4,2) and substituting into (4,2), yields

$$R_n(\beta^2\Omega,\eta_1)\left\{1 + \left(\frac{\beta^2\Omega}{2}\right)^2 \frac{2}{(n+1)n(n-1)}\right\} = A\left\{\frac{\sin n\eta_1}{n} - \right.$$

$$\left. - \frac{i\beta^2\Omega}{2}\left[\frac{\sin(n+1)\eta_1}{(n+1)n} - \frac{\sin(n-1)\eta_1}{n(n-1)}\right]\right\} - \left(\frac{\beta^2\Omega}{2}\right)^2\left[\frac{R_{n+2}(\beta\Omega,\eta_1)}{(n+1)n} - \right.$$

$$\left. - \frac{R_{n-2}(\beta^2\Omega,\eta_1)}{n(n-1)}\right]. \qquad (4,3)$$

It is clear that after performing this process $[{}^2\log n+1] = p$ times, the expansion will diverge because of the factors in the denominator. Using only m steps ($m < p$) one gets a good approximation for $R_n(\beta,\Omega,\eta_1)$, when n is supposed to be large and it is even possible to state the order of the remainderterm, viz. $O(n^{-2^m-1}(\frac{\beta^2\Omega}{2})^{2^m+1}.\eta_1)$.

Neglecting the terms of orders lower than n^{-2} one finds the following expansions:

$$S_n(\beta,\Omega,\eta_1) = \int_0^{\eta_1} e^{i\beta^2\Omega\cos\eta}\, se_n(\eta)\sin\eta\, d\eta = \frac{1}{2}\sum_{r=1}^{\infty} B_r^{(n)}$$

$$\left\{R_{r-1}(\beta^2,\Omega,\eta_1) - R_{r+1}(\beta^2,\Omega,\eta_1)\right\} =$$

$$= \frac{A}{2}\left[\left\{\frac{\sin(n-1)\eta_1}{n-1} - \frac{\sin(n+1)\eta_1}{n+1}\right\} + \frac{i\beta^2\Omega}{2}\left\{\frac{\sin(n-2)\eta_1}{(n-2)(n-1)} - \frac{2\sin n\eta_1}{(n-1)(n+1)} + \right.\right.$$

$$+ \left.\frac{\sin(n+2)\eta_1}{(n+1)(n+2)}\right\} + \tau\left\{\frac{\sin(n-3)\eta_1}{(n-3)(n-1)} - \frac{\sin(n-1)\eta_1}{(n-1)^2} - \frac{\sin(n+1)\eta_1}{(n+1)^2} + \right.$$

$$\left.\left. + \frac{\sin(n+3)\eta_1}{(n+1)(n+3)}\right\}\right] + O(n^{-3}), \qquad (4,4)$$

and

$$T_n(\beta,\Omega,\eta_1) = \int_0^{\eta_1} e^{i\beta^2\Omega\cos\eta}\, se_n(\eta)\cos\eta\sin\eta\, d\eta =$$

$$= \frac{1}{4}\sum_{r=1}^{\infty} B_r^{(n)}\left\{R_{r-2}(\beta^2\Omega,\eta_1) - R_{r+2}(\beta^2\Omega,\eta_1)\right\} = \qquad (4,5)$$

$$= \frac{A}{4}\left[\left\{\frac{\sin(n-2)\eta_1}{n-2} - \frac{\sin(n+2)\eta_1}{n+2}\right\} + \frac{i\beta^2\Omega}{2}\left\{\frac{\sin(n-3)\eta_1}{(n-3)(n-2)} - \frac{\sin(n-1)\eta_1}{(n-2)(n-1)} - \right.\right.$$

$$\left. - \frac{\sin(n+1)\eta_1}{(n+1)(n+2)} + \frac{\sin(n+3)\eta_1}{(n+2)(n+3)}\right\} + \tau\left\{\frac{\sin(n-4)\eta_1}{(n-4)(n-1)} - \frac{2\sin n\eta_1}{(n-1)(n+1)} + \frac{\sin(n+4)\eta_1}{(n+1)(n+4)}\right.$$

One needs also

$$\frac{Ne_n^{(2)}(0)}{Ne_n^{(2)\prime}(0)} = -\frac{1}{n} - \frac{4\tau}{(n-1)n(n+1)} - \frac{(5n^2-11)4\tau^2}{(n-2)(n-1)^2n(n+1)^2(n+2)} \cdots, \tag{4,6}$$

which result is obtained by means of section 6 of the report R 53 Int 4 and the well-known series for $J_n(x)$ and $Y_n(x)$.

From (4,6) and the expansions of R 53 Int 5 (3,5) it follows that

$$\left\{se_n(\eta_1)\frac{Ne_n^{(2)}(0)}{Ne_n^{(2)\prime}(0)} + \sum_{m=1}^{\infty} B_m^{(n)}\frac{\sin m\eta_1}{m}\right\} = \tau\left\{\frac{2\sin(n-2)\eta_1}{(n-2)(n-1)n} - \frac{4\sin n\eta_1}{(n-1)n(n+1)} + \frac{2\sin(n+2)\eta_1}{n(n+1)(n+2)}\right\} +$$
$$+\tau^2\left\{\frac{2\sin(n-4)\eta_1}{(n-4)(n-2)(n-1)n} - \frac{4\sin(n-2)\eta_1}{(n-1)^2n(n+1)} + \frac{4\sin(n+2)\eta_1}{(n-1)n(n+1)^2} \cdots - \frac{2\sin(n+4)\eta_1}{n(n+1)(n+2)(n+4)}\right\} + O(n^{-5}). \tag{4,7}$$

Indeed, it is clear that

$$\Lambda_n(\Omega) = \int_0^\infty e^{-i\Omega\cosh\xi - n\xi}\,d\xi =$$
$$= \frac{1}{n} - \frac{i\Omega}{1!2}\left(\frac{1}{n-1}+\frac{1}{n+1}\right) - \frac{\Omega^2}{2!2^2}\left(\frac{1}{n-2}+\frac{2}{n}+\frac{1}{n+2}\right) +$$
$$+\frac{i\Omega^3}{3!2^3}\left(\frac{1}{n-3}+\frac{3}{n-1}+\frac{3}{n+1}+\frac{1}{n+3}\right)+\ldots+\frac{(-i\Omega)^N}{N!2^N}\left(\frac{1}{n-N}+\frac{N}{n-N+2}+\ldots\frac{1}{n+N}\right)+$$
$$+R_N(\Omega), \tag{4,8}$$

where $N < n$, N is a positive integer, and

$$R_N(\Omega) = \int_0^\infty \sum_{\ell=N}^{\infty}\frac{(-i\Omega\cosh\xi)^\ell}{\ell!}\,e^{-n\xi}\,d\xi, \tag{4,9}$$

because

$$\Lambda_n(\Omega) = \int_0^\infty \sum_{\ell=0}^{\infty}\frac{(-i\Omega\cosh\xi)^\ell}{\ell!}\,e^{-n\xi}\,d\xi =$$
$$= \sum_{\ell=0}^{N}\frac{(i\Omega)^\ell}{\ell!}\int_0^\infty e^{-n\xi}(\cosh\xi)^\ell\,d\xi + R_n(\Omega),$$

and evaluating the integrals in the last line yields readily formula (4,7). Again it may be noticed that all these expressions diverge when $N\to\infty$.

Finally it will be proved that

$$\Lambda_n(\beta,\Omega) = (-1)^{n+1}\left\{\sum_{m=1}^{\infty} B_m^{(n)}\Lambda_m(\Omega) - O\left(\tau^{\frac{1}{2}n}\cdot\left(\frac{n}{2}\right)^{-\frac{n}{2}-1}\cdot\frac{1}{\Omega}\right)\right. . \qquad (4,10)$$

Therefore one needs (5,5) and (5,6) of R 53, Int 4,7. First is to be shown

$$\vartheta^{(n)} = \frac{n!(n-1)!\tau^{-n}}{4} + \dots , \qquad (4,11)$$

In this order only the case of even n is dealt with. The case of odd n is similar.

Use is made of some formula's of R 53, Int 2 (9,2).

$$T_{2r}^{(2n)} = -\frac{1}{2}(-\tau)^r \frac{d}{dr}\left\{(-\tau)^{-r} B_{2r}^{(2n)}\right\} , \qquad (4,12)$$

and

$$\vartheta^{(2n)} = \frac{b_{2n}T_0^{(2n)} - 2q\, T_2^{(2n)}}{2q\, B_2^{(2n)}} = \frac{b_{2n}T_0^{(2n)} - 8\tau\, T_2^{(2n)}}{8\tau B_2^{(2n)}} . \qquad (4,13)$$

Furthermore, $T_{2r}^{(2n)}$ is the solution of the recurrence-relations of $T_m^{(2n)}$ that becomes small for large values of r. Hence, one has only to substitute the expressions (see R 53, Int 5, (3,5).

$$B_{2r}^{(2n)} = \frac{(-1)^{r-n}(2n)!\tau^{r-n}}{(r-n)!(r+n)!} + \dots ,$$

and

$$T_{2r}^{(2n)} = -\frac{1}{2}(-\tau)^{r-n}(2n)!\,\frac{\Psi(r-n)+\Psi(r+n)}{(r-n)!(r+n)!}$$

into (4,12) and (4,13), and (4,10) results.

By means of relation (3,6) of R 53, Int 4 it is shown that

$$|I_m'(\Omega)| < (n-1)!\ 2^{+n+1}\Omega^{\,n} .$$

So the largest term in (5,5) of Int 4 is

$$T_0^{(n)}\quad H_0^{*}(\Omega)$$

that has the order $\tau^{-\frac{1}{2}n}\, n!\left[\frac{1}{2}n\right]!$, where $\left[\frac{1}{2}n\right]$ denotes the greatest integer contained in $\frac{1}{2}n$.

Also,

$$\left|\frac{K_n(\beta,\Omega)}{Se_n'(0)}\right| < K_1\,\frac{\tau^{\frac{1}{2}n}}{\Omega}\left(\tfrac{1}{2}n\right)^{-\frac{1}{2}n+1},$$

where K_1 is some constant independent of n, τ and Ω, what yields the proof of (4,9).

If at last

$$\Lambda_N(\Omega) = \sum_{h=-N+1}^{h=N-1} \frac{\theta_h}{n+h} + R_N(\Omega), \qquad (4,14)$$

and one sees that except for (1,2), all series, mentioned in section 1, can be built up by the parts described above, and the method of section 3 is available to produce the results correct in 5 decimals using only the mentioned first two terms of the expansions.

For N = 9 the highest value of N taken in the computations, the functions θ_h are tabulated below

$$\begin{aligned}
\theta_0 &= 1 - \frac{\Omega^2}{4} + \frac{\Omega^4}{64} - \frac{\Omega^6}{2304} + \frac{\Omega^8}{147456}, \\
\theta_1 = \theta_{-1} &= i\left\{ -\frac{\Omega}{2} + \frac{\Omega^3}{16} - \frac{\Omega^5}{384} + \frac{\Omega^7}{18432} \right\}, \\
\theta_2 = \theta_{-2} &= -\frac{\Omega^2}{8} + \frac{\Omega^4}{96} - \frac{\Omega^6}{3072} + \frac{\Omega^8}{184320}, \\
\theta_3 = \theta_{-3} &= i\left\{ \frac{\Omega^3}{48} - \frac{\Omega^5}{768} + \frac{\Omega^7}{30720} \right\}, \\
\theta_4 = \theta_{-4} &= \frac{\Omega^4}{384} - \frac{\Omega^6}{7680} + \frac{\Omega^8}{368640}, \\
\theta_5 = \theta_{-5} &= i\left\{ \frac{-\Omega^5}{3840} + \frac{\Omega^7}{92160} \right\}, \\
\theta_6 = \theta_{-6} &= -\frac{\Omega^6}{46080} + \frac{\Omega^8}{1290240}, \\
\theta_7 = \theta_{-7} &= i\,\frac{\Omega^7}{645120}, \\
\theta_8 = \theta_{-8} &= \frac{\Omega^8}{10321920}.
\end{aligned} \qquad (4,15)$$

Now $G_n(\beta,\Omega,\eta_1)$ has to be treated, defined by

$$G_n(\beta,\Omega,\eta_1) = (-1)^n\, se_n'(0) \int_0^\infty e^{-i\Omega\cosh\xi} \left\{ se_n(\eta_1)\, \frac{Ne_n^{(2)}(\xi)}{Ne_n^{(2)\prime}(0)} + \sum_{m=1}^{\infty} B_m^{(n)} \frac{e^{-m\xi}\sin m\eta_1}{m} \right\} d\xi = se_n(\eta_1)\,\Lambda_n(\beta,\Omega) +$$

$$+ (-1)^n\, se_n'(0) \sum_{m=1}^{\infty} B_m^{(n)} \frac{\sin m\eta_1}{m}\, \Lambda_m(\Omega). \qquad (4,16)$$

Substituting (4,9), (4,14) and

$$se_n'(0) = \sum_{m=1}^{\infty} m\, B_m^{(n)},$$

one readily gets

$$G_n(\beta,\Omega,\eta_1) = (-1)^n \left\{ \sum_{\ell=1}^{\infty} B_h^{(n)} \, c_h^n(\eta_1) . \sum_{k=-\ell+1}^{\ell-1} \frac{\theta_k}{n+k} \right\} , \tag{4,17}$$

where

$$c_h^n(\eta_1) = \sum_{m=1}^{\infty} m \, B_m^{(n)} \left\{ \frac{\sin h\eta_1}{h} - \frac{\sin m\eta_1}{m} \right\} . \tag{4,18}$$

Because of the poor convergence of the series (4,14) it is necessary to take in (4,17) n equal to 8 and more. Then one has only to use the $B_n^{(n)}$, $B_{n-2}^{(n)}$ and $B_{n+2}^{(n)}$ multiplied with enough terms of the series for $\Lambda_m(\Omega)$ in order to get out results correct in 5 decimals when $\tau \leqslant \frac{1}{4}$.

