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R 80

The Transversal Conical Wave in an Elastic Medium

1951

1. Introduction.

In an earlier publication, viz. "Application of Cagniard's method to the calculation of shock-waves in elastic media", Report R 06, 25, 26 by the Computation Department of the Mathematical Centre, Amsterdam, 1949, the displacements due to longitudinal shockwaves in stratified elastic or liquid media are calculated. In this report a more complicated case is considered in that the displacements due to the transversal conical wave are calculated.

These calculations were carried out on behalf of the Geological Department of the B.P.M. at the Hague.

2. Statement of the problem.

The whole space be filled by two elastic media each filling a half space. Using the notations of Report G.A. 22363, "Reflection and refraction of seismic waves" by B.L. v.d. Waerden, the interface between the two media is defined in cylindercoordinates r, φ, z by $z = -H$. In view of the geophysical applications, this interface is referred to as horizontal, the direction of the positive z-axis as upwards, the medium for $z > -H$ as upper medium and the medium for $z < -H$ as lower medium. Moreover $Z = -z$.

The velocity of propagation of longitudinal waves in the upper resp. lower medium be $1/a$ resp. $1/a'$, that of transversal waves $1/b$ resp. $1/b'$, the density be ρ resp. ρ', Lamé's constants λ and μ resp. λ' and μ'.

From the origin O a spherically symmetric longitudinal wave of prescribed shape is emitted at the time $t = 0$. This is refracted in the lower medium. As $1/a' > 1/a$ the longitudinal part of it causes a conical wave to be originated in the upper medium. Whereas in R 06, 25, 26 the longitudinal conical wave was considered, here the transversal conical wave comes into consideration.

As in the foregoing computations $\sigma = \rho/\rho'$ is taken to be equal to 1. Furthermore, $a = 1$, $b = 1/\sqrt{3}$, $a' = \sqrt{2}$, $b' = \sqrt{2/3}$.

The displacements due to the primary wave can be derived from a potential $\Phi_o = F(t-R/a)$, where $R = \sqrt{r^2 + z^2}$ is the distance to O and t is the time. Especially, F(t) has been taken equal to the unit function, i.e. $F(t) = 0$, when $t < 0$ and $F(t) = 1$, when $t > 0$.

Often reference is made to L. Cagniard: "Réflexion et Réfraction des Ondes Seismiques Progressives".

3. Analytical solution of the problem.

After checking the formulae in GA 22363, the correct formula for the Laplace-transform $\omega(p)$ of the transmission-function $\Omega(t)$ appears to be

$$\omega(p) = -\frac{2ip}{\pi}\int_0^\infty e^{-p\{H\alpha+(H-Z)\beta\}} f(u)\, u^2 du \int_0^\pi e^{-ipur\cos\varphi}\cos\varphi\, d\varphi, \qquad (3,1)$$

where $\alpha = \sqrt{u^2+a^2}$ and $\beta = \sqrt{u^2+b^2}$, and $f(u) = p\,K/(2uN\alpha)$, K and N being defined by the formulae (35) and (38) of GA 22363, p. 8.

The transformation to a new variable t:

$$t = H\alpha + (H-Z)\beta + i\,u\,r\cos\varphi \tag{3,2}$$

leads, with $\Theta = Ha + (H-z)b$, to

$$\omega(p) = -\frac{2p}{\pi}\int_{\Theta}^{\infty} e^{-pt}\,dt \int_{u_0}^{u_\pi} \frac{f(u)u\cos\varphi}{r\sin\varphi}\,du. \tag{3,3}$$

The limits of integration u_0 and u_π are the roots of the equations

$$t = H\alpha + (H-Z)\beta \pm i\,u\,r. \tag{3,4}$$

The transformation (3,2) is justified at great length by Cagniard.

As the discussion of the displacements will be restricted to the time-interval before the arrival of the directly reflected transversal wave, u_0 and u_π are purely imaginary, and moreover $u_0 = -u_\pi$. If, therefore, Im $u_0 \geqslant 0$, then

$$t = H\sqrt{u_0^2 + a^2} + (H-Z)\sqrt{u_0^2 + b^2} - i\,u_0\,r \tag{3,5}$$

because r increases with t.

