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An Approximation Connected with cos n and sin n

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1. Introduction.

Some attention has been paid to the effect of truncating a certain power series at its maximum term.

Ramanujan [1] conjectured that if n is a positive integer and

$$1 + \frac{n}{1!} + \frac{n^2}{2!} + \dots + \frac{n^{n-1}}{(n-1)!} + \frac{n^n}{n!}\vartheta_n = \frac{1}{2}e^n, \tag{1,1}$$

then ϑ_n is represented asymptotically by

$$\vartheta_n \sim \frac{1}{3} + \frac{4}{135n} - \frac{8}{2835n^2} - \dots . \tag{1,2}$$

Proofs were published by Szegö [2] and Watson [3].

A similar result connected with e^{-n} was conjectured by Aitken and proved by Copson. Defining φ_n by the equation

$$1 - \frac{n}{1!} + \frac{n^2}{2!} - \dots + (-1)^n\frac{n^n}{n!}\varphi_n = e^{-n}, \tag{1,3}$$

he proved that φ_n is represented asymptotically by

$$\varphi_n \sim \frac{1}{2} + \frac{1}{8n} + \frac{1}{32n^2} + \frac{1}{128n^3} - \frac{13}{256n^4} + \dots . \tag{1,4}$$

This report is concerned with the effect of truncating the series for cos n and sin n in a similar way to the following results:

If γ_{2n+1} is defined by

$$\sum_{k=0}^{n-1} (-1)^k\frac{(2n+1)^{2k}}{(2k)!} + (-1)^n\frac{(2n+1)^{2n}}{(2n)!}\gamma_{2n+1} = \cos(2n+1), \tag{1,5}$$

then it is given asymptotically by

$$\gamma_{2n+1} = \frac{1}{2} - \frac{1}{4(2n+1)} - \frac{1}{8(2n+1)^2} + \frac{3}{16(2n+1)^3} + \frac{55}{32(2n+1)^4} + \frac{599}{64(2n+1)^5} \dots, \tag{1,6}$$

In the same way, if γ_{2n} is defined by

$$\sum_{k=0}^{n-1} (-1)^k\frac{(2n)^{2k}}{(2k)!} + (-1)^n\frac{(2n)^{2n}}{(2n)!}\gamma_{2n} = \cos 2n, \tag{1,7}$$

then

$$\gamma_{2n} \sim \frac{1}{2} + \frac{1}{4(2n)} + \frac{3}{8(2n)^2} + \frac{13}{16(2n)^3} - \frac{59}{32(2n)^4} + \frac{185}{64(2n)^5} \cdots \qquad (1,8)$$

Also, defining σ_{2n+1} and σ_{2n} by the equations

$$\sum_{k=0}^{n-1} (-1)^k \frac{(2n+1)^{2k+1}}{(2k+1)!} + (-1)^n \frac{(2n+1)^{2n+1}}{(2n+1)!} \sigma_{2n+1} = \sin(2n+1), \qquad (1,9)$$

$$\sum_{k=0}^{n-2} (-1)^k \frac{(2n)^{2k+1}}{(2k+1)!} + (-1)^{n-1} \frac{(2n)^{2n-1}}{(2n-1)!} \sigma_{2n} = \sin 2n, \qquad (1,10)$$

the asymptotic expansions for σ_{2n} resp. σ_{2n+1} are again (1,6) resp. (1,8) if only 2n resp. 2n+1 is changed into 2n+1 resp. 2n in the right hand side of the equations.

It has to be noticed that these asymptotic expansions can not be used to compute the values of cos n and sin n, for large n. The maximum term in the series is of the order $n^n/n! \sim e^n/\sqrt{n}$, whereas the order of the error in γ_n and σ_n in n^{-m}, and evidently the order of the error in the left hand side of the equations (1,5), (1,7), (1,9) and (1,10) exceeds the order of the right hand side viz. one.

The same argument holds, moreover, a fortiori to the case of e^{-n}, treated by Copson.