5. $\sum_{12}^{\infty} G_n(\beta,\Omega,\eta_1)$.

$$\theta_0 . q/4 \; x$$

$$\Big\{ \mathrm{Im}\Big[G(\eta_1,4)\{ e^{2i\eta_1} + 2 + e^{-2i\eta_1} \} - G(\eta_1,2)\{ 1 + e^{2i\eta_1} \} -$$

$$- G(\eta_1,6)\{ 1 + e^{-2i\eta_1} \} \Big] + 2\{ \bar{S}_2^{(9)}(\eta_1) - S_2^{(13)}(\eta_1) \} \Big\}$$

$$- q/4 \, \frac{\theta_1}{2} \; x$$

$$\Big\{ \mathrm{Im}\Big[G(\eta_1,5)\{ 4e^{i\eta_1} - e^{i\eta_1} - e^{-3i\eta_1} \} + G(\eta_1,3)\{ 4\, e^{-i\eta_1} - e^{i\eta_1} - e^{3i\eta_1} \} +$$

$$+ G(\eta_1,7)\{ e^{-3i\eta_1} - 3\, e^{-i\eta_1} \} + G(\eta_1,1)\{ e^{3i\eta_1} - 3\, e^{i\eta_1} \} \Big] +$$

$$+ 8 \cos \eta_1 \{ - \bar{S}_2^{(10)}(\eta_1) + \bar{S}_2^{(12)}(\eta_1) \} \Big\}$$

$$- q/4 \, \frac{\theta_2}{3} \; x$$

$$\Big\{ \mathrm{Im}\Big[- 6\, G(\eta_1,4) + G(\eta_1,6)\{ 3 + 2\, e^{-2i\eta_1} - e^{-4i\eta_1} \} +$$

$$+ G(\eta_1,+2)\{ 3 + 2\, e^{2i\eta_1} - e^{4i\eta_1} \} + G(\eta_1,8)\{ e^{-4i\eta_1} - 2e^{-2i\eta_1} \} +$$

$$+ G(\eta_1,0)\{ e^{4i\eta_1} - 2\, e^{2i\eta_1} \} \Big] \Big\}$$

$$+ q/4 \, \frac{\theta_3}{12} \; x$$

$$\Big\{ \mathrm{Im}\Big[G(\eta_1,5)\{ 6\, e^{-i\eta_1} - 2\, e^{-3i\eta_1} \} + G(\eta_1,3)\{ 6\, e^{i\eta_1} - 2\, e^{3i\eta_1} \} +$$

$$+ G(\eta_1,7)\{ - 6\, e^{-i\eta_1} - 3\, e^{-3i\eta_1} + 3\, e^{-5i\eta_1} \}$$

$$+ G(\eta_1,9)\{ + 5\, e^{-3i\eta_1} - 3\, e^{-5i\eta_1} \} + G(\eta_1,-1)\{ 5\, e^{3i\eta_1} - 3\, e^{5i\eta_1} \} +$$

$$+ G(\eta_1,1)\{ - 6\, e^{i\eta_1} - 3\, e^{3i\eta_1} + 3\, e^{5i\eta_1} \} \Big] \Big\}$$

$$+ \,{}^{q}/_{4}\, \frac{\theta_4}{30} \times$$

$$\Big\{ \mathrm{Im}\big[G(\eta_1,-2)\{-6\,e^{6i\eta_1} + 9\,e^{4i\eta_1}\} + G(\eta_1,0)\{-10\,e^{2i\eta_1} - 4e^{4i\eta_1} + 6e^{6i\eta_1}\} +$$

$$+ G(\eta_1,+2)\{10\,e^{2i\eta_1} - 5\,e^{4i\eta_1}\} + G(\eta_1,10)\{-6\,e^{-6i\eta_1} + 9\,e^{-4i\eta_1}\} +$$

$$+ G(\eta_1,8)\{-10\,e^{-2i\eta_1} - 4\,e^{-4i\eta_1} + 6\,e^{6i\eta_1}\} +$$

$$+ G(\eta_1,6)\{10\,e^{-2i\eta_1} - 5\,e^{-4i\eta_1}\}\big] \Big\}$$

$$- \,{}^{q}/_{4}\, \frac{\theta_5}{60} \times$$

$$\Big\{ \mathrm{Im}\big[G(\eta_1,-3)\{-14\,e^{5i\eta_1} + 10\,e^{7i\eta_1}\} +$$

$$+ G(\eta_1,-1)\{15\,e^{3i\eta_1} + 5\,e^{5i\eta_1} - 10\,e^{+7i\eta_1}\} +$$

$$+ G(\eta_1,1)\{-15\,e^{3i\eta_1} + 9\,e^{5i\eta_1}\} +$$

$$+ G(\eta_1,11)\{-14\,e^{-5i\eta_1} + 10\,e^{-7i\eta_1}\} +$$

$$+ G(\eta_1,9)\{15\,e^{-3i\eta_1} + 5\,e^{-5i\eta_1} - 10\,e^{-7i\eta_1}\} +$$

$$+ G(\eta_1,7)\{-15\,e^{-3i\eta_1} + 9\,e^{-5i\eta_1}\}\big] \Big\}$$

$$- \,{}^{q}/_{4}\, \frac{\theta_6}{105} \times$$

$$\Big\{ \mathrm{Im}\big[G(\eta_1,-4)\{-20\,e^{6i\eta_1} + 15\,e^{8i\eta_1}\} +$$

$$+ G(\eta_1,-2)\{21\,e^{4i\eta_1} + 6\,e^{6i\eta_1} - 15\,e^{8i\eta_1}\} +$$

$$+ G(\eta_1,0)\{-21\,e^{4i\eta_1} + 14\,e^{6i\eta_1}\} +$$

$$+ G(\eta_1,12)\{-20\,e^{-6i\eta_1} + 15\,e^{-8i\eta_1}\} +$$

$$+ G(\eta_1,10)\{21\,e^{-4i\eta_1} + 6\,e^{-6i\eta_1} - 15\,e^{-8i\eta_1}\} +$$

$$+ G(\eta_1,8)\{-21\,e^{-4i\eta_1} + 14\,e^{-6i\eta_1}\}\big] \Big\}$$

$${}^{q}/_{4}\, \frac{\theta_7}{168} \times$$

$$\Big\{ \mathrm{Im}\big[G(\eta_1,-5)\{+27\,e^{+7i\eta_1} - 21\,e^{9i\eta_1}\} +$$

$$+ G(\eta_1,-3)\{-28\,e^{5i\eta_1} - 7\,e^{7i\eta_1} + 21\,e^{9i\eta_1}\} +$$

$$+ G(\eta_1,-1)\{28\,e^{5i\eta_1} - 20\,e^{7i\eta_1}\} +$$

$$+ G(\eta_1,13)\{27\,e^{-7i\eta_1} - 21\,e^{-9i\eta_1}\} +$$

$$+ G(\eta_1,11)\{-28\,e^{-5i\eta_1} - 7\,e^{-7i\eta_1} + 21\,e^{-9i\eta_1}\} +$$

$$+ G(\eta_1,9)\{28\,e^{-5i\eta_1} - 20\,e^{-7i\eta_1}\}\big] \Big\}$$

$$q/4 \ \frac{\theta_8}{252} \ x$$

$$\Big\{ \mathrm{Im}\big[G(\eta_1,-6)\{35\ e^{8i\eta_1} - 28\ e^{10i\eta_1}\} +$$

$$+ G(\eta_1,-4)\{-36\ e^{6i\eta_1} - 8\ e^{8i\eta_1} + 28\ e^{10i\eta_1}\} +$$

$$+ G(\eta_1,-2)\{36\ e^{6i\eta_1} - 27\ e^{8i\eta_1}\} +$$

$$+ G(\eta_1,14)\{35\ e^{-8i\eta_1} - 28\ e^{-10i\eta_1}\} +$$

$$+ G(\eta_1,12)\{-36\ e^{-6i\eta_1} - 8\ e^{-8i\eta_1} + 28\ e^{-10i\eta_1}\} +$$

$$+ G(\eta_1,10)\{36\ e^{-6i\eta_1} - 27\ e^{-8i\eta_1}\}\big]\Big\}$$

6. $$-2\sum_{5}^{\infty}{}_{n} \Big\{ se_n(\eta_1)\ \frac{Ne_n^{(2)}(0)}{Ne_n^{(2)\prime}(0)} + \sum_{1}^{\infty}{}_{m}\ B_m^{(n)}\ \frac{\sin m\eta_1}{m} \Big\}\ S_n(\beta,\Omega,\eta_1).$$

$$q/4\ A\Big\{-\frac{1}{3}\Big[\mathrm{Re}\{e^{3i\eta_1}F(2\eta_1,-4) - 3\ e^{i\eta_1}F(2\eta_1,-2) + 3\ e^{-i\eta_1}F(2\eta_1,0) -$$

$$- e^{-3i\eta_1}F(2\eta_1,2)\} - \frac{1}{60}\cos 3\eta_1\Big]$$

$$+ \Big[\mathrm{Re}\{e^{i\eta_1}F(2\eta_1,-4) - e^{-3i\eta_1}F(2\eta_1,0)\} -$$

$$- 4\cos\eta_1\{C_2^{(6)}(2,\eta_1) + \frac{119}{288} - \frac{\pi^2}{24}\} - 4\sin\eta_1 . S_2^{(6)}(2,\eta_1)\Big]$$

$$+ \Big[\mathrm{Re}\{e^{3i\eta_1}F(2\eta_1,-2) - 4\ e^{i\eta_1}F(2\eta_1,0) + 3\ e^{-i\eta_1}F(2\eta_1,2)\} +$$

$$+ 4\cos\eta_1\{C_2^{(10)}(2,\eta_1) + \frac{5899}{14400} - \frac{\pi^2}{24}\} + 4\sin\eta_1 . S_2^{(10)}(2,\eta_1)\Big]$$

$$+ \Big[\mathrm{Re}\{3\ e^{i\eta_1}F(2\eta_1,-2) + e^{-3i\eta_1}F(2\eta_1,2) - 4\ e^{-i\eta_1}F(2\eta_1,0) -$$

$$- 4\cos\eta_1\{C_2^{(6)}(2,\eta_1) - \frac{\pi^2}{24} + \frac{589}{1440}\} + 4\sin\eta_1 . S_2^{(5)}(2,\eta_1)\Big]$$

$$- \Big[\mathrm{Re}\{e^{3i\eta_1}F(2\eta_1,0) - e^{-i\eta_1}F(2\eta_1,4)\} -$$

$$- 4\cos\eta_1\{C_2^{(10)}(2,\eta_1) - \frac{\pi^2}{24} + \frac{5929}{14400}\} + 4\sin\eta_1 . S_2^{(10)}(2,\eta_1)\Big]$$

$$+ \frac{1}{3}\Big[\mathrm{Re}\{e^{3i\eta_1}F(2\eta_1,-2) - 3\ e^{i\eta_1}F(2\eta_1,0) +$$

$$+ 3\ e^{-i\eta_1}F(2\eta_1,2) - e^{-3i\eta_1}F(2\eta_1,4)\} - \frac{1}{120}\cos 3\eta_1\Big]\Big\}$$

$$+ i\, q/4\, \beta^2 \Omega A$$

$$\Big\{ \frac{1}{12}\Big[\mathrm{Re}\big\{e^{4i\eta_1} F(2\eta_1,-4) - 4\, e^{2i\eta_1} F(2\eta_1,-2) + 6\, F(2\eta_1,0) -$$

$$- 4\, e^{-2i\eta_1} F(2\eta_1,+2) + e^{-4i\eta_1} F(2\eta_1,4)\big\} - \frac{1}{120} \cos 4\eta_1\Big]$$

$$- \frac{1}{3}\Big[\mathrm{Re}\big\{2\, e^{2i\eta_1} F(2\eta_1,-4) + 3\, F(2\eta_1,-2) - 6\, e^{-2i\eta_1} F(2\eta_1,0) +$$

$$+ e^{-4i\eta_1} F(2\eta_1,2)\big\} - 12\, C_2^{(6)}(2,\eta_1) + \cos 2\eta_1\left(\frac{\pi^2}{2} - \frac{593}{120}\right)\Big]$$

$$+ \frac{1}{4}\Big[\mathrm{Re}\big\{-5\, F(2\eta_1,-4) + 4\, e^{-2i\eta_1} F(2\eta_1,-2) + e^{-4i\eta_1} F(2\eta_1,0)\big\}$$

$$+ 4\, C_2^{(4)}(2,\eta_1) + 8 \cos 2\eta_1 . C_2^{(6)}(2,\eta_1) + 8 \sin 2\eta_1 . S_2^{(6)}(2,\eta_1) -$$

$$- \frac{\pi^2}{2} + \frac{355}{72}\Big]$$

$$- \frac{1}{3}\Big[\mathrm{Re}\big\{e^{4i\eta_1} F(2\eta_1,-2) - 6\, e^{2i\eta_1} F(2\eta_1,0) + 3\, F(2\eta_1,2) + 2 e^{-2i\eta_1} F(2\eta_1,4)\big\}$$

$$+ 12\, C_2^{(10)}(2,\eta_1) - \cos 2\eta_1\left(\frac{\pi^2}{2} - \frac{5919}{1200}\right)\Big]$$

$$+ \Big[\mathrm{Re}\big\{-2\, e^{2i\eta_1} F(2\eta_1,-2) + 4\, F(2\eta_1,0) - 2\, e^{-2i\eta_1} F(2\eta_1,2)\big\} +$$

$$+ 2 \cos 2\eta_1\left(\frac{\cos 8\eta_1}{8^2} + \frac{\cos 10\eta_1}{10^2}\right)$$

$$- 2 \sin 2\eta_1 \Big\{2\, S_2(2,\eta_1) - \frac{\sin 2\eta_1}{2} - \frac{\sin 4\eta_1}{8} - \frac{\sin 6\eta_1}{18} -$$

$$- \frac{\sin 8\eta_1}{8^2} - \frac{\sin 10\eta_1}{10^2}\Big\} - \frac{1}{800}\Big]$$

$$+ \frac{1}{4}\Big[\mathrm{Re}\big\{e^{4i\eta_1} F(2\eta_1,0) + 4\, e^{2i\eta_1} F(2\eta_1,2) - 5\, F(2\eta_1,4)\big\} -$$

$$- 4\, C_2^{(12)}(2,\eta_1) - 8 \cos 2\eta_1 . C_2^{(10)}(2,\eta_1) +$$

$$+ 8 \sin 2\eta_1 . S_2^{(10)}(2\eta_1) + \frac{\pi^2}{2} - \frac{17767}{3600}\Big] \Big\}.$$

$$+ q^2/16\, A\, x$$

$$\Big\{ - \Big[\mathrm{Re}\big\{e^{i\eta_1} F(2\eta_1,-4) - 2\, e^{-i\eta_1} F(2\eta_1,-2) + e^{-3i\eta_1} F(2\eta_1,0)\big\} -$$

$$- 8 \cos\eta_1 \Big\{C_3^{(6)}(2,\eta_1) - \frac{1}{8}\zeta(3) + \frac{65}{432}\Big\} - 8 \sin\eta_1 . S_3^{(6)}(2,\eta_1)\Big]$$

$$+ \frac{1}{6}\Big[\mathrm{Re}\Big\{ e^{i\eta_1} F(2\eta_1,-6) - 6\, e^{-i\eta_1} F(2\eta_1,-4) + 3\, e^{-3i\eta_1} F(2\eta_1,-2) +$$

$$+ 2\, e^{-5i\eta_1} F(2\eta_1,0)\Big\} + 12 \cos 3\eta_1 . C_2^{(6)}(2,\eta_1) +$$

$$+ 12 \sin 3\eta_1 .\ S_2^{(6)}(2,\eta_1) - \cos\eta_1 \Big(\frac{\pi^2}{2} - \frac{59}{12} \Big)\Big]$$

$$+ \frac{1}{60}\Big[\mathrm{Re}\Big\{ 4\, e^{5i\eta_1} F(2\eta_1,-4) - 15\, e^{3i\eta_1} F(2\eta_1,-2) + 20\, e^{i\eta_1} F(2\eta_1,0) -$$

$$- 10\, e^{-i\eta_1} F(2\eta_1,2) + e^{-5i\eta_1} F(2\eta_1,6)\Big\} - \frac{31}{840} \cos 5\eta_1 \Big]$$

$$- \frac{1}{18}\Big[\mathrm{Re}\Big\{ 2\, e^{3i\eta_1} F(2\eta_1,-4) - 9\, e^{i\eta_1} F(2\eta_1,-2) + 18\, e^{-i\eta_1} F(2\eta_1,0) -$$

$$- 11\, e^{-3i\eta_1} F(2\eta_1,2)\Big\} - 12 \cos 3\eta_1 \Big\{ C_2^{(10)}(2,\eta_1) - \frac{\pi^2}{24} + \frac{5939}{14400} \Big\} -$$

$$- 12 \sin 3\eta_1 . S_2^{(10)}(2,\eta_1) \Big]$$

$$+ \frac{1}{2}\Big[\mathrm{Re}\Big\{ 7\, e^{i\eta_1} F(2\eta_1,-2) - 8\, e^{-i\eta_1} F(2\eta_1,0) + e^{-3i\eta_1} F(2\eta_1,2)\Big\} +$$

$$+ 48 \cos\eta_1 \Big\{ \frac{1}{3} C_3^{(6)}(2,\eta_1) - \frac{1}{4} C_2^{(6)}(2,\eta_1) - \frac{1}{24}\zeta(3) + \frac{\pi^2}{96} - \frac{2737}{5184} \Big\} -$$

$$- 48 \sin\eta_1 \Big\{ \frac{1}{3} S_3^{(6)}(2,\eta_1) - \frac{1}{4} S_2^{(6)}(2,\eta_1) \Big\}$$