Hence,

$$\omega(p) = \frac{2p}{\pi r}\int_{\Theta}^{\infty} e^{-pt}\,dt \int_{-u_0}^{u_0} f(u)u\cot\varphi\,du. \tag{3,6}$$

The inversion of the Laplace transformation yields the transmission-function

$$\Omega(t) = 0 \qquad \text{when } t < \Theta,$$

$$\Omega(t) = \frac{2i}{\pi r}\int_{-u_0}^{u_0} \frac{f(u)u\{t-H\alpha-(H-Z)\beta\}}{\sqrt{u^2r^2+\{t-H\alpha-(H-Z)\beta\}^2}}\,du, \quad \text{when } t \geqslant \Theta. \tag{3,7}$$

This formula is in agreement with the one given by Cagniard. The corresponding formula (165) of GA 22363 is incorrect.

It should be noticed that all square roots, entering in these and following expressions, are so defined that they take real positive values on the positive real axis of the u-plane.

Moreover, f(u), α and β are functions of u^2 only. If, therefore,

$$F(u) = 2\,f(u)\left\{ t - H\alpha - (H-Z)\beta \right\}, \tag{3.8}$$

then $F(u) = F(-u)$, and according to (3,7), $\Omega(t)$ vanishes as long as u_0, that moves outward along the imaginary axis with increasing t, has not passed the first branchpoint ia'. If, however, u_0 has passed the branchpoint, i.e. $a' \leqslant |u_0| < a$, the sign of the root $\sqrt{u^2 + a'^2}$ is changed in F(-u). When ε, δ, ε' and δ' stand for functions of u^2 only that do no contain the said root, then

$$F(u) = \frac{\varepsilon + \delta\sqrt{u^2+a'^2}}{\varepsilon' + \delta'\sqrt{u^2+a'^2}}, \tag{3,9}$$

and

$$F(u) - F(-u) = \frac{2(\delta\varepsilon' - \delta'\varepsilon)}{\varepsilon'^2 - \delta'^2(u^2+a'^2)}\sqrt{u^2+a'^2} = P(u^2,t,Z)\sqrt{u^2+a'^2}. \tag{3,10}$$

Hence,

$$\Omega(t) = \frac{i}{2\pi r}\int_{ia'}^{u_o} \frac{P(u^2,t,Z)\sqrt{u^2+a'^2}\,du^2}{\sqrt{u^2r^2+\{t-H\alpha-(H-Z)\beta\}^2}}. \tag{3,11}$$

From (3,4) it follows that both u_o and $-u_o$ satisfy the equation

$$u^2r^2 + \left\{t - H\alpha - (H-z)\beta\right\}^2 = 0 .$$

Hence, it is appropriate to define the function $N^2(u^2)$ by

$$u^2r^2 + \left\{t - H\alpha - (H-z)\beta\right\}^2 = N^2(u^2)(u^2-u_o^2), \tag{3,12}$$

and a new variable ψ by

$$u^2 = u_o^2 - (a'^2 + u_o^2)\sin^2\psi = u_o^2\cos^2\psi - a'^2\sin^2\psi . \tag{3,13}$$

Then (3,11) can be written in the form

$$\Omega(t) = -\frac{a'^2+u_o^2}{\pi r}\int_0^{\pi/2} \frac{P(u^2)}{N(u^2)}\cos^2\psi\,d\psi, \tag{3,14}$$

if the sign of the root is taken in accordance with:

$$\sqrt{\frac{u^2+a'^2}{u^2-u_o^2}} = i\cot\psi. \tag{3,15}$$

From $\Omega(t)$ the displacements in radial direction U_r and in vertical direction U_z can be found by differentiation with respect to r or z.

$$U_r = -\frac{\partial\Omega(t)}{\partial r} = +\frac{\Omega(t)}{r} + \frac{(a'^2+u_o^2)}{\pi r}\frac{\partial u_o^2}{\partial r}\int_0^{\pi/2}\frac{\partial P(u^2)}{\partial u^2}\frac{\cos^4\psi}{N(u^2)}\,d\psi + $$
$$- \frac{(a'^2+u_o^2)}{\pi r}\int_0^{\pi/2}\frac{P(u^2)}{N^2(u^2)}\frac{\partial N(u^2)}{\partial r}\cos^2\psi\,d\psi + \frac{1}{\pi r}\frac{\partial u_o^2}{\partial r}\int_0^{\pi/2}\frac{P(u^2)}{N(u^2)}\cos^2\psi\,d\psi . \tag{3,16}$$