2. Derivation of the expansions.

By means of (1,5), rewritten as

$$\gamma_{2n+1} = \frac{(-1)^n (2n)!}{(2n+1)^{2n}} \left\{ \sum_{k=0}^{n-1} (-1)^{k+1} \frac{(2n+1)^{2k}}{(2k)!} + \cos(2n+1) \right\}$$

$$= \frac{2n(2n-1)}{(2n+1)^2} - \frac{2n(2n-1)(2n-2)(2n-3)}{(2n+1)^4} + \cdots + (-1)^n \frac{(2n)!}{(2n+1)^{2n}} \cos(2n+1)$$

it is easily checked that

$$\gamma_{2n+1} = 1 - \int_0^{2n+1} \cos u \left(1 - \frac{u}{2n+1}\right)^{2n+1} du \,. \qquad (2,1)$$

In the same way the following expressions hold:

$$\gamma_{2n} = 1 - \int_0^{2n} \sin u \left(1 - \frac{u}{2n}\right)^{2n} du \qquad (2,2)$$

$$\sigma_{2n+1} = 1 - \int_0^{2n+1} \sin u \left(1 - \frac{u}{2n+1}\right)^{2n+1} du \qquad (2,3)$$

$$\sigma_{2n} = 1 - \int_0^{2n} \cos u \left(1 - \frac{u}{2n}\right)^{2n} du \qquad (2,4)$$

and one sees that the asymptotic expansions of γ_{2n+1} and σ_{2n}, γ_{2n} and σ_{2n+1} must be the same.

Introducing the function

$$f(t,u) = (1 - tu)^{1/t} \qquad (2,5)$$

one has only to consider two types of integrals:

$$I_1(t) = \int_0^{1/t} \cos u \, f(t,u) du \qquad (2,6)$$

and

$$I_2(t) = \int_0^{1/t} \sin u \, f(t,u) du . \qquad (2,7)$$

In order to get the expressions (2,1) till (2,4) one has to replace t by 1/2n or by 1/(2n+1).

By means of

$$f(t,u) = \exp \frac{\log(1-tu)}{t}$$

$$= \exp \left(- \sum_{k=1}^{\infty} t^{k-1} u^k\right)$$

one sees that f(t,u) is a regular function of the variable t in the neighbourhood of t = 0 and has also an expansion converging in a circle with radius $|t| = 1/|u|$. As a function of the variable u one has the same expansion converging in the region $|u| < 1/|t|$.

Putting

$$f(t,u) = \sum_{k=0}^{N} \frac{f^{(k)}(o,u)}{k!} t^k + t^{N+1} R_N\{f(t,u)\}, \qquad (2,8)$$

and choosing $|t| < 1$, one knows

$$\lim_{u \to 1/t} f(t,u) = 0 .$$

Moreover,

$$I_1(t) = \int_0^{1/t} \cos u \, f(t,u) du =$$

$$= \sum_{k=0}^{N} \frac{t^k}{k!} \int_0^{1/t} \cos u \, f^{(k)}(o,u) du + t^{N+1} \int_0^{1/t} \cos u \, R_N f(t,u) du \qquad (2,9)$$

This can be brought into the form:

$$I_1(t)= \sum_{k=0}^{N} \frac{t^k}{k!} \int_0^{\infty} \cos u \; f^{(k)}(o,u)du + O(t^N). \qquad (2,10)$$

The error, made by changing the upper bound of the integrals in the sum on the right hand side of equation (2,9) is equal to

$$\sum_{k=0}^{N} \frac{t^k}{k!} \int_{1/t}^{\infty} \cos u \; f^{(k)}(o,u)du \; . \qquad (2,11)$$

As is shown below all $f^{(k)}(o,u)$, $(k = 0,1,\ldots,N)$ contain a factor e^{-u}. The integrals do converge therefore and the error is of the order $e^{-1/t}\, t^k$. Because of the fact that the expansion of $f(t,u)$ does not converge out side the region $|u| < 1/|t|$, the limit for $N \to \infty$ of the right hand side of equation (2,10), is not possible.