$$- \frac{1}{12}\Big[\mathrm{Re}\Big\{ e^{3i\eta_1} F(2\eta_1,-6) + 12\, e^{-i\eta_1} F(2\eta_1,-2) - 16\, e^{-3i\eta_1} F(2\eta_1,0) +$$

$$+ 3\, e^{-5i\eta_1} F(2\eta_1,2)\Big\} - 24 \cos\eta_1 . C_2^{(6)}(2,\eta_1) -$$

$$- 24 \sin\eta_1 .\ S_2^{(6)}(2,\eta_1) - \cos 3\eta_1 \Big(\frac{1189}{120} - \pi^2 \Big)\Big]$$

$$- \frac{1}{12}\Big[\mathrm{Re}\Big\{ 3\, e^{5i\eta_1} F(2\eta_1,-2) - 16\, e^{3i\eta_1} F(2\eta_1,0) + 12\, e^{i\eta_1} F(2\eta_1,2) +$$

$$+ e^{-3i\eta_1} F(2\eta_1,6)\Big\} + 24 \cos\eta_1 . C_2^{(10)}(2,\eta_1) -$$

$$- 24 \sin\eta_1 . S_2^{(10)}(2,\eta_1) - \cos 3\eta_1 \Big(\pi^2 - \frac{20709}{2100} \Big)\Big]$$

$$+ \frac{1}{2}\Big[\mathrm{Re}\Big\{ e^{3i\eta_1} F(2\eta_1,-2) - 8\, e^{i\eta_1} F(2\eta_1,0) + 7\, e^{-i\eta_1} F(2\eta_1,2)\Big\} +$$

$$+ 48 \cos\eta_1 \Big\{ \frac{1}{3} C_3^{(10)}(2,\eta_1) + \frac{1}{4} C_2^{(10)}(2,\eta_1) - \frac{1}{24}\zeta(3) - \frac{\pi^2}{96} +$$

$$+ \frac{792413}{5184000} \Big\} +$$

$$+ 48 \sin\eta_1 \Big\{ \frac{1}{3} S_3^{(10)}(2,\eta_1) + \frac{1}{4} S_2^{(10)}(2,\eta_1) \Big\}\Big]$$

$$- \frac{1}{18}\Big[\mathrm{Re}\Big\{ -11\, e^{3i\eta_1} F(2\eta_1,-2) + 18\, e^{i\eta_1} F(2\eta_1,0) - 9\, e^{-i\eta_1} F(2\eta_1,2) + 2\, e^{-3i\eta_1} F(2\eta_1,4)\Big\} +$$

$$+ 12 \cos 3\eta_1 \Big\{ C_2^{(6)}(2,\eta_1) - \frac{\pi^2}{24} + \frac{591}{1440} \Big\} - 12 \sin 3\eta_1 . S_2^{(6)}(2,\eta_1) \Big]$$

$$+\frac{1}{60}\Big[\mathrm{Re}\Big\{e^{5i\eta_1}F(2\eta_1,-6)-10\,e^{i\eta_1}F(2\eta_1,-2)+20\,e^{-i\eta_1}F(2\eta_1,0)-$$

$$-15\,e^{-3i\eta_1}F(2\eta_1,2)+4\,e^{-5i\eta_1}F(2\eta_1,4)\Big\}-\frac{7\cos 5\eta_1}{120}\Big]$$

$$+\frac{1}{6}\Big[\mathrm{Re}\Big\{2\,e^{5i\eta_1}F(2\eta_1,0)+3\,e^{3i\eta_1}F(2\eta_1,2)-6\,e^{i\eta_1}F(2\eta_1,4)+e^{-i\eta_1}F(2\eta_1,6)\Big\}-$$

$$-12\cos 3\eta_1.C_2^{(10)}(2,\eta_1)-\cos\eta_1\Big(\frac{41463}{8400}-\frac{\pi^2}{2}\Big)+12\sin 3\eta_1.S_2^{(10)}(2,\eta_1)\Big]$$

$$-\Big[\mathrm{Re}\Big\{e^{3i\eta_1}F(2\eta_1,0)-2\,e^{i\eta_1}F(2\eta_1,2)+e^{-i\eta_1}F(2\eta_1,4)\Big\}-$$

$$-8\cos\eta_1\Big\{C_3^{(10)}(2,\eta_1)+\frac{259703}{1728000}-\frac{1}{8}\zeta(3)\Big\}+8\sin\eta_1.S_3^{(10)}(2,\eta_1)\Big]$$

$$-\frac{1}{60}\Big[\mathrm{Re}\Big\{e^{5i\eta_1}F(2\eta_1,-8)-10\,e^{i\eta_1}F(2\eta_1,-4)+20\,e^{-i\eta_1}F(2\eta_1,-2)-$$

$$-15\,e^{-3i\eta_1}F(2\eta_1,0)+4\,e^{-5i\eta_1}F(2\eta_1,2)\Big\}-\frac{9\cos 5\eta_1}{40}\Big]$$

$$+\Big[\mathrm{Re}\Big\{-2\,e^{i\eta_1}F(2\eta_1,-2)+4\,e^{-i\eta_1}F(2\eta_1,0)-2\,e^{-3i\eta_1}F(2\eta_1,2)\Big\}+$$

$$+2\cos\eta_1.C_2^{(6)}(2,\eta_1)-2\cos 3\eta_1\Big\{C_2^{(10)}(2,\eta_1)+\frac{1}{1600}\Big\}-$$

$$-2\sin\eta_1.S_2^{(6)}(2,\eta_1)-2\sin 3\eta_1.S_2^{(10)}(2,\eta_1)\Big]$$

$$-\frac{1}{2}\Big[\mathrm{Re}\Big\{e^{5i\eta_1}F(2\eta_1,-2)-8\,e^{3i\eta_1}F(2\eta_1,0)+7\,e^{i\eta_1}F(2\eta_1,2)\Big\}+$$

$$+48\cos\eta_1\Big\{\frac{1}{3}C_3^{(10)}(2,\eta_1)+\frac{1}{4}C_2^{(10)}(2,\eta_1)-\frac{1}{24}\zeta(3)-\frac{\pi^2}{96}+$$

$$+\frac{792413}{5184000}\Big\}-$$

$$-48\sin\eta_1\Big\{\frac{1}{3}S_3^{(10)}(2,\eta_1)+\frac{1}{4}S_2^{(10)}(2,\eta_1)\Big\}$$

$$+\frac{1}{36}\Big[\mathrm{Re}\Big\{9\,e^{5i\eta_1}F(2\eta_1,0)+8\,e^{3i\eta_1}F(2\eta_1,2)-18\,e^{i\eta_1}F(2\eta_1,4)+$$

$$+e^{-3i\eta_1}F(2\eta_1,8)\Big\}-$$

$$-48\cos 3\eta_1\Big\{C_2^{(10)}(2,\eta_1)+\frac{55289}{134400}-\frac{\pi^2}{24}\Big\}+48\sin 3\eta_1 S_2^{(10)}(2,\eta_1)\Big]$$

$$+\frac{1}{36}\Big[\mathrm{Re}\Big\{e^{3i\eta_1}F(2\eta_1,-8)-18\,e^{-i\eta_1}F(2\eta_1,-4)+8\,e^{-3i\eta_1}F(2\eta_1,-2)+$$

$$+9\,e^{-5i\eta_1}F(2\eta_1,0)\Big\}+$$

$$+48\cos 3\eta_1\Big\{C_2^{(6)}(2,\eta_1)-\frac{\pi^2}{24}+\frac{469}{1152}\Big\}+48\sin 3\eta_1.S_2^{(6)}(2,\eta_1)\Big]$$

$$-\frac{1}{2}\Big[\mathrm{Re}\Big\{7e^{-i\eta_1}F(2\eta_1,-2)-8e^{-3i\eta_1}F(2\eta_1,0)+e^{-5i\eta_1}F(2\eta_1,2)\Big\}+48\cos\eta_1\Big\{\frac{1}{3}C_3^{(6)}(2,\eta_1)-$$

$$-\frac{1}{4}C_2^{(6)}(2,\eta_1)-\frac{1}{24}\zeta(3)+\frac{\pi^2}{96}-\frac{2737}{51840}\Big\}+48\sin\eta_1\Big\{\frac{1}{3}S_3^{(6)}(2,\eta_1)-\frac{1}{4}S_2^{(6)}(2,\eta_1)\Big\}\Big]$$

$$+ \Big[\mathrm{Re}\Big\{-2e^{3i\eta_1} F(2\eta_1,-2)+4\ e^{i\eta_1} F(2\eta_1,0)-2\ e^{-i\eta_1} F(2\eta_1,2)\Big\}+$$

$$+2\cos 3\eta_1\Big\{C_2^{(6)}(2,\eta_1)-\frac{1}{1600}\Big\} - 2\cos\eta_1 . C_2^{(10)}(2,\eta_1) -$$

$$-2\sin 3\eta_1 . S_2^{(6)}(2,\eta_1)-2\sin\eta_1\ S_2^{(10)}(2,\eta_1)\Big]$$

$$-\frac{1}{60}\Big[\mathrm{Re}\Big\{4\ e^{5i\eta_1} F(2\eta_1,-2)-15\ e^{3i\eta_1} F(2\eta_1,0)+20\ e^{i\eta_1} F(2\eta_1,2)-$$

$$-10\ e^{-i\eta_1} F(2\eta_1,4)+e^{-5i\eta_1} F(2\eta_1,8)\Big\}-\frac{9\cos 5\eta_1}{560}\Big]\Big\}$$

7. $$-4\sum_{5}^{\infty}{}_{n}\Big\{ se_n(\eta_1)\frac{Ne_n^{(2)}(0)}{Ne_n^{(2)'}(0)} + \sum_{1}^{\infty}{}_{m} B_m^{(n)}\frac{\sin m\eta_1}{m}\Big\} T_n(\beta,\Omega,\eta_1).$$

$q/4$ A x

$$\Big\{-\frac{1}{12}\Big[\mathrm{Re}\Big\{+3\ e^{+4i\eta_1} F(2\eta_1,-4)-8\ e^{2i\eta_1} F(2\eta_1,-2)+6\ F(2\eta_1,0)-e^{-4i\eta_1} F(2\eta_1,4)\Big\}-$$

$$-\frac{7}{120}\cos 4\eta_1\Big]$$

$$+\frac{2}{3}\Big[\mathrm{Re}\ e^{4i\eta_1} F(2\eta_1,-2)-3\ e^{2i\eta_1} F(2\eta_1,0)+3\ F(2\eta_1,2)-e^{-2i\eta_1} F(2\eta_1,4)\Big\}-$$

$$-\frac{1}{120}\cos 2\eta_1\Big]$$

$$-\frac{1}{4}\Big[\mathrm{Re}\Big\{2\ e^{4i\eta_1} F(2\eta_1,0)-8\ e^{2i\eta_1} F(2\eta_1,2)+6\ F(2\eta_1,4)\Big\}+$$

$$+8\ C_2^{(12)}(2,\eta_1)-\frac{\pi^2}{3}+\frac{5909}{1800}\Big]$$

$$+\frac{1}{2}\Big[\mathrm{Re}\Big\{-3\ F(2\eta_1,-4)+4\ e^{-2i\eta_1} F(2\eta_1,-2)-e^{-4i\eta_1} F(2\eta_1,0)\Big\}+$$

$$+4\ C_2^{(4)}(2,\eta_1)+\frac{13}{8}-\frac{\pi^2}{6}\Big]$$

$$-\frac{2}{3}\Big[\mathrm{Re}\Big\{e^{2i\eta_1} F(2\eta_1,-4)-3\ F(2\eta_1,-2)+3\ e^{-2i\eta_1} F(2\eta_1,0)-e^{-4i\eta_1} F(2\eta_1,2)\Big\}-$$

$$-\frac{1}{60}\cos 2\eta_1\Big]$$

$$+\frac{1}{12}\Big[\mathrm{Re}\Big\{e^{4i\eta_1} F(2\eta_1,-4)-6\ F(2\eta_1,0)+8\ e^{-2i\eta_1} F(2\eta_1,2)-3\ e^{-4i\eta_1} F(2\eta_1,4)\Big\}-$$

$$-\frac{1}{24}\cos 4\eta_1\Big]\Big\}$$

$$+ i\, {}^{q}/4\, \beta^2 \Omega . A \; x$$

$$\Big\{ - \frac{1}{12} \Big[\mathrm{Re}\big\{ e^{5i\eta_1} F(2\eta_1,-2) - 4\, e^{3i\eta_1} F(2\eta_1,0) + 6\, e^{i\eta_1} F(2\eta_1,2) -$$

$$-4\, e^{-i\eta_1} F(2\eta_1,4) + e^{-3i\eta_1} F(2\eta_1,6) \big\} - \frac{1}{280} \cos 3\eta_1 \Big]$$

$$+ \frac{1}{12} \Big[\mathrm{Re}\big\{ e^{5i\eta_1} F(2\eta_1,0) - 6\, e^{3i\eta_1} F(2\eta_1,2) + 3\, e^{i\eta_1} F(2\eta_1,4) + 2 e^{-i\eta_1} F(2\eta_1,6) \big\} +$$

$$+12 \cos\eta_1 . C_2^{(12)}(2,\eta_1) - 12 \sin\eta_1 . S_2^{(12)}(2,\eta_1) - \cos\eta_1 \left(\frac{\pi^2}{2} - \frac{41443}{8400}\right) \Big]$$

$$- \frac{1}{24} \Big[\mathrm{Re}\big\{ e^{3i\eta_1} F(2\eta_1,-4) - 4\, e^{i\eta_1} F(2\eta_1,-2) + 6\, e^{-i\eta_1} F(2\eta_1,0) -$$

$$-4\, e^{-3i\eta_1} F(2\eta_1,2) + e^{-5i\eta_1} F(2\eta_1,4) \big\} - \frac{1}{120} \cos 3\eta_1 \Big]$$

$$+ \frac{1}{6} \Big[\mathrm{Re}\big\{ e^{3i\eta_1} F(2\eta_1,-2) - 6\, e^{i\eta_1} F(2\eta_1,0) + 3\, e^{-i\eta_1} F(2\eta_1,2) + 2 e^{-3i\eta_1} F(2\eta_1,4) \big\} +$$

$$+12 \cos\eta_1 . C_2^{(10)}(2,\eta_1) + 12 \sin\eta_1 . S_2^{(10)}(2,\eta_1) - \cos\eta_1 \left(\frac{\pi^2}{2} - \frac{1973}{400}\right) \Big]$$

$$- \frac{1}{4} \Big[\mathrm{Re}\big\{ e^{3i\eta_1} F(2\eta_1,0) + 4\, e^{i\eta_1} F(2\eta_1,2) - 5\, e^{-i\eta_1} F(2\eta_1,4) \big\} -$$

$$-8 \cos\eta_1 . C_2^{(10)}(2,\eta_1) + 8 \sin\eta_1\, S_2^{(10)}(2,\eta_1) - 4 \cos\eta_1 . C_2^{(12)}(2,\eta_1) -$$

$$- 4 \sin\eta_1 . S_2^{(12)}(2,\eta_1) - \cos\eta_1 \left(\frac{17767}{3600} - \frac{\pi^2}{2}\right) \Big]$$

$$- \frac{1}{4} \Big[\mathrm{Re}\big\{ -5\, e^{i\eta_1} F(2\eta_1,-4) + 4\, e^{-i\eta_1} F(2\eta_1,-2) + e^{-3i\eta_1} F(2\eta_1,0) \big\} +$$

$$+8 \cos\eta_1 . C_2^{(6)}(2,\eta_1) + 8 \sin\eta_1 . S_2^{(6)}(2,\eta_1) + 4 \cos\eta_1 . C_2^{(4)}(2,\eta_1) -$$

$$-4 \sin\eta_1 . S_2^{(4)}(2,\eta_1) - \cos\eta_1 \left(\frac{\pi^2}{2} - \frac{355}{72}\right) \Big]$$