Differentiating (3,12) with respect to r in order to compute $\frac{\partial N(u^2)}{\partial r}$ gives:

$$N(u^2)\,\frac{\partial N(u^2)}{\partial r} = \frac{-2u^2r-\cos^2\psi\,\frac{\partial u_o^2}{\partial r}\left[Q(u^2)-N^2(u^2)\right]-N^2(u^2)\frac{\partial u_o^2}{\partial r}}{2(a'^2+u_o^2)\sin^2\psi}\,, \tag{3,17}$$

with

$$Q(u^2)= r^2 - \left\{t - H\alpha - (H-Z)\beta\right\}\left\{\frac{H}{\alpha}+\frac{H-Z}{\beta}\right\}. \tag{3,18}$$

One can show that $N\frac{\partial N}{\partial r}$ stays finite even for the value $\psi = 0$ or $u = u_o$, and therefore

$$U_r = +\frac{\Omega(t)}{r} + \frac{(a'^2+u_o^2)}{\pi r}\,\frac{\partial u_o^2}{\partial r}\int_0^{\pi/2}\frac{\partial P(u^2)}{\partial u^2}\,\frac{\cos^4\psi}{N(u^2)}\,\partial\psi\, +$$

$$+\frac{1}{2\pi r}\,\frac{\partial u_o^2}{\partial r}\int_0^{\pi/2}\frac{P(u^2)}{N(u^2)}\,\frac{\cos^2\psi}{\sin^2\psi}\,d\psi + \frac{1}{2\pi r}\int_0^{\pi/2}\frac{P(u^2)}{N^3(u^2)}\Big\{2u^2r\, +$$

$$+\cos^2\psi\,\frac{\partial u_o^2}{\partial r}\left[Q(u^2)-N^2(u^2)\right]\Big\}\,\frac{\cos^2\psi}{\sin^2\psi}\,d\psi\, +$$

$$+\frac{1}{\pi r}\,\frac{\partial u_o^2}{\partial r}\int_0^{\pi/2}\frac{P(u^2)}{N(u^2)}\,\cos^2\psi\,d\psi\,. \tag{3,19}$$

In the same way U_z is found as follows

$$U_z = -\frac{\partial\Omega(t)}{\partial z} = \frac{\partial\Omega(t)}{\partial Z} =$$

$$-\frac{1}{\pi r}\,\frac{\partial u_o^2}{\partial Z}\int_0^{\pi/2}\frac{P(u^2,t,Z)}{N(u^2)}\,\cos^2\psi\,d\psi$$

$$-\frac{(a'^2+u_o^2)}{\pi r}\,\frac{\partial u_o^2}{\partial Z}\int_0^{\pi/2}\frac{\partial P(u^2,t,Z)}{\partial u^2}\,\frac{\cos^4\psi}{N(u^2)}\,d\psi$$

$$+\frac{(a'^2+u_o^2)}{\pi r}\int_0^{\pi/2}\frac{P(u^2,t,Z)}{N^2(u^2)}\,\frac{\partial N(u^2)}{\partial Z}\,\cos^2\psi\,d\psi$$

$$-\frac{(a'^2+u_o^2)}{\pi r}\int_0^{\pi/2}\frac{P(u^2,t,Z)\,\beta(u^2)\cos^2\psi}{\left\{(t-H\alpha-H-Z)\beta\right\}\,N(u^2)}\,d\psi\,. \tag{3,20}$$

4. A check.

The following method is used to check the results.
From (3,11) follows

$$\omega(p) = -\frac{2pi}{\pi}\int_0^{\infty}\int_0^{\pi} e^{-p\left\{H\alpha+(H-Z)\beta+iur\cos\varphi\right\}}\,u^2\,f(u)\,du\,\cos\varphi\,d\varphi\,.$$

If $\omega(p)$ is first differentiated with respect to r, and then the inverse Laplace transformation is applied $\partial\Omega(t)/\partial r$ is found.

Now

$$\frac{\partial\omega(p)}{\partial r} = -\frac{2p^2}{\pi}\int_0^\infty\int_0^\pi e^{-p\{H\alpha+(H-Z)\beta+iur\cos\varphi\}}u^3 f(u)\cos^2\varphi\, du\, d\varphi$$

$$= -\frac{2p^2 i}{\pi r}\int_\theta^\infty e^{-pt}\, dt\int_{-u_0}^{u_0}\frac{u^2 f(u)\cos^2\varphi}{\sin\varphi}\, du =$$

$$= \frac{2pi}{\pi r}\int_\theta^\infty de^{-pt}\int_{-u_0}^{u_0}\frac{u^2 f(u)\cos^2\varphi}{\sin\varphi}\, du =$$

$$= -\frac{2pi}{\pi r}\int_\theta^\infty e^{-pt}\, dt\,\frac{\partial}{\partial t}\int_{-u_0}^{u_0}\frac{u^2 f(u)\cos^2\varphi}{\sin\varphi}\, du .$$