The result for $I_2(t)$ follows in an equivalent way:

$$I_2(t)= \sum_{k=0}^{N} \frac{t^k}{k!} \int_0^{\infty} \sin u \; f^{(k)}(o,u)du + O(t^N). \qquad (2,12)$$

In order to determine the derivatives $f^{(k)}(o,u)$ we put

$$\log f(t,u)= g(t,u)= \frac{1}{t} \log (1-tu)= - \sum_{k=1}^{\infty} \frac{t^{k-1}u^k}{k} \; . \qquad (2,13)$$

Hence,

$$g^{(k)}(o,u)= - \frac{k!}{k+1} \; u^{k+1} \; . \qquad (2,14)$$

If $^{(n)}$ denotes n times differentiation with respect to the variable t, then by use of

$$f^{(m+1)}(o,u)= \sum_{k=0}^{m} \binom{m}{k} f^{(m-k)}(o,u) \; g^{(k+1)}(o,u) \qquad (2,15)$$

it easily follows that

$$f(o,u)= e^{-u},$$

$$f^{(1)}(o,u)= e^{-u} \left\{- \frac{1}{2} u^2 \right\},$$

$$f^{(2)}(o,u)= e^{-u} \left\{ \frac{1}{4} u^4 - \frac{2}{3} u^3 \right\},$$

$$f^{(3)}(o,u)= e^{-u} \left\{- \frac{1}{8} u^8 + u^5 - \frac{3}{2} u^4 \right\},$$

$$f^{(4)}(o,u)= e^{-u} \left\{ \frac{1}{16} u^8 - u^7 + \frac{13}{3} u^6 - \frac{24}{5} u^5 \right\},$$

$$f^{(5)}(o,u)= e^{-u} \left\{- \frac{1}{32} u^{10} + \frac{5}{6} u^9 - \frac{85}{12} u^8 + 22 u^7 - 20 u^6 \right\}, \qquad (2,16)$$

. . . .

By means of (2,15) it can be proved that all $f^{(k)}(o,u)$ contain the factor e^{-u}.

In order to prove the relations (1,6) and (1,8) one has only to compute the values of the integrals

$$C_n = \int_0^\infty e^{-x} x^n \cos x \, dx, \quad (n > 0), \qquad (2,17)$$

$$S_n = \int_0^\infty e^{-x} x^n \sin x \, dx, \quad (n > 0). \qquad (2,18)$$

This is done in the following chapter.

By means of the relations (2,10), (2,12) and (2,16) the proof is completed.

3. Computation of the integrals C_n and S_n.

From the defining formulae (2,17) and (2,19) one obtains after integration by parts the recurrence relations

$$C_n = \frac{n}{2} (C_{n-1} - S_{n-1}) , \quad (n > 0), \qquad (3,1)$$

$$S_n = \frac{n}{2} (C_{n-1} + S_{n-1}) , \quad (n > 0). \qquad (3,2)$$

Moreover, obviously

$$C_o = S_o = 1/2. \qquad (3,3)$$

The solution of this system is

$$C_n = 2^{-\frac{n+1}{2}} n! \quad \cos \frac{(n+1)\pi}{4} , \qquad (3,4)$$

$$S_n = 2^{-\frac{n+1}{2}} n! \quad \sin \frac{(n+1)\pi}{4} , \qquad (3,5)$$

when n is a positive integer. Indeed, these formulae hold for n = 0, and from the validity for a given n-1 the validity for n follows, as then

$$\begin{matrix} C_n \\ S_n \end{matrix} = \frac{n}{2} (C_{n-1} \mp S_{n-1}) = \frac{n}{2} 2^{-\frac{n}{2}} (n-1)! \left\{ \frac{\cos n\pi}{4} \mp \sin \frac{n\pi}{4} \right\}$$

$$= 2^{-\frac{n+1}{2}} n! \begin{matrix} \cos \\ \sin \end{matrix} \frac{(n+1)\pi}{4} .$$

4. Numerical values.

For checking purposes values of γ_{2n}, γ_{2n+1}, σ_{2n} and σ_{2n+1} were computed in two different ways. The results are given in table I. The values in a left hand side column are computed from the defining relations (1,5) and (1,10), whereas those in a right hand side column are computed from the asymptotic expansions. The results are seen to agree in five decimals practically for n = 10 already.