$$+ \frac{1}{6} \Big[\mathrm{Re}\big\{ 2\, e^{3i\eta_1} F(2\eta_1,-4) + 3\, e^{i\eta_1} F(2\eta_1,-2) - 6\, e^{-i\eta_1} F(2\eta_1,0) +$$

$$+ e^{-3i\eta_1} F(2\eta_1,2) \big\} -$$

$$- 12 \cos\eta_1 . C_2^{(6)}(2,\eta_1) + 12 \sin\eta_1 . S_2^{(6)}(2,\eta_1) - \cos\eta_1 \left(\frac{593}{120} - \frac{\pi^2}{2}\right) \Big]$$

$$- \frac{1}{24} \Big[\mathrm{Re}\big\{ e^{5i\eta_1} F(2\eta_1,-4) - 4\, e^{3i\eta_1} F(2\eta_1,-2) + 6\, e^{i\eta_1} F(2\eta_1,0) -$$

$$-4\, e^{-i\eta_1} F(2\eta_1,2) + e^{-3i\eta_1} F(2\eta_1,4) \big\} - \frac{1}{120} \cos 3\eta_1 \Big]$$

$$+ \frac{1}{12}\Big[\mathrm{Re}\Big\{2\ e^{i\eta_1} F(2\eta_1,-6)+3\ e^{-i\eta_1} F(2\eta_1,-4)-6\ e^{-3i\eta_1} F(2\eta_1,-2)+$$

$$+e^{-5i\eta_1} F(2\eta_1,0)\Big\}-$$

$$- 12 \cos\eta_1 . C_2^{(4)}(2,\eta_1)-12 \sin\eta_1 . S_2^{(4)}(2,\eta_1)-\cos\eta_1\left(\frac{119}{24}-\frac{\pi^2}{2}\right)\Big]$$

$$- \frac{1}{12}\Big[\mathrm{Re}\Big\{e^{3i\eta_1} F(2\eta_1,-6)-4\ e^{i\eta_1} F(2\eta_1,-4)+6\ e^{-i\eta_1} F(2\eta_1,-2)-$$

$$-4\ e^{-3i\eta_1} F(2\eta_1,0)+e^{-5i\eta_1} F(2\eta_1,+2)\Big\} - \frac{1}{40} \cos 5\eta_1\Big]$$

$$+ \frac{1}{120}\Big[\mathrm{Re}\Big\{2\ e^{5i\eta_1} F(2\eta_1,-6)-5\ e^{3i\eta_1} F(2\eta_1,-4)+10\ e^{-i\eta_1} F(2\eta_1,0)-$$

$$-10\ e^{-3i\eta_1} F(2\eta_1,2)+3\ e^{-5i\eta_1} F(2\eta_1,4)\Big\} - \frac{3}{40} \cos 5\eta_1\Big]$$

$$+ \frac{1}{120}\Big[\mathrm{Re}\Big\{3\ e^{5i\eta_1} F(2\eta_1,-4)-10\ e^{3i\eta_1} F(2\eta_1,-2)+ 10\ e^{i\eta_1} F(2\eta_1,0)-$$

$$-5\ e^{-3i\eta_1} F(2\eta_1,4)+2\ e^{-5i\eta_1} F(2\eta_1,6)\Big\} - \frac{9}{280} \cos 5\eta_1\Big]\Big\}$$

$$+ q^2/16.A\ x$$

$$\Big\{ - \frac{1}{72}\Big[\mathrm{Re}\Big\{e^{6i\eta_1} F(2\eta_1,-8)-9\ e^{2i\eta_1} F(2\eta_1,-4)+16\ F(2\eta_1,-2)-$$

$$-9\ e^{-2i\eta_1} F(2\eta_1,0)+e^{-6i\eta_1} F(2\eta_1,4)\Big\} - \frac{7}{30} \cos 6\eta_1\Big]$$

$$+ \frac{1}{9}\Big[\mathrm{Re}\Big\{-11\ e^{2i\eta_1} F(2\eta_1,-2)+18\ F(2\eta_1,0)-9\ e^{-2i\eta_1} F(2\eta_1,2)+$$

$$+2\ e^{-4i\eta_1} F(2\eta_1,4)\Big\} +$$

$$+12 \cos 2\eta_1 C_2^{(6)}(2,\eta_1)-12 \sin 2\eta_1 S_2^{(6)}(2,\eta_1)-\cos 4\eta_1\left(\frac{\pi^2}{2}-\frac{591}{120}\right)\Big]$$

$$- \frac{1}{6}\Big[\mathrm{Re}\Big\{2\ e^{6i\eta_1} F(2\eta_1,-2)-12\ e^{4i\eta_1} F(2\eta_1,8)+6\ e^{2i\eta_1} F(2\eta_1,2)+4F(2\eta_1,4)\Big\}+$$

$$+24 \cos 2\eta_1\ C_2^{(10)}(2,\eta_1)-24 \sin 2\eta_1\ S_2^{(10)}(2,\eta_1)-\left(\pi^2-\frac{1973}{200}\right)\Big]$$

$$+ \frac{P}{24}\Big[\mathrm{Re}\Big\{3\ e^{6i\eta_1} F(2\eta_1,0)-16\ e^{4i\eta_1} F(2\eta_1,2)+12\ e^{2i\eta_1} F(2\eta_1,4)+$$

$$+e^{-2i\eta_1} F(2\eta_1,8)\Big\} +$$

$$+24 \cos 2\eta_1 . C_2^{(12)}(2,\eta_1)-24 \sin 2\eta_1 S_2^{(12)}(2,\eta_1)-\cos 2\eta_1 .$$

$$\left(\pi^2-\frac{82871}{8400}\right)\Big]$$

$$+\frac{1}{24}\Big[\mathrm{Re}\Big\{e^{2i\eta_1}F(2\eta_1,-8)+12\,e^{-2i\eta_1}F(2\eta_1,-4)-16\,e^{-4i\eta_1}F(2\eta_1,-2)+$$

$$+3\,e^{-6i\eta_1}F(2\eta_1,0)\Big\}-$$

$$-24\cos 2\eta_1.C_2^{(4)}(2,\eta_1)-24\sin 2\eta_1\,S_2^{(4)}(2,\eta_1)\cdot\cos 2\eta_1\left(\frac{241}{24}-\pi^2\right)\Big]$$

$$+\frac{1}{6}\Big[\mathrm{Re}\Big\{-4\,F(2\eta_1,-4)-6\,e^{-2i\eta_1}F(2\eta_1,-2)+2\,e^{-4i\eta_1}F(2\eta_1,0)-$$

$$-2\,e^{-6i\eta_1}F(2\eta_1,2)\Big\}+$$

$$+24\cos 2\eta_1.C_2^{(6)}(2,\eta_1)+24\sin 2\eta_1.S_2^{(6)}(2,\eta_1)-\left(\pi^2-\frac{593}{60}\right)\Big]$$

$$+\frac{1}{9}\Big[\mathrm{Re}\Big\{2\,e^{4i\eta_1}F(2\eta_1,-4)-9\,e^{2i\eta_1}F(2\eta_1,-2)+18\,F(2\eta_1,0)-$$

$$-11\,e^{-2i\eta_1}F(2\eta_1,2)\Big\}-$$

$$-12\cos 2\eta_1.C_2^{(10)}(2,\eta_1)-12\sin 2\eta_1.S_2^{(10)}(2,\eta_1)-$$

$$-\cos 4\eta_1\left(-\frac{\pi^2}{2}+\frac{5939}{1200}\right)\Big]$$

$$-\frac{1}{72}\Big[\mathrm{Re}\Big\{e^{6i\eta_1}F(2\eta_1,-4)-9\,e^{2i\eta_1}F(2\eta_1,0)+16\,F(2\eta_1,2)-9\,e^{-2i\eta_1}F(2\eta_1,4)+$$

$$+e^{-6i\eta_1}F(2\eta_1,8)\Big\}-\frac{41}{1680}\cos 6\eta_1\Big]$$

$$-\frac{1}{2}\Big[\mathrm{Re}\Big\{e^{2i\eta_1}F(2\eta_1,-4)+4\,F(2\eta_1,-2)-5\,e^{-2i\eta_1}F(2\eta_1,0)\Big\}-$$

$$-4\cos 2\eta_1.C_2^{(8)}(2,\eta_1)-4\sin 2\eta_1 S_2^{(8)}(2,\eta_1)-8\,C_2^{(6)}(2,\eta_1)-$$

$$-\cos 2\eta_1\left(\frac{79}{16}-\frac{\pi^2}{2}\right)\Big]$$

$$+\Big[\mathrm{Re}\Big\{-5\,e^{2i\eta_1}F(2\eta_1,-2)+4\,F(2\eta_1,0)+2\,e^{-2i\eta_1}F(2\eta_1,2)\Big\}+$$

$$+8\,C_2^{(8)}(2,\eta_1)+4\cos 2\eta_1.C_2^{(6)}(2,\eta_1)-4\sin 2\eta_1.S_2^{(6)}(2,\eta_1)-$$

$$-\left(\frac{\pi^2}{2}-\frac{74}{15}\right)\Big]$$

$$-\frac{1}{6}\Big[\mathrm{Re}\Big\{2\,e^{4i\eta_1}F(2\eta_1,-2)+3\,e^{2i\eta_1}F(2\eta_1,0)-6\,F(2\eta_1,2)+e^{-2i\eta_1}F(2\eta_1,4)\Big\}-$$

$$-12\cos 2\eta_1.C_2^{(8)}(2,\eta_1)+12\sin 2\eta_1 S_2^{(8)}(2,\eta_1)-\cos 2\eta_1\left(-\frac{\pi^2}{2}+\frac{79}{15}\right)\Big]$$

$$+ \frac{1}{36} \Big[\mathrm{Re} \Big\{ e^{2i\eta_1} F(2\eta_1,-8) - 18\, e^{-2i\eta_1} F(2\eta_1,-4) + 8\, e^{-4i\eta_1} F(2\eta_1,-2) + 9\, e^{-6i\eta_1} F(2\eta_1,0) \Big\}$$

$$+48 \cos 4\eta_1 \cdot C_2^{(6)}(2,\eta_1) + 48 \sin 4\eta_1 \cdot S_2^{(6)}(2,\eta_1) - \cos 2\eta_1 (2\pi^2 - \frac{469}{24}) \Big]$$

$$- \frac{1}{45} \Big[\mathrm{Re} \Big\{ e^{4i\eta_1} F(2\eta_1,-8) + 35\, e^{-2i\eta_1} F(2\eta_1,-2) - 45\, e^{-4i\eta_1} F(2\eta_1,0) + 9\, e^{-6i\eta_1} F(2\eta_1,2) \Big\} -$$

$$-60 \cos 2\eta_1 \cdot C_2^{(6)}(2,\eta_1) - 60 \sin 2\eta_1 \cdot S_2^{(6)}(2,\eta_1) - \cos 4\eta_1 (\frac{374}{15} - \frac{5\pi^2}{2}) \Big]$$

$$+ \frac{1}{180} \Big[\mathrm{Re} \Big\{ e^{6i\eta_1} F(2\eta_1,-8) - 20\, F(2\eta_1,-2) + 45\, e^{-2i\eta_1} F(2\eta_1,0) -$$

$$-36\, e^{-4i\eta_1} F(2\eta_1,2) + 10\, e^{-6i\eta_1} F(2\eta_1,4) \Big\} - \frac{37}{120} \cos 6\eta_1 \Big]$$

$$+ \frac{1}{180} \Big[\mathrm{Re} \Big\{ 10\, e^{6i\eta_1} F(2\eta_1,-4) - 36\, e^{4i\eta_1} F(2\eta_1,-2) + 45\, e^{2i\eta_1} F(2\eta_1,0) -$$

$$-20\, F(2\eta_1,2) + e^{-6i\eta_1} F(2\eta_1,8) \Big\} - \frac{167}{1680} \cos 6\eta_1 \Big]$$

$$- \frac{1}{45} \Big[\mathrm{Re} \Big\{ 9\, e^{6i\eta_1} F(2\eta_1,-2) - 45\, e^{4i\eta_1} F(2\eta_1,0) + 35\, e^{2i\eta_1} F(2\eta_1,2) + e^{-4i\eta_1} F(2\eta_1,8) \Big\} +$$

$$+60 \cos 2\eta_1 \cdot C_2^{(10)}(2,\eta_1) - 60 \sin 2\eta_1 \cdot S_2^{(10)}(2,\eta_1) - \cos 4\eta_1 (\frac{5\pi^2}{2} - \frac{6901}{280}) \Big]$$

$$+ \frac{1}{36} \Big[\mathrm{Re} \Big\{ 9\, e^{6i\eta_1} F(2\eta_1,0) + 8\, e^{4i\eta_1} F(2\eta_1,2) - 18\, e^{2i\eta_1} F(2\eta_1,4) + e^{-2i\eta_1} F(2\eta_1,8) \Big\} -$$

$$-48 \cos 4\eta_1 \cdot C_2^{(10)}(2,\eta_1) + 48 \sin 4\eta_1 \cdot S_2^{(10)}(2,\eta_1) - \cos 2\eta_1 (\frac{55289}{2800} - 2\pi^2) \Big]$$

$$- \frac{1}{6} \Big[\mathrm{Re} \Big\{ e^{2i\eta_1} F(2\eta_1,-4) - 6\, F(2\eta_1,-2) + 3\, e^{-2i\eta_1} F(2\eta_1,0) + 2\, e^{-4i\eta_1} F(2\eta_1,2) \Big\} +$$

$$+12 \cos 2\eta_1 \cdot C_2^{(8)}(2,\eta_1) + 12 \sin 2\eta_1 \cdot S_2^{(8)}(2,\eta_1) - \cos 2\eta_1 (\frac{\pi^2}{2} - \frac{1183}{240}) \Big]$$

$$+\Big[\mathrm{Re}\Big\{e^{2i\eta_1}F(2\eta_1-2)+4\,F(2\eta_1,0)-5\,e^{-2i\eta_1}F(2\eta_1,2)\Big\}-8\,C_2^{(8)}(2,\eta_1)-$$

$$-4\cos 2\eta_1\cdot C_2^{(10)}(2,\eta_1)-4\sin 2\eta_1\cdot S_2^{(10)}(2,\eta_1)-\Big(\frac{5923}{1200}-\frac{\pi^2}{2}\Big)\Big]$$

$$-\frac{1}{2}\Big[\mathrm{Re}\Big\{-5\,e^{2i\eta_1}F(2\eta_1,0)+4\,F(2\eta_1,2)+e^{-2i\eta_1}F(2\eta_1,4)\Big\}+8\,C_2^{(10)}(2,\eta_1)+$$

$$+4\cos\eta_1\cdot C_2^{(8)}(2,\eta_1)-4\sin\eta_1\cdot S_2^{(8)}(2,\eta_1)-\cos 2\eta_1\Big(\frac{\pi^2}{2}-\frac{5923}{1200}\Big)\Big]$$

8. $$-4\sum_{5}^{\infty} n\,\frac{Ne_n^{(2)}(0)}{Ne_n^{(2)\prime}(0)}\,S_n(\beta,\Omega,\eta_1)\,S_n^{*}(\beta,\Omega,\eta_1).$$

$$+\mathrm{Re}\Big\{F(2\eta_1,2)-e^{2i\eta_1}F(2\eta_1,0)\Big\}+2\,C_2^{(10)}(2,\eta_1)-\frac{\pi^2}{12}+\frac{5989}{7200}$$

$$+\mathrm{Re}\Big\{e^{-2i\eta_1}F(2\eta_1,2)-2\,F(2\eta_1,0)+e^{2i\eta_1}F(2\eta_1,-2)\Big\}-\frac{1}{40}\cos 2\eta_1$$

$$+\mathrm{Re}\Big\{-e^{-2i\eta_1}F(2\eta_1,0)+F(2\eta_1,-2)\Big\}-2\,C_2^{(6)}(2,\eta_1)+\frac{\pi^2}{12}-\frac{29}{36}$$

$+\ ^{q}/_{2}\ x$

$$+\mathrm{Re}\Big\{e^{-2i\eta_1}F(2\eta_1,2)-2\,F(2\eta_1,0)+e^{2i\eta_1}F(2\eta_1,-2)\Big\}-$$

$$-\cos 2\eta_1\Big\{\frac{\cos 8\eta_1}{64}+\frac{\cos 10\eta_1}{100}-\frac{1}{1600}\Big\}+\sin 2\eta_1\Big(S_2^{(10)}(2,\eta_1)+S_2^{(6)}(2,\eta_1)\Big)$$

$$+\mathrm{Re}\Big\{-\frac{1}{8}e^{-4i\eta_1}F(2\eta_1,+2)+\frac{1}{3}e^{-2i\eta_1}F(2\eta_1,0)-\frac{1}{4}F(2\eta_1,-2)+$$

$$+\frac{1}{24}e^{4i\eta_1}F(2\eta_1,-6)\Big\}+\frac{11}{2880}\cos 4\eta_1$$

$$+\mathrm{Re}\Big\{-\frac{1}{12}e^{-2i\eta_1}F(2\eta_1,6)-\frac{1}{4}e^{2i\eta_1}F(2\eta_1,2)+\frac{1}{3}e^{4i\eta_1}F(2\eta_1,0)\Big\}-$$

$$-\cos 2\eta_1\Big\{C_2^{(10)}(2,\eta_1)-\frac{1}{24}\pi^2+\frac{41543}{100800}\Big\}+\sin 2\eta_1\,S_2^{(10)}(\eta_1)$$

$$+\mathrm{Re}\Big\{F(2\eta_1,2)-e^{+2i\eta_1}F(2\eta_1,0)\Big\}+4\,C_3^{(10)}(2,\eta_1)+2\,C_2^{(10)}(2,\eta_1)-$$

$$-\frac{1}{2}\zeta(3)-\frac{\pi^2}{12}+\frac{615443}{432000}$$

$$+\mathrm{Re}\Big\{-e^{-2i\eta_1}F(2\eta_1,0)+F(2\eta_1,-2)\Big\}+4\,C_3^{(6)}(2,\eta_1)-2\,C_2^{(6)}(2,\eta_1)-$$

$$-\frac{1}{2}\zeta(3)+\frac{\pi^2}{12}-\frac{97}{432}$$

$$+ \operatorname{Re}\left\{ \frac{1}{3} e^{-4i\eta_1} F(2\eta_1,0) - \frac{1}{4} e^{-2i\eta_1} F(2\eta_1,-2) - \frac{1}{12} e^{2i\eta_1} F(2\eta_1,-6) \right\} +$$

$$+ \cos 2\eta_1 \left\{ C_2^{(6)}(2,\eta_1) - \frac{\pi^2}{24} + \frac{5}{12} \right\} + \sin 2\eta_1 \, S_2^{(6)}(2,\eta_1)$$

$$+ \operatorname{Re}\left\{ \frac{1}{24} e^{-4i\eta_1} F(2\eta_1,6) - \frac{1}{4} F(2\eta_1,2) + \frac{1}{3} e^{2i\eta_1} F(2\eta_1,0) - \frac{1}{8} e^{4i\eta_1} F(2\eta_1,-2) \right\} +$$

$$+ \frac{5}{4032} \cos 4\eta_1 \Big\}$$

9. $$8 \sum_{n=5}^{\infty} - \frac{Ne_n^{(2)}(0)}{Ne_n^{(2)\prime}(0)} \; T_n(\beta,\Omega,\eta_1)^* S_n(\beta,\eta_1) \; .$$

$$\operatorname{Re}\left\{ \frac{1}{2} F(2\eta_1,4) e^{-i\eta_1} - F(2\eta_1,2) e^{i\eta_1} + \frac{1}{2} F(2\eta_1,0) e^{3i\eta_1} \right\} - \frac{1}{120} \cos\eta_1 +$$

$$+ \operatorname{Re}\left\{ - \frac{1}{6} F(2\eta_1,4) e^{-3i\eta_1} + \frac{1}{2} F(2\eta_1,0) e^{i\eta_1} - \frac{1}{3} F(2\eta_1,-2) e^{3i\eta_1} \right\} +$$

$$+ \frac{1}{90} \cos 3\eta_1 +$$

$$+ \operatorname{Re}\left\{ - \frac{1}{3} F(2\eta_1,2) e^{-3i\eta_1} + \frac{1}{2} F(2\eta_1,0) e^{-i\eta_1} - \frac{1}{6} F(2\eta_1,-4) e^{3i\eta_1} \right\} +$$

$$+ \frac{11}{720} \cos 3\eta_1 +$$

$$+ \operatorname{Re}\left\{ \frac{1}{2} F(2\eta_1,0) e^{-3i\eta_1} - F(2\eta_1,-2) e^{-i\eta_1} + \frac{1}{2} F(2\eta_1,-4) e^{i\eta_1} - \right.$$

$$- \frac{1}{48} \cos\eta_1 +$$

$$+ i \times \frac{\beta^2 \Omega}{2}$$

$$\Big[\operatorname{Re}\left\{ \frac{3}{4} F(2\eta_1,4) - F(2\eta_1,2) e^{2i\eta_1} + \frac{1}{4} F(2\eta_1,0) e^{4i\eta_1} \right\} + C_2^{(12)}(2,\eta_1) -$$

$$- \frac{\pi^2}{24} + \frac{5909}{14400} +$$

$$+ \operatorname{Re}\left\{ \frac{1}{3} F(2\eta_1,4) e^{-2i\eta_1} - F(2\eta_1,2) + F(2\eta_1,0) e^{2i\eta_1} - \frac{1}{3} F(2\eta_1,-2) e^{4i\eta_1} \right\} +$$

$$+ \frac{1}{360} \cos 2\eta_1 +$$

$$+ \operatorname{Re}\left\{ - \frac{1}{24} F(2\eta_1,4) e^{-4i\eta_1} + \frac{1}{4} F(2\eta_1,0) - \frac{1}{3} F(2\eta_1,-2) e^{2i\eta_1} + \frac{1}{8} F(2\eta_1,-4) e^{4i\eta_1} \right\} -$$

$$- \frac{7}{2880} \cos 4\eta_1 +$$

$$+ \mathrm{Re}\left\{\frac{1}{8} F(2\eta_1,4)e^{-4i\eta_1} - \frac{1}{3} F(2\eta_1,2)e^{-2i\eta_1} + \frac{1}{4} F(2\eta_1,0) - \frac{1}{24}F(2\eta_1,-4)e^{4i\eta_1}\right\} +$$

$$+ \frac{1}{576} \cos 4\eta_1 +$$

$$+ \mathrm{Re}\left\{- \frac{1}{3} F(2\eta_1,2)e^{-4i\eta_1} + F(2\eta_1,0)e^{-2i\eta_1} - F(2\eta_1,-2) + \frac{1}{3} F(2\eta_1,-4)e^{2i\eta_1}\right\} -$$

$$- \frac{1}{180} \cos 2\eta_1 +$$

$$+ \mathrm{Re}\left\{\frac{1}{4} F(2\eta_1,0)e^{-4i\eta_1} - F(2\eta_1,-2)e^{-2i\eta_1} + \frac{3}{4} F(2\eta_1,-4)\right\} -$$

$$- C_2^{(4)}(2,\eta_1) + \frac{\pi^2}{24} - \frac{13}{32} +$$

$$+ \mathrm{Re}\left\{\frac{1}{6} F(2\eta_1,6)e^{-2i\eta_1} - \frac{1}{2}F(2\eta_1,4) + \frac{1}{2} F(2\eta_1,2)e^{2i\eta_1} - \frac{1}{6} F(2\eta_1,0)e^{4i\eta_1}\right\} +$$

$$+ \frac{1}{1260} \cos 2\eta_1 +$$

$$+ \mathrm{Re}\left\{- \frac{1}{2} F(2\eta_1,4)e^{-2i\eta_1} + \frac{1}{2}F(2\eta_1,0)e^{2i\eta_1}\right\} - 2 C_2^{(10)}(2,\eta_1) + \frac{\pi^2}{12} - \frac{5929}{7200} +$$

$$+ \mathrm{Re}\left\{- \frac{1}{6} F(2\eta_1,2)e^{-2i\eta_1} + \frac{1}{2} F(2\eta_1,0) - \frac{1}{2} F(2\eta_1,-2)e^{2i\eta_1} +\right.$$

$$\left.+ \frac{1}{6} F(2\eta_1,-4)e^{4i\eta_1}\right\} - \frac{1}{360} \cos 2\eta_1 +$$

$$+ \mathrm{Re}\left\{\frac{1}{12} F(2\eta_1,2)e^{-4i\eta_1} - \frac{1}{6} F(2\eta_1,0)e^{-2i\eta_1} + \frac{1}{6} F(2\eta_1,-4)e^{2i\eta_1} -\right.$$

$$\left.- \frac{1}{12} F(2\eta_1,-6)e^{4i\eta_1}\right\} + \frac{7}{1440} \cos 4\eta_1 +$$

$$+ \mathrm{Re}\left\{- \frac{1}{12} F(2\eta_1,6)e^{-4i\eta_1} + \frac{1}{6} F(2\eta_1,4)e^{-2i\eta_1} - \frac{1}{6} F(2\eta_1,0)e^{2i\eta_1} +\right.$$

$$\left.+ \frac{1}{12} F(2\eta_1,-2)e^{4i\eta_1}\right\} - \frac{11}{10080} \cos 4\eta_1 +$$

$$+ \mathrm{Re}\left\{\frac{1}{6} F(2\eta_1,4)e^{-4i\eta_1} - \frac{1}{2} F(2\eta_1,2)e^{-2i\eta_1} + \frac{1}{2} F(2\eta_1,0) - \frac{1}{6} F(2\eta_1,-2)e^{2i\eta_1}\right\} +$$

$$+ \frac{1}{720} \cos 2\eta_1 +$$

$$+ \mathrm{Re}\left\{\frac{1}{2} F(2\eta_1,0)e^{-2i\eta_1} - \frac{1}{2} F(2\eta_1,-4)e^{2i\eta_1}\right\} + 2 C_2^{(6)}(2,\eta_1) - \frac{\pi^2}{12} + \frac{119}{144} +$$

$$+ \mathrm{Re}\left\{- \frac{1}{6} F(2\eta_1,0)e^{-4i\eta_1} + \frac{1}{2} F(2\eta_1,-2)e^{-2i\eta_1} + \frac{1}{2} F(2\eta_1,-4) +\right.$$

$$\left.\left.+ \frac{1}{6} F(2\eta_1,-6)e^{2i\eta_1}\right\} - \frac{1}{144} \cos 2\eta_1\right] +$$

$$+ {}^{q}/4 \; x$$

$$\Big[\mathrm{Re}\left\{- \frac{4}{9} F(2\eta_1,2)e^{-3i\eta_1} + \frac{1}{2} F(2\eta_1,0)e^{-i\eta_1} - \frac{1}{18} F(2\eta_1,-4)e^{3i\eta_1}\right\} -$$

$$-\cos 3\eta_1\left[\frac{2}{3} C_2^{(10)}(2,\eta_1) - \frac{\pi^2}{36} + \frac{5879}{21600}\right] - \frac{2}{3} S_2^{(10)}(2\eta_1)\sin 3\eta_1 +$$

$$\mathrm{Re}\left\{\frac{3}{2}\ F(2\eta_1,2)e^{-i\eta_1} - 2\ F(2\eta_1,0)e^{i\eta_1} + \frac{1}{2}\ F(2\eta_1,-2)e^{3i\eta_1}\right\} -$$

$$- \cos\eta_1\left(-2\ C_2^{(10)}(2,\eta_1) + \frac{\pi^2}{12} - \frac{5899}{7200}\right) + 2\ S_2^{(10)}(2,\eta_1)\sin\eta_1 +$$

$$\mathrm{Re}\left\{\frac{1}{36}\ F(2\eta_1,8)e^{-3i\eta_1} + \frac{2}{9}\ F(2\eta_1,2)e^{3i\eta_1} - \frac{1}{4}\ F(2\eta_1,0)e^{5i\eta_1}\right\} -$$

$$- \cos 3\eta_1\left(-\frac{2}{3}\ C_2^{(10)}(2,\eta_1) + \frac{\pi^2}{36} - \frac{9239}{33600}\right) - \frac{2}{3}\ S_2^{(10)}(2,\eta_1)\sin 3\eta_1 +$$

$$\mathrm{Re}\left\{\frac{1}{10}\ F(2\eta_1,2)e^{-5i\eta_1} - \frac{1}{4}\ F(2\eta_1,0)e^{-3i\eta_1} + \frac{1}{6}\ F(2\eta_1,-2)e^{-i\eta_1} -\right.$$

$$\left.- \frac{1}{60}\ F(2\eta_1,8)e^{5i\eta_1}\right\} + \frac{47}{7200}\ \cos 5\eta_1 +$$

$$\mathrm{Re}\left\{\frac{1}{2}\ F(2\eta_1,2)e^{-3i\eta_1} - 2\ F(2\eta_1,0)e^{-i\eta_1} + \frac{3}{2}\ F(2\eta_1,-2)e^{i\eta_1}\right\} -$$

$$- \cos\eta_1\left(2\ C_2^{(6)}(2,\eta_1) - \frac{\pi^2}{12} + \frac{589}{720}\right) + 2\ S_2^{(6)}(2,\eta_1)\sin\eta_1 +$$

$$\mathrm{Re}\left\{-\frac{1}{60}\ F(2\eta_1,8)e^{-5i\eta_1} + \frac{1}{6}\ F(2\eta_1,2)e^{i\eta_1} - \frac{1}{4}\ F(2\eta_1,0)e^{3i\eta_1} +\right.$$

$$\left.+ \frac{1}{10}\ F(2\eta_1,-2)e^{5i\eta_1}\right\} - \cos 5\eta_1 \cdot \frac{113}{100800} +$$

$$\mathrm{Re}\left\{-\frac{1}{4}\ F(2\eta_1,0)e^{-5i\eta_1} + \frac{2}{9}\ F(2\eta_1,-2)e^{-3i\eta_1} + \frac{1}{36}\ F(2\eta_1,-8)e^{3i\eta_1}\right\} -$$

$$- \cos 3\eta_1\left(\frac{2}{3}\ C_2^{(6)}(2,\eta_1) - \frac{\pi^2}{36} + \frac{245}{864}\right) - \frac{2}{3}\ S_2^{(6)}(2,\eta_1)\sin 3\eta_1 +$$

$$\mathrm{Re}\left\{-\frac{1}{18}\ F(2\eta_1,4)e^{-3i\eta_1} + \frac{1}{2}\ F(2\eta_1,0)e^{i\eta_1} - \frac{4}{9}\ F(2\eta_1,-2)e^{3i\eta_1}\right\} -$$

$$- \cos 3\eta_1\left(-\frac{2}{3}\ C_2^{(6)}(2,\eta_1) + \frac{\pi^2}{36} - \frac{49}{180}\right) - \frac{2}{3}\ S_2^{(6)}(2\eta_1)\sin 3\eta_1 +$$

$$\mathrm{Re}\left\{\frac{1}{30}\ F(2\eta_1,4)e^{-5i\eta_1} - \frac{1}{6}\ F(2\eta_1,0)e^{-i\eta_1} + \frac{1}{6}\ F(2\eta_1,-2)e^{i\eta_1} -\right.$$

$$\left.- \frac{1}{30}\ F(2\eta_1,-6)e^{5i\eta_1}\right\} + \frac{1}{300}\ \cos 5\eta_1 +$$

$$\mathrm{Re}\left\{\frac{1}{6}\ F(2\eta_1,6)e^{-i\eta_1} - \frac{1}{2}\ F(2\eta_1,4)e^{i\eta_1} + \frac{1}{2}\ F(2\eta_1,2)e^{3i\eta_1} - \frac{1}{6}F(2\eta_1,0)e^{5i\eta_1}\right\} +$$

$$+ \frac{1}{1260}\ \cos\eta_1 +$$

$$\mathrm{Re}\left\{-\frac{1}{2}\ F(2\eta_1,4)e^{-i\eta_1} + \frac{1}{2}\ F(2\eta_1,0)e^{3i\eta_1}\right\} - \cos\eta_1\left(+2\ C_2^{(10)}(2,\eta_1) -\right.$$

$$\left.- \frac{\pi^2}{12} + \frac{5929}{7200}\right) + 2\ S_2^{(10)}(2,\eta_1)\ \sin\eta_1 +$$

$$\mathrm{Re}\left\{\frac{1}{2}\ F(2\eta_1,0)e^{-3i\eta_1} - \frac{1}{2}\ F(2\eta_1,-4)e^{i\eta_1}\right\} - \cos\eta_1\left(-2C_2^{(6)}(2,\eta_1) + \frac{\pi^2}{12} - \frac{119}{144}\right) +$$

$$+ 2\ S_2^{(6)}(2,\eta_1)\sin\eta_1 +$$

$$\mathrm{Re}\Big\{-\tfrac{1}{6}F(2\eta_1,0)e^{-5i\eta_1}+\tfrac{1}{2}F(2\eta_1,-2)e^{-3i\eta_1}-\tfrac{1}{2}F(2\eta_1,-4)e^{-i\eta_1}+$$

$$+\tfrac{1}{6}F(2\eta_1,-6)e^{i\eta_1}\Big\}-\tfrac{1}{144}\cos\eta_1+$$

$$\mathrm{Re}\Big\{-\tfrac{1}{30}F(2\eta_1,6)e^{-5i\eta_1}+\tfrac{1}{6}F(2\eta_1,2)e^{-i\eta_1}-\tfrac{1}{6}F(2\eta_1,0)e^{i\eta_1}+$$

$$+\tfrac{1}{30}F(2\eta_1,-4)e^{5i\eta_1}\Big\}-\tfrac{13}{8400}\cos 5\eta_1\ \Big]$$