In tables II and III the values of C_n, S_n, $C_n/n!$ and $S_n/n!$ for n = 0(1)30 are given.

5. References.

(1) Collected Papers of Srinivasa Ramanujan, (1927), XXVI, VII. Theorems on approximate integration and summation of series.

(2) G. Szegö, Ueber einige von Ramanujan gestellte Aufgaben. J.London Math.Soc., (3) 31 (1928), 225-232.

(3) G.N. Watson, Theorems stated by Ramanujan. Proc.London Math.Soc., (2) 29 (1928), 293-308.

(4) E.T. Copson, An Approximation connected with e^{-x}. Proc.Edinb.Math.Soc., (2) 3 (1933), 201-206.

TABLE I

n	γ_{2n+1}		σ_{2n}	
1	0,442221	0,466307	0,454649	0,367188
2	0,452493	0,452245	0,445950	0,432617
3	0,463814	0,463555	0,458651	0,455729
4	0,472381	0,471357	0,468020	0,467868
5	0,476581	0,476556	0,474054	0,474203
10	0,487844	0,487843	0,487226	0,487225

n	γ_{2n}		σ_{2n+1}	
1	0,708074	0,795411	0,635307	0,617133
2	0,594971	0,594256	0,571177	0,569475
3	0,556170	0,554745	0,546107	0,544797
4	0,538995	0,538334	0,533744	0,533291
5	0,529602	0,529408	0,526633	0,526328
10	0,513541	0,513529	0,512853	0,512839

TABLE II

n	C_n	$C_n/n!$
0	1/2	1/2
1	0	0
2	- 1/2	- 1/4
3	- 3/2	- 1/4
4	- 3	- 1/8
5	0	0
6	45	1/16
7	315	1/16
8	1260	1/32
9	0	0
10	- 56700	- 1/64
11	- 6 23700	- 1/64
12	- 37 42200	- 1/128
13	0	0
14	3405 04200	1/256
15	51081 03000	1/256
16	4 08648 24000	1/512
17	0	0
18	- 625 23180 72000	- 1/1024
19	- 11879 40433 68000	- 1/1024
20	- 1 18794 04336 80000	- 1/2048
21	0	0
22	274 41424 01800 80000	1/4096
23	6311 52752 41418 40000	1/4096
24	75738 33028 97020 80000	1/8192
25	0	0
26	- 246 14957 34415 31760 00000	- 1/16384
27	- 6646 03848 29213 57520 00000	- 1/16384
28	- 93044 53876 08990 05280 00000	- 1/32768
29	0	0
30	809 48748 72198 21345 93600 00000	1/65536

TABLE III

	S_n	$S_n/n!$
0	1/2	1/2
1	1/2	1/2
2	1/2	1/4
3	0	0
4	- 3	- 1/8
5	- 15	- 1/8
6	- 45	- 1/16
7	0	0
8	1260	1/32
9	11340	1/32
10	56700	1/64
11	0	0
12	- 37 42200	- 1/128
13	- 486 48600	- 1/128
14	- 3405 40200	- 1/256
15	0	0
16	4 08648 24000	1/512
17	69 47020 08000	1/512
18	625 23180 72000	1/1024
19	0	0
20	- 1 18794 04336 80000	- 1/2048
21	- 24 94674 91072 80000	- 1/2048
22	- 274 41424 01800 80000	- 1/4096
23	0	0
24	75738 33028 97020 80000	1/8192
25	18 93458 25724 25520 00000	1/8192
26	246 14957 34415 31760 00000	1/16384
27	0	0
28	- 93044 53876 08990 05280 00000	- 1/32768
29	- 26 98291 62406 60711 53120 00000	- 1/32768
30	- 809 48748 72198 21345 93600 00000	- 1/65536