The complex conjugated of this result gives the result of

$$8.\sum_{5}^{\infty}n\ \frac{Ne_n^{(2)}(0)}{Ne_n^{(2)\prime}(0)}\ S_n(\beta,\Omega,\eta_1)\ T_n^*(\beta,\Omega,\eta_1)\ .$$

10: $$-16\times\sum_{5}^{\infty}n\ \frac{Ne_n^{(2)}(0)}{Ne_n^{(2)\prime}(0)}\ T_n(\beta,\Omega,\eta_1)\ T_n^*(\beta,\Omega,\eta_1)\ .$$

$$\mathrm{Re}\Big\{\tfrac{1}{4}F(2\eta_1,4)-\tfrac{1}{4}e^{4i\eta_1}F(2\eta_1,0)\Big\}+C_2^{(12)}(2,\eta_1)-\frac{\pi^2}{24}+\frac{6029}{14400}+$$

$$+\mathrm{Re}\Big\{\tfrac{1}{4}e^{-4i\eta_1}F(2\eta_1,4)-\tfrac{1}{2}F(2\eta_1,0)+\tfrac{1}{4}e^{4i\eta_1}F(2\eta_1,-4)\Big\}-\tfrac{13}{480}\cos 4\eta_1+$$

$$+\mathrm{Re}\Big\{-\tfrac{1}{4}e^{-4i\eta_1}F(2\eta_1,0)+\tfrac{1}{4}F(2\eta_1,-4)\Big\}-C_2^{(4)}(2,\eta_1)+\frac{\pi^2}{24}-\frac{37}{96}$$

$+\ 9/4\ \times$

$$\Big[+\mathrm{Re}\Big\{\tfrac{2}{3}e^{-2i\eta_1}F(2\eta_1,4)-2F(2\eta_1,2)+2e^{2i\eta_1}F(2\eta_1,0)-\tfrac{2}{3}e^{4i\eta_1}F(2\eta_1,-2)\Big\}+$$

$$+\tfrac{1}{180}\cos 2\eta_1+$$

$$+\mathrm{Re}\Big\{-\tfrac{1}{18}e^{-6i\eta_1}F(2\eta_1,4)+\tfrac{1}{4}e^{-2i\eta_1}F(2\eta_1,0)-\tfrac{2}{9}F(2\eta_1,-2)+$$

$$+\tfrac{1}{36}e^{6i\eta_1}F(2\eta_1,-8)\Big\}-\tfrac{49}{4320}\cos 6\eta_1+$$

$$+\mathrm{Re}\Big\{-\tfrac{1}{12}e^{-2i\eta_1}F(2\eta_1,8)+\tfrac{1}{2}e^{2i\eta_1}F(2\eta_1,4)-\tfrac{2}{3}e^{4i\eta_1}F(2\eta_1,2)+$$

$$+\tfrac{1}{4}e^{6i\eta_1}F(2\eta_1,0)\Big\}-\tfrac{29}{20160}\cos 2\eta_1+$$

$$+\mathrm{Re}\Big\{-\tfrac{2}{3}e^{-4i\eta_1}F(2\eta_1,2)+2e^{-2i\eta_1}F(2\eta_1,0)-2F(2\eta_1,-2)+\tfrac{2}{3}e^{2i\eta_1}F(2\eta_1,-4)\Big\}-$$

$$-\tfrac{1}{90}\cos 2\eta_1\ +$$

$$+ \mathrm{Re}\left\{+ \frac{1}{4} e^{-6i\eta_1} F(2\eta_1,0) - \frac{2}{3} e^{-4i\eta_1} F(2\eta_1,-2) + \frac{1}{2} e^{-2i\eta_1} F(2\eta_1,-4) - \right.$$

$$\left. - \frac{1}{12} e^{2i\eta_1} F(2\eta_1,-8)\right\} + \frac{7}{288} \cos 2\eta_1 +$$

$$+ \mathrm{Re}\left\{\frac{1}{36} e^{-6i\eta_1} F(2\eta_1,8) - \frac{2}{9} F(2\eta_1,2) + \frac{1}{4} e^{2i\eta_1} F(2\eta_1,0) - \frac{1}{18} e^{6i\eta_1} F(2\eta_1,-4)\right\} +$$

$$\left. + \frac{169}{60480} \cos 6\eta_1 \right]$$

11: $\displaystyle 2 \sum_{8}^{\infty} {}_n \Lambda_n(\beta,\Omega)\, S_n^*(\beta,\Omega,\eta_1)$.

$A^*\Big[\ \theta_0 \times$

$$\mathrm{Im}\left\{ G(\eta_1,0).(e^{i\eta_1} + e^{-i\eta_1}) - G(\eta_1,1) - G(\eta_1,-1)\right\}.$$

$- \frac{1}{2}\theta_1 \times$

$$\mathrm{Im}\left\{ G(\eta_1,-1).(1-e^{2i\eta_1}) + G(\eta_1,1).(1-e^{-2i\eta_1})\right\} + 2\left\{\bar{S}_2^{(8)}(\eta_1) - \bar{S}_2^{(6)}(\eta_1)\right\}.$$

$- \frac{1}{3}\theta_2 \times$

$$\mathrm{Im}\left\{ -2\, G(\eta_1,1) - 2\, G(\eta_1,-1) + G(\eta_1,2).(3\, e^{-i\eta_1} - e^{-3i\eta_1}) + G(\eta_1,-2).\right.$$

$$\left.(3\, e^{i\eta_1} - e^{3i\eta_1})\right\}.$$

$+ \frac{1}{4}\theta_3 \times$

$$\mathrm{Im}\left\{ G(\eta_1,1) + G(\eta_1,-1) + G(\eta_1,3).(-2e^{-2i\eta_1} + e^{-4i\eta_1}) + G(\eta_1,-3).\right.$$

$$\left.(-2e^{2i\eta_1} + e^{4i\eta_1})\right\}.$$

$+ \frac{1}{15}\theta_4 \times$

$$\mathrm{Im}\left\{ 2\, G(\eta_1,1) + 2\, G(\eta_1,-1) + G(\eta_1,4).(-5\, e^{-3i\eta_1} + 3e^{-5i\eta_1}) + \right.$$

$$\left. + G(\eta_1,-4).(-5\, e^{3i\eta_1} + 3\, e^{5i\eta_1})\right\}.$$

$- \frac{1}{12}\theta_5 \times$

$$\mathrm{Im}\left\{ -G(\eta_1,1) - G(\eta_1,-1) + G(\eta_1,5).(3\, e^{-4i\eta_1} - 2\, e^{-6i\eta_1}) + G(\eta_1,-5).\right.$$

$$\left.(3e^{4i\eta_1} - 2\, e^{6i\eta_1})\right\}$$

$- \frac{1}{35}\theta_6 \times$

$$\mathrm{Im}\left\{ -2\, G(\eta_1,1) - 2\, G(\eta_1,-1) + G(\eta_1,6).(7\, e^{-5i\eta_1} - 5\, e^{-7i\eta_1}) + \right.$$

$$\left. + G(\eta_1,-6)(7\, e^{5i\eta_1} - 5\, e^{7i\eta_1})\right\}$$

$$+ \frac{1}{24}\Theta_7 \times$$

$$\mathrm{Im}\Big\{G(\eta_1,1)+G(\eta_1,-1)+G(\eta_1,7).(-4\ e^{-6i\eta_1}+3\ e^{-8i\eta_1})+$$

$$+G(\eta_1,-7).(-4\ e^{6i\eta_1}+3\ e^{8i\eta_1})\Big\}\Big]$$

$$+\frac{i\beta^2\Omega}{2}A^*\Big[\frac{1}{2}\Theta_o \times$$

$$\mathrm{Im}\Big\{G(\eta_1,-2)+G(\eta_1,2)+G(\eta_1,-1).(-2\ e^{-i\eta_1}-2\ e^{i\eta_1})+$$

$$+G(\eta_1,1).(-2\ e^{i\eta_1}-2\ e^{-i\eta_1})+G(\eta_1,0).(e^{2i\eta_1}+4+e^{-2i\eta_1})\Big\}$$

$$+\frac{1}{6}\Theta_1 \times$$

$$\mathrm{Im}\Big\{8\ G(\eta_1,-2)+G(\eta_1,-1).(-9\ e^{-i\eta_1}+e^{3i\eta_1})$$

$$+8\ G(\eta_1,2)+G(\eta_1,1).(-9\ e^{i\eta_1}+e^{-3i\eta_1})\Big\}+12\cos\eta_1\cdot\Big(\overline{S}\,{}_2^{(6)}(\eta_1)-\overline{S}\,{}_2^{(8)}(\eta_1)\Big).$$

$$-\frac{1}{12}\Theta_2 \times$$

$$\mathrm{Im}\Big\{G(\eta_1,-2).(9+8e^{2i\eta_1}-e^{4i\eta_1})+G(\eta_1,-1).(-8\ e^{-i\eta_1}-8\ e^{i\eta_1})+$$

$$+G(\eta_1,2).(9+8\ e^{-2i\eta_1}-e^{-4i\eta_1})+G(\eta_1,1).(-8\ e^{i\eta_1}-8\ e^{-i\eta_1})\Big\}+$$

$$+12\Big(\overline{S}\,{}_2^{(9)}(\eta_1)-\overline{S}\,{}_{(2)}^{(5)}(\eta_1)\Big).$$

$$-\frac{1}{20}\Theta_3 \times$$

$$\mathrm{Im}\Big\{G(\eta_1,-3).(-10\ e^{i\eta_1}+5\ e^{3i\eta_1}-e^{5i\eta_1})+16\ G(\eta_1,-2)+$$

$$+G(\eta_1,-1).(-5\ e^{i\eta_1}-5\ e^{-i\eta_1})+$$

$$G(\eta_1,3).(-10\ e^{-i\eta_1}+5\ e^{-3i\eta_1}-e^{-5i\eta_1})+16\ G(\eta_1,2)+$$

$$+G(\eta_1,1).(-5\ e^{-i\eta_1}-5\ e^{i\eta_1})\Big\}.$$

$$+\frac{1}{30}\Theta_4 \times$$

$$\mathrm{Im}\Big\{G(\eta_1,-4).(+5e^{2i\eta_1}-4e^{+4i\eta_1}+e^{6i\eta_1})-10\ G(\eta_1,-2)+$$

$$+G(\eta_1,-1).(4\ e^{-i\eta_1}+4\ e^{i\eta_1})+$$

$$G(\eta_1,4).(+5e^{-2i\eta_1}-4e^{-4i\eta_1}+e^{-5i\eta_1})-10\ G(\eta_1,2)+$$

$$+G(\eta_1,1).(4\ e^{-i\eta_1}+4\ e^{i\eta_1})\Big\}.$$

$$+ \frac{1}{84}\Theta_5 \times$$

$$\mathrm{Im}\Big\{ G(\eta_1,-5)\cdot(7\ e^{3i\eta_1} - 7\ e^{5i\eta_1} + 2\ e^{7i\eta_1}) - 16\ G(\eta_1,-2) +$$

$$+ G(\eta_1,-1)\cdot(7\ e^{i\eta_1} + 7\ e^{-i\eta_1}) +$$

$$+ G(\eta_1,5)\cdot(7\ e^{-3i\eta_1} - 7\ e^{-5i\eta_1} + 2\ e^{-7i\eta_1}) - 16\ G(\eta_1,2) +$$

$$+ G(\eta_1,1)\cdot(7\ e^{i\eta_1} + 7\ e^{-i\eta_1})\Big\}\Big]$$

$$+ {}^{c}/4\ A^{*}\Big[\ \frac{1}{6}\Theta_6 \times$$

$$\mathrm{Im}\Big\{ - G(\eta_1,-3) + G(\eta_1,-2)\cdot(-6\ e^{i\eta_1} + 2\ e^{3i\eta_1}) + G(\eta_1,-1)(3e^{-2i\eta_1} + 1 - 3e^{2i\eta_1}) +$$

$$- G(\eta_1,3) + G(\eta_1,2)\cdot(-6e^{-i\eta_1} + 2e^{-3i\eta_1}) + G(\eta_1,1)\cdot(3e^{2i\eta_1} + 1 - 3e^{-2i\eta_1}) +$$

$$G(\eta_1,0)\cdot(6e^{-i\eta_1} - 2e^{-3i\eta_1} - 2\ e^{3i\eta_1} + 6\ e^{i\eta_1})\Big\} + 12\big(\bar{S}\,{}^{(8)}_{2}(\eta_1) - \bar{S}\,{}^{(6)}_{2}(\eta_1)\big).$$

$$- \frac{1}{8}\Theta_1 \times$$

$$\mathrm{Im}\Big\{\ G(\eta_1,-1)\cdot(-3-4e^{-2i\eta_1} + 2e^{2i\eta_1} + e^{4i\eta_1}) + G(\eta_1,-3)(3+2e^{2i\eta_1} - e^{4i\eta_1})$$

$$+ G(\eta_1,1)\cdot(-3-4\ e^{2i\eta_1} + 2\ e^{-2i\eta_1} + \ e^{-4i\eta_1}) + G(\eta_1,3)\cdot(3+2e^{-2i\eta_1} - 1e^{-4i\eta_1})\Big\} +$$

$$+ 8(1+\cos 2\eta_1)\cdot\big(\bar{S}\,{}^{(6)}_{2}(\eta_1) - \bar{S}\,{}^{(8)}_{2}(\eta_1)\big)\ .$$

$$- \frac{1}{45}\Theta_2 \times$$

$$\mathrm{Im}\Big\{ G(\eta_1,-4)\cdot(5e^{3i\eta_1} - 3e^{5i\eta_1}) + 27\ G(\eta_1,-3) +$$

$$+ G(\eta_1,-2)\cdot(-45e^{-i\eta_1} - 45e^{i\eta_1} - 5\ e^{3i\eta_1} + 3\ e^{5i\eta_1}) +$$

$$G(\eta_1,4)\cdot(5e^{-3i\eta_1} - 3e^{-5i\eta_1}) + 27\ G(\eta_1,3) +$$

$$+ G(\eta_1,+2)\cdot(-45e^{i\eta_1} - 45e^{-i\eta_1} - 5e^{-3i\eta_1} + 3e^{-5i\eta_1}) +$$

$$+ G(\eta_1,-1)\cdot(15e^{-2i\eta_1} - 27 - 15e^{2i\eta_1}) + (90\,e^{i\eta_1} + 90\,e^{-i\eta_1})\cdot G(\eta_1,0) +$$

$$+ G(\eta_1,1)\cdot(-15e^{-2i\eta_1} - 27 + 15e^{2i\eta_1})\Big\} + 60\,\big(\bar{S}^{(8)}_2(\eta_1) - \bar{S}^{(6)}_2(\eta_1)\big)$$

$$+ \frac{1}{48}\Theta_3 \times$$

$$\mathrm{Im}\Big\{ G(\eta_1,-5)\cdot(-3\ e^{4i\eta_1} + 2\ e^{6i\eta_1}) + G(\eta_1,-3)\cdot(8+12\ e^{2i\eta_1} + 3e^{4i\eta_1} - 2e^{6i\eta_1}) +$$

$$+ G(\eta_1,-1)\cdot(-8-6e^{-2i\eta_1} - 6\ e^{2i\eta_1}) +$$

$$+ G(\eta_1,5)\cdot(-3e^{-4i\eta_1} + 2e^{-6i\eta_1}) + G(\eta_1,+3)\cdot(8+12e^{-2i\eta_1} + 3e^{-4i\eta_1} - 2e^{-6i\eta_1}) +$$

$$+ G(\eta_1,1)\cdot(-8-6e^{-2i\eta_1} - 6e^{2i\eta_1})\Big\} + 24\big(\bar{S}\,{}^{(4)}_{2}(\eta_1) - \bar{S}^{(10)}_{2}(\eta_1)\big)$$

$$+\frac{1}{3150}\Theta_4 \times$$

$$\mathrm{Im}\Big\{ G(\eta_1,-6)\cdot\left(-126\,e^{5i\eta_1}+90e^{7i\eta_1}\right)+$$

$$+G(\eta_1,-4)\cdot\left(-1050e^{i\eta_1}+350e^{3i\eta_1}+126e^{5i\eta_1}-90e^{7i\eta_1}\right)+1350\;G(\eta_1,-3)+$$

$$+\quad G\;(\eta_1,6)\cdot\left(-126e^{-5i\eta_1}+90e^{-7i\eta_1}\right)+$$

$$+G(\eta_1,4)\cdot\left(-1050e^{-i\eta_1}+350e^{-3i\eta_1}+126e^{-5i\eta_1}-90e^{-7i\eta_1}\right)+1350\;G(\eta_1,3)+$$

$$+\;G(\eta_1,-2)\cdot\left(+1050e^{i\eta_1}-350e^{3i\eta_1}\right)+G(\eta_1,-1)\cdot\left(210e^{2i\eta_1}-1350-210e^{-2i\eta_1}\right)$$

$$+\;G(\eta_1,2)\cdot\left(1050e^{-i\eta_1}-350e^{-3i\eta_1}\right)+G(\eta_1,1)\cdot\left(+210e^{-2i\eta_1}-1350-210e^{2i\eta_1}\right)\Big\}+$$

$$+840\left(\bar{S}_2^{(6)}(\eta_1)-\bar{S}_2^{(8)}(\eta_1)\right).$$

$$-\frac{1}{144}\Theta_5 \times$$

$$\mathrm{Im}\Big\{ G(\eta_1,-7)\cdot\left(4e^{6i\eta_1}-3e^{8i\eta_1}\right)+G(\eta_1,-5)\cdot\left(18e^{2i\eta_1}-9e^{4i\eta_1}-4e^{6i\eta_1}+3e^{8i\eta_1}\right)+$$

$$+\,G(\eta_1,7)\cdot\left(4e^{-6i\eta_1}-3e^{-8i\eta_1}\right)+G(\eta_1,5)\cdot\left(18e^{-2i\eta_1}-9e^{-4i\eta_1}-4e^{-6i\eta_1}+3e^{-8i\eta_1}\right)+$$

$$+\;G(\eta_1,-3)\cdot\left(-27-18e^{2i\eta_1}+9e^{4i\eta_1}\right)+G(\eta_1,-1)\cdot\left(6e^{-2i\eta_1}+27-6e^{2i\eta_1}\right)\;+$$

$$+\;G(\eta_1,3)\cdot\left(-27-18e^{-2i\eta_1}+9e^{-4i\eta_1}\right)+G(\eta_1,1)\cdot\left(6e^{2i\eta_1}+27-6e^{-2i\eta_1}\right)\Big\}\;+$$

$$+24\left(\bar{S}_2^{(8)}(\eta_1)-\;\bar{S}_2^{(6)}(\eta_1)\right)\Big].$$

12: $4\sum_{8}^{\infty} n\;\Lambda_n(\beta,\Omega)\;\;T_n^{*}(\beta,\Omega,\eta_1).$

$$\frac{i\beta^2\Omega}{12}\;A^{*}\Big[\Theta_0\times$$

$$\mathrm{Im}\Big\{ G(\eta_1,0)\cdot\left(-e^{3i\eta_1}+3e^{i\eta_1}+3e^{-i\eta_1}-e^{-3i\eta_1}\right)-6\;G(\eta_1,1)-6\;G(\eta_1,-1)\;+$$

$$+\;\left(G(\eta_1,2)+\;G(\eta_1,-2)\right)\cdot\left(3e^{i\eta_1}+3e^{-i\eta_1}\right)-2G(\eta_1,3)-2\;G(\eta_1,-3)\Big\}$$

$$+\;\frac{1}{2}\Theta_1\times$$

$$\mathrm{Im}\Big\{ G(\eta_1,1)\cdot\left(-6e^{2i\eta_1}-18+2e^{-2i\eta_1}-e^{-4i\eta_1}\right)+G(\eta_1,2)\cdot\left(16e^{i\eta_1}+16e^{-i\eta_1}\right)-$$

$$-9\;G(\eta_1,3)+$$

$$+\;G(\eta_1,-1)\cdot\left(-6e^{-2i\eta_1}-18+2e^{2i\eta_1}-e^{4i\eta_1}\right)+G(\eta_1,-2)\cdot\left(16e^{i\eta_1}+16e^{-i\eta_1}\right)\;-$$

$$-9\;G(\eta_1,-3)\;\Big\}\;+12\left(\bar{S}_2^{(6)}(\eta_1)+\bar{S}_2^{(8)}(\eta_1)\right).$$

$- \frac{1}{10}\Theta_2 \times$

$\mathrm{Im}\Big\{ -40\ G(\eta_1,1) + G(\eta_1,2)\cdot\big(3e^{-5i\eta_1} - 5e^{-3i\eta_1} + 45e^{-i\eta_1} - 75e^{i\eta_1}\big) + 72\ G(\eta_1,3) +$

$-40\ G(\eta_1,-1) + G(\eta_1,-2)\cdot\big(3e^{5i\eta_1} - 5e^{3i\eta_1} + 45e^{i\eta_1} - 75e^{-i\eta_1}\big) +$

$+72\ G(\eta_1,-3)\Big\} + 120\ \cos\eta_1\big(\bar{S}_2^{(9)}(\eta_1) - \bar{S}_2^{(5)}(\eta_1)\big)$

$- \frac{1}{10}\theta_3 \times$

$\mathrm{Im}\Big\{ -15\ G(\eta_1,1) + G(\eta_1,2)\cdot\big(48e^{i\eta_1} + 48e^{-i\eta_1}\big) + G(\eta_1,3)\cdot\big(-50 - 30e^{-2i\eta_1} - 3e^{-4i\eta_1} + 2e^{-6i\eta_1}\big) +$

$-15\ G(\eta_1,-1) + G(\eta_1,-2)\cdot\big(48e^{i\eta_1} + 48e^{-i\eta_1}\big) +$

$+G(\eta_1,-3)\cdot\big(-50 - 30e^{2i\eta_1} - 3e^{4i\eta_1} + 2e^{6i\eta_1}\big) + 60\big(\bar{S}_2^{(10)}(\eta_1) - \bar{S}_2^{(4)}(\eta_1)\big)$

$+ \frac{1}{35}\Theta_4 \times$

$\mathrm{Im}\Big\{ 28\ G(\eta_1,1) + G(\eta_1,2)\cdot\big(-70e^{i\eta_1} - 70e^{-i\eta_1}\big) +$

$+180\ G(\eta_1,3) + \big(-105e^{-i\eta_1} + 35e^{-3i\eta_1} + 7e^{-5i\eta_1} - 5e^{-7i\eta_1}\big)\ G(\eta_1,4) +$

$+28\ G(\eta_1,-1) + G(\eta_1,-2)\cdot\big(-70e^{i\eta_1} - 70e^{-i\eta_1}\big) +$

$+180\ G(\eta_1,-3) + \big(-105e^{i\eta_1} + 35e^{3i\eta_1} + 7e^{5i\eta_1} - 5e^{7i\eta_1}\big) G(\eta_1,-4)\Big\}$

$+\frac{1}{28}\Theta_5 \times$

$\mathrm{Im}\Big\{ 14\ G(\eta_1,-1) + G(\eta_1,-2)\cdot\big(-32e^{i\eta_1} - 32e^{-i\eta_1}\big) + 63\ G(\eta_1,-3) +$

$+ G(\eta_1,-5)\cdot\big(-28e^{2i\eta_1} + 14e^{4i\eta_1} + 4e^{6i\eta_1} - 3e^{8i\eta_1}\big)$

$+14\,G(\eta_1,1) + G(\eta_1,2)\cdot\big(-32e^{i\eta_1} - 32e^{-i\eta_1}\big) + 63\ G(\eta_1,3) + G(\eta_1,5)\cdot\big(-28e^{-2i\eta_1} + 14e^{-4i\eta_1} + 4e^{-6i\eta_1} - 3e^{-8i\eta_1}\big)\Big\}\Big]$

$A^*\Big[\ \frac{1}{2}\Theta_0 \times$

$\mathrm{Im}\Big\{ G(\eta_1,0)\cdot\big(e^{2i\eta_1} + e^{-2i\eta_1}\big) - G(\eta_1,2) - G(\eta_1,-2)\Big\}.$

$- \Theta_1 \times$

$\mathrm{Im}\Big\{ G(\eta_1,1)\cdot\big(-e^{i\eta_1} - \frac{1}{3}e^{-3i\eta_1}\big) + G(\eta_1,-1)\cdot\big(-e^{-i\eta_1} - \frac{1}{3}e^{3i\eta_1}\big) +$

$+ \frac{4}{3}G(\eta_1,2) + \frac{4}{3}G(\eta_1,-2)\Big\}.$

$- \Theta_2 \times$

$\mathrm{Im}\Big\{ G(\eta_1,2)\cdot\big(\frac{1}{4} - \frac{1}{4}e^{-4i\eta_1}\big) + G(\eta_1,-2)\cdot\big(\frac{1}{4} - \frac{1}{4}e^{4i\eta_1}\big)\Big\} + \bar{S}_2^{(5)}(\eta_1) - \bar{S}_2^{(9)}(\eta_1).$

$+ \Theta_3 \times$

$\mathrm{Im}\Big\{ \frac{4}{5}G(\eta_1,2) + \frac{4}{5}G(\eta_1,-2) + G(\eta_1,3)\cdot\big(-e^{-i\eta_1} + \frac{1}{5}e^{-5i\eta_1}\big) +$

$+ G(\eta_1,-3)\cdot\big(-e^{i\eta_1} + \frac{1}{5}e^{5i\eta_1}\big)\Big\}.$

$+\Theta_4 \; x$

$$\mathrm{Im}\Big\{\tfrac{1}{3}\, G(\eta_1,2) + \tfrac{1}{3}\, G(\eta_1,-2) + G(\eta_1,4)\cdot\big(-\tfrac{1}{2}\, e^{-2i\eta_1} + \tfrac{1}{6}\, e^{-6i\eta_1}\big) +$$

$$+ G(\eta_1,-4)\cdot\big(-\tfrac{1}{2}\, e^{2i\eta_1} + \tfrac{1}{6}\, e^{6i\eta_1}\big)\Big\}.$$

$-\Theta_5 \; x$

$$\mathrm{Im}\Big\{-\tfrac{4}{21}\, G(\eta_1,2) - \tfrac{4}{21}\, G(\eta_1,-2) + G(\eta_1,5)\cdot\big(\tfrac{1}{3}\, e^{-3i\eta_1} - \tfrac{1}{7}\, e^{-7i\eta_1}\big) +$$

$$+ G(\eta_1,-5)\cdot\big(\tfrac{1}{3}\, e^{3i\eta_1} - \tfrac{1}{7}\, e^{7i\eta_1}\big)\Big\}$$

$-\Theta_6 \; x$

$$\mathrm{Im}\Big\{-\tfrac{1}{8}\, G(\eta_1,2) - \tfrac{1}{8}\, G(\eta_1,-2) + G(\eta_1,6)\cdot\big(\tfrac{1}{4}\, e^{-4i\eta_1} - \tfrac{1}{8}\, e^{-8i\eta_1}\big) +$$

$$+ G(\eta_1,-6)\cdot\big(\tfrac{1}{4}\, e^{4i\eta_1} - \tfrac{1}{8}\, e^{8i\eta_1}\big)\Big\}$$

$+\Theta_7 \; x$

$$\mathrm{Im}\Big\{\tfrac{4}{45}\, G(\eta_1,2) + \tfrac{4}{45}\, G(\eta_1,-2) + G(\eta_1,7)\cdot\big(-\tfrac{1}{5}\, e^{-5i\eta_1} + \tfrac{1}{9}\, e^{-9i\eta_1}\big) +$$

$$+ G(\eta_1,-7)\cdot\big(-\tfrac{1}{5}\, e^{5i\eta_1} + \tfrac{1}{9}\, e^{9i\eta_1}\big)\Big\}\Big].$$

$+ \, {}^{q}/4 \; A^{*} \Big[\Theta_0 \, x$

$$\mathrm{Im}\Big\{-G(\eta_1,0)\cdot\big(\tfrac{1}{4}\, e^{4i\eta_1} + 2 + \tfrac{1}{4}\, e^{-4i\eta_1}\big) + G(\eta_1,1)\cdot\big(-\tfrac{1}{3}e^{-3i\eta_1} + e^{-i\eta_1} - e^{i\eta_1} + \tfrac{1}{3}e^{3i\eta_1}\big) +$$

$$+ G(\eta_1,-1)\cdot\big(+\tfrac{1}{3}\, e^{-3i\eta_1} - e^{-i\eta_1} + e^{i\eta_1} - \tfrac{1}{3}e^{3i\eta_1}\big)$$

$$+ G(\eta_1,2)\cdot\big(\tfrac{13}{12} + \tfrac{1}{4}e^{-4i\eta_1}\big) + G(\eta_1,-2)\cdot\big(\tfrac{13}{12} + \tfrac{1}{4}\, e^{4i\eta_1}\big) -$$

$$- \tfrac{1}{12}\, G(\eta_1,4) - \tfrac{1}{12}\, G(\eta_1,-4)\Big\} + \bar{S}_2^{(9)}(\eta_1) - \bar{S}_2^{(5)}(\eta_1) \; .$$

$-\Theta_1 \; x$

$$\mathrm{Im}\Big\{G(\eta_1,1)\cdot\big(\tfrac{1}{10}\, e^{-5i\eta_1} + \tfrac{5}{18}\, e^{-3i\eta_1} - \tfrac{1}{2}\, e^{i\eta_1} - \tfrac{5}{18}\, e^{3i\eta_1}\big) +$$

$$+ G(\eta_1,-1)\cdot\big(\tfrac{1}{10}e^{5i\eta_1} + \tfrac{5}{18}e^{3i\eta_1} - \tfrac{1}{2}\, e^{-i\eta_1} - \tfrac{5}{18}\, e^{-3i\eta_1}\big) +$$

$$- \tfrac{8}{45}\, G(\eta_1,2) - \tfrac{8}{45}\, G(\eta_1,-2) + G(\eta_1,3)\cdot\big(-\tfrac{1}{10}\, e^{-5i\eta_1} + \tfrac{1}{2}\, e^{-i\eta_1}\big) +$$

$$+ G(\eta_1,-3)\cdot\big(-\tfrac{1}{10}e^{5i\eta_1} + \tfrac{1}{2}e^{i\eta_1}\big) + \tfrac{8}{45}\, G(\eta_1,4) + \tfrac{8}{45}\, G(\eta_1,-4)\Big\}$$

$$+ 2\cos\eta_1\Big(\bar{S}_2^{(6)}(\eta_1) - \bar{S}_2^{(8)}(\eta_1)\Big) + \tfrac{2}{3}\cos 3\eta_1\Big(\bar{S}_2^{(6)}(\eta_1) - \bar{S}_2^{(8)}(\eta_1)\Big).$$

$-\theta_2$ x

$$\operatorname{Im}\Big\{G(\eta_1,0)\big(e^{2i\eta} + e^{-2i\eta}\big) + G(\eta_1,1)\big(\tfrac{2}{9}e^{3i\eta} - \tfrac{2}{3}e^{i\eta} + \tfrac{2}{3}e^{-i\eta} - \tfrac{2}{9}e^{-3i\eta}\big) +$$

$$+G(\eta_1,-1)\big(\tfrac{2}{9}e^{-3i\eta} - \tfrac{2}{3}e^{-i\eta} + \tfrac{2}{3}e^{+i\eta} - \tfrac{2}{9}e^{3i\eta}\big)$$

$$+ G(\eta_1,2)\big(-\tfrac{1}{2}e^{2i\eta} - \tfrac{2}{9} - \tfrac{2}{3}e^{-2i\eta} + \tfrac{1}{18}e^{-6i\eta}\big) +$$

$$+G(\eta_1,-2)\big(-\tfrac{1}{2}e^{-2i\eta} - \tfrac{2}{9} - \tfrac{2}{3}e^{2i\eta} + \tfrac{1}{18}e^{6i\eta}\big) +$$

$$G(\eta_1,4)\big(\tfrac{2}{9} + \tfrac{1}{6}e^{-2i\eta} - \tfrac{1}{18}e^{-6i\eta}\big) + G(\eta_1,-4)\big(\tfrac{2}{9} + \tfrac{1}{6}e^{2i\eta} - \tfrac{1}{18}e^{6i\eta}\big)\Big\}.$$

$+\theta_3$ x

$$\operatorname{Im}\Big\{\big(-\tfrac{1}{12}e^{-3i\eta} - \tfrac{1}{4}e^{-i\eta} - \tfrac{1}{4}e^{i\eta} - \tfrac{1}{12}e^{3i\eta}\big)\big(G(\eta_1,1) + G(\eta_1,-1)\big) +$$

$$+ \tfrac{8}{21}\big(G(\eta_1,2) + G(\eta_1,-2)\big) +$$

$$+ G(\eta_1,3)\big(\tfrac{1}{2}e^{i\eta} + \tfrac{1}{4}e^{-3i\eta} - \tfrac{1}{28}e^{-7i\eta}\big) + G(\eta_1,-3)\big(\tfrac{1}{2}e^{-i\eta} + \tfrac{1}{4}e^{3i\eta} - \tfrac{1}{28}e^{7i\eta}\big) -$$

$$- \tfrac{8}{21}\big(G(\eta_1,4) + G(\eta_1,-4)\big)$$

$$+ G(\eta_1,5)\big(-\tfrac{1}{12}e^{-3i\eta} + \tfrac{1}{28}e^{-7i\eta}\big) + G(\eta_1,-5)\big(-\tfrac{1}{12}e^{3i\eta} + \tfrac{1}{28}e^{7i\eta}\big)\Big\}$$

$+\theta_4$ x

$$\operatorname{Im}\Big\{\big(-\tfrac{2}{45}e^{3i\eta} + \tfrac{2}{15}e^{i\eta} - \tfrac{2}{15}e^{-i\eta} + \tfrac{2}{45}e^{-3i\eta}\big)\big(G(\eta_1,1) - G(\eta_1,-1)\big) +$$

$$+G(\eta_1,2)\big(-\tfrac{5}{72} - \tfrac{1}{12}e^{-4i\eta}\big) + G(\eta_1,-2)\big(-\tfrac{5}{72} - \tfrac{1}{12}e^{4i\eta}\big)$$

$$+ G(\eta_1,4)\big(\tfrac{5}{72} + \tfrac{2}{15}e^{-4i\eta} - \tfrac{1}{40}e^{-8i\eta}\big) + G(\eta_1,-4)\big(\tfrac{5}{72} + \tfrac{2}{15}e^{4i\eta} - \tfrac{1}{40}e^{8i\eta}\big) +$$

$$+ G(\eta_1,6)\big(\tfrac{1}{40}e^{-8i\eta} - \tfrac{1}{20}e^{-4i\eta}\big) + G(\eta_1,-6)\big(\tfrac{1}{40}e^{+8i\eta} - \tfrac{1}{20}e^{4i\eta}\big)\Big\} +$$

$$+ \tfrac{1}{3}\Big(\bar{S}_2^{(11)}(\eta_1) - \bar{S}_2^{(9)}(\eta_1) + \bar{S}_2^{(5)}(\eta_1) - \bar{S}_2^{(3)}(\eta_1)\Big).$$

$-\theta_5$ x

$$\operatorname{Im}\Big\{\big(\tfrac{1}{3}e^{3i\eta} - e^{i\eta} + e^{-i\eta} - \tfrac{1}{3}e^{-3i\eta}\big)\tfrac{1}{12}\big(G(\eta_1,1) - G(\eta_1,-1)\big) + \tfrac{8}{27}\big(G(\eta_1,2) + G(\eta_1,-2)\big) +$$

$$+G(\eta_1,3)\big(-\tfrac{1}{4}e^{-i\eta} + \tfrac{1}{20}e^{-5i\eta}\big) + G(\eta_1,-3)\big(-\tfrac{1}{4}e^{i\eta} + \tfrac{1}{20}e^{5i\eta}\big) -$$

$$- \tfrac{8}{27}\big(G(\eta_1,4) + G(\eta_1,-4)\big) +$$

$$+G(\eta_1,5)\big(\tfrac{1}{4}e^{-i\eta} - \tfrac{1}{12}e^{-5i\eta} + \tfrac{1}{54}e^{-9i\eta}\big) + G(\eta_1,-5)\big(\tfrac{1}{4}e^{i\eta} - \tfrac{1}{12}e^{5i\eta} + \tfrac{1}{54}e^{9i\eta}\big) +$$

$$+G(\eta_1,7)\big(\tfrac{1}{30}e^{-5i\eta} - \tfrac{1}{54}e^{-9i\eta}\big) + G(\eta_1,-7)\big(\tfrac{1}{30}e^{5i\eta} - \tfrac{1}{54}e^{9i\eta}\big)\Big\}\Big]$$

Errata R 53, Int 7.

Page	Line	For:	read:
2	9	$se_m(\eta_1)$	$se_n(\eta_1)$
3	23	$\frac{\eta_1}{\sin\ 1}$	$\frac{\eta_1}{\sin\eta_1}$
	24	$\frac{\eta_1}{\sin\ 1}$	$\frac{\eta_1}{\sin\eta_1}$
4	9	$\left\{Jc(\frac{1}{c},\frac{\Omega}{c}) - Ys(\frac{1}{c},\frac{\Omega}{c})\right\}$	$\frac{1}{c}\left\{Jc(\frac{1}{c},c\Omega)-Ys(\frac{1}{c},c\Omega)\right\}$.
	10	$\left\{Js(\frac{1}{c},\frac{\Omega}{c}) + Yc(\frac{1}{c},\frac{\Omega}{c})\right\}$	$\frac{1}{c}\left\{Js(\frac{1}{c},c\Omega)+Yc(\frac{1}{c},c\Omega)\right\}$.
5	21	$\sum_{j=0}^{\infty} x^{-j-1}\,dx$	$\sum_{j=0}^{\infty} A_j\, x^{-j-\ell}\,dx$.
6	4	$\frac{(i\alpha)^j}{J!}$	$\frac{(i\alpha)^j}{j!}$.
	9	$\frac{(i\alpha)^j}{j-1}$	$\frac{(i\alpha)^j}{j!}$.
7	7	$-\frac{\cos 4\zeta}{16} \dots \frac{\cos 2k\zeta}{(2k)^2}$	$-\frac{\cos 4\zeta}{16} - \dots - \frac{\cos 2k\zeta}{(2k)^2}$.
	10	$\frac{\cos 2n\zeta}{(2n)^2}$	$\frac{\cos 2n\zeta}{(2n)^3}$.
	13	$+\frac{\pi^2}{24}$	$+\frac{\pi^2\zeta}{24}$.
	15	$+\frac{\cos\zeta}{1}\ \frac{\cos 2\zeta}{2^2} \dots +$	$+\frac{\cos\zeta}{1} - \frac{\cos 2\zeta}{2^2} + \dots +$
	17	$(2\cos\zeta)$	$(2\cos\zeta/2)$.
	20	$(-1)\ \frac{\sin n\zeta}{n^3}$	$(-1)^n\ \frac{\sin n\zeta}{n^3}$.
8	8	"1 being a positive"	"ℓ being a non-negative"
	10		when $\alpha = -8 + 2\ell$;
	11	$\sum_{K=1}^{\ell}$	$\sum_{K=0}^{\ell}$
	22	$R_n(\beta^2,\Omega,\eta_1)$	$R_n(\beta^2\Omega,\eta_1)$.
9	3	(β^2,Ω,η_1)	$(\beta^2\Omega,\eta_1)$ (twice).
	6	$\frac{R_{n+2}(\beta\Omega,\eta_1)}{(n+1)n}$	$\frac{R_{n+2}(\beta^2\Omega,\eta_1)}{(n+1)n}$.

Page	Line	For:	Read:
9	11	$R_n(\beta,\Omega,\eta_1)$	$R_n(\beta^2\Omega,\eta_1)$
	16	(β^2,Ω,η_1)	$(\beta^2\Omega,\eta_1)$ (twice)
	18	$+\frac{\sin(n+2)\eta_1}{(n+1)(n+2)}+$	$+\frac{\sin(n+2)\eta_1}{(n+1)(n+2)}\}+$
	21	$-R_{r+2}(\beta^2\Omega,\eta_1)=$	$-R_{r+2}(\beta^2\Omega,\eta_1)\}=$
10	9	$O(\frac{-5}{n})$	$O(n^{-5})$
	19	$R_n(\Omega)$	$R_N(\Omega)$
11	2	$-O(\tau^{\frac{1}{2}n}.(\frac{n}{2})^{-\frac{n}{2}-1}.\frac{1}{\Omega})$	$-O\left[\tau^{\frac{1}{2}n}.(\frac{n}{2})^{-\frac{n}{2}-1}.\frac{1}{\Omega}\right]\}.$
12	2	$\frac{\theta_h}{n+h}$	$\frac{\theta_h}{N+h}$
	20	$\frac{e^{-m\zeta}\sin m\eta_1}{m}d\zeta$	$\frac{e^{-m\zeta}\sin m\eta_1}{m}\}d\zeta$
13	13	$-S_2^{(13)}(\eta_1)$	$-\bar{S}_2^{(13)}(\eta_1)$
	15	$\{4e^{i\eta_1}-e^{i\eta_1}-e^{-3i\eta_1}\}$	$\{4e^{i\eta_1}-e^{-i\eta_1}-e^{-3i\eta_1}\}$
14	20	$q/4\ \frac{\theta_7}{168}\ x$	$+\ q/4\ \frac{\theta_7}{168}\ x$
15	1	$q/4\ \frac{\theta_8}{252}\ x$	$+\ q/4\ \frac{\theta_8}{252}\ x$
	15	$-4e^{-i\eta_1}F(2\eta_1,0)-$	$-4e^{-i\eta_1}F(2\eta_1,0)\}-$
16	1	$+\ i\ q/4\ \beta^2\Omega\ A$	$+\ i\ q/4\ \beta^2\Omega\ A\ x$
	17	$S_2^{(10)}(2\eta_1)$	$S_2^{(10)}(2,\eta_1)$
17	11	$-\frac{1}{4}S_2^{(6)}(2,\eta_1)\}$	$-\frac{1}{4}S_2^{(6)}(2,\eta_1)\}]$
18	15	$+\frac{1}{4}S_2^{(10)}(2,\eta_1)\}$	$+\frac{1}{4}S_2^{(10)}(2,\eta_1)\}]$
19	10	$+\frac{2}{3}[\mathrm{Re}\ e^{4i\eta_1}\ldots$	$+\frac{2}{3}[\mathrm{Re}\{e^{4i\eta_1}\ldots$
21	16	$-12\ e^{4i\eta_1}F(2\eta_1,8)$	$-12\ e^{4i\eta_1}F(2\eta_1,0)$

Page	Line	For:	Read:
22	4	$2\ e^{-4i\eta_1}\ F(2\eta_1,0)$	$12\ e^{-4i\eta_1}\ F(2\eta_1,0)$
23	3	η should not be there.	
24	4	$-\ \frac{5921}{1200})]$	$-\ \frac{5921}{1200})]\}$
	10	$+\ \mathrm{Re}\ \{e^{-2i\eta_1}\ \ldots$	$\{+\ \mathrm{Re}\ \{e^{-2i\eta_1}\ \ldots$
	15	$S_2^{(10)}(\eta_1)$	$S_2^{(10)}(2,\eta_1)$
25	5	$8\sum_{n=5}^{\infty} - \frac{Ne_n^{(2)}(0)}{Ne_n^{(2)'}(0)}\ X$	$8\sum_{n=5}^{\infty} + \frac{Ne_n^{(2)}(0)}{Ne_n^{(2)'}(0)}\ X$
		$X\ T_n(\beta,\Omega,\eta_1)^*\ \ S_n(\beta,\eta_1)$	$X\ T_n(\beta,\Omega,\eta_1)\ \ S_n^*(\beta,\Omega,\eta_1)$
	11	$+\ \frac{1}{2}\ F(2\eta_1,-4)e^{i\eta_1}\ -$	$+\ \frac{1}{2}\ F(2\eta_1,-4)e^{i\eta_1}\}\ -$
26	19	$+\ \frac{1}{2}\ F(2\eta_1,-4)+$	$-\ \frac{1}{2}\ F(2\eta_1,-4)+$
	23	$\frac{2}{3}\ S_2^{(10)}(2\eta_1)\ldots$	$\frac{2}{3}\ S_2^{(10)}(2,\eta_1)\ \ldots$
27	6	$-\ \frac{1}{60}\ F(2\eta_1,8)\ \ldots$	$-\ \frac{1}{60}\ F(2\eta_1,-8)\ldots$
	14	$-\ \frac{2}{3}\ S_2^{(6)}(2\eta_1)\ldots$	$-\ \frac{2}{3}\ S_2^{(6)}(2,\eta_1)\ldots$
30	13	$-\ \bar{S}_{(2)}^{(5)}(\eta_1))$	$-\ \bar{S}_2^{(5)}(\eta_1))$.
32	16	$\frac{i\beta^2\Omega}{12}\ A^*$	$-\ \frac{i\beta^2\Omega}{12}\ A^*$.
33	8	$+\ 2\ e^{6i\eta_1})+\ 60\ldots$	$+\ 2\ e^{6i\eta_1})\}+\ 60\ldots$
	19	$A^*[\frac{1}{2}\theta_o\ x$	$+\ A^*[\frac{1}{2}\theta_o\ x$
	35	$G(\eta_1,4)(\ldots$	$+\ G(\eta_1,4)(\ldots$