Hence,

$$\frac{\partial\Omega(t)}{\partial r} = 0 \qquad , \text{ when } t<\theta,$$

$$\frac{\partial\Omega(t)}{\partial r} = -\frac{2i}{\pi r}\frac{\partial}{\partial t}\int_{-u_0}^{u_0}\frac{u^2 f(u)\cos^2\varphi}{\sin\varphi}\, du, \text{ when } t\geqslant\theta,$$

which result can be rewritten as

$$\frac{\partial\Omega(t)}{\partial r} = \frac{2i}{\pi r^2}\frac{\partial}{\partial t}\int_{-u_0}^{u_0}\frac{u\, f(u)\{t-H\alpha-(H-Z)\beta\}^2}{\sqrt{u^2r^2+\{t-H\alpha-(H-Z)\beta\}^2}}\, du =$$

$$= \frac{i}{\pi r^2}\frac{\partial}{\partial t}\int_0^{u_0}\frac{[F(u)-F(-u)]\{t-H\alpha-(H-Z)\beta\}u\, du}{\sqrt{u^2r^2+\{t-H\alpha-(H-Z)\beta\}^2}} =$$

$$= \frac{i}{2\pi r^2}\frac{\partial}{\partial t}\int_{ia'}^{u_0}\frac{P(u^2)}{N(u^2)}\sqrt{\frac{u^2+a'^2}{u^2-u_0^2}}\{t-H\alpha-(H-Z)\beta\}\, du^2 =$$

$$= +\frac{1}{\pi r^2}\frac{\partial}{\partial t}(a'^2+u_0^2)\int_0^{\pi/2}\frac{P(u^2)}{N(u^2)}\{t-H\alpha-(H-Z)\beta\}\cos^2\psi\, d\psi =$$

$$= \frac{2}{\pi r^2}(a'^2+u_0^2)\int_0^{\pi/2}\frac{P(u^2)}{N(u^2)}\cos^2\psi\, d\psi + \frac{1}{\pi r^2}\frac{\partial u_0^2}{\partial t}\cdot$$

$$\int_0^{\pi/2}\frac{P(u^2)}{N(u^2)}\{t-H\alpha-(H-Z)\beta\}\cos^2\psi\, d\psi + \frac{1}{\pi r^2}(a'^2+u_0^2)\frac{\partial u_0^2}{\partial t}\cdot$$

$$\int_0^{\pi/2}\frac{\partial}{\partial u^2}\Big[P(u^2)\{t-H\alpha-(H-Z)\beta\}\Big]\frac{\cos^4\psi}{N(u^2)}\, d\psi$$

$$-\frac{a'^2+u_0^2}{\pi r^2}\int_0^{\pi/2}\frac{P(u^2)}{[N(u^2)]^2}\frac{\partial N(u^2)}{\partial t}\{t-H\alpha-(H-Z)\beta\}\cos^2\psi\, d\psi .$$

Hence,

$$U_r = -\frac{\partial\Omega(t)}{\partial r} = \frac{2\Omega(t)}{r} \cdot \frac{1}{\pi r^2}\frac{\partial u_o^2}{\partial t}\int_0^{\pi/2} \frac{P(u^2)}{N(u^2)}\left\{t-H\alpha-(H-Z)\beta\right\}\cos^2\psi\, d\psi -$$

$$-\frac{(a'^2+u_o^2)}{\pi r^2}\frac{\partial u_o^2}{\partial t}\int_0^{\pi/2}\frac{\partial}{\partial u^2}\left[P(u^2)\left\{t-H\alpha-(H-Z)\beta\right\}\right]\frac{\cos^4\psi}{N(u^2)}\, d\psi +$$

$$+\frac{a'^2+u_o^2}{\pi r^2}\int_0^{\pi/2}\frac{P(u^2)}{N^2(u^2)}\cdot\frac{\partial N(u^2)}{\partial t}\left\{t-H\alpha-(H-Z)\beta\right\}\cos^2\psi\, d\psi . \qquad (4,1)$$

After the carrying out of the differentiation $\partial N/\partial t$, this leads to

$$U_r = \frac{2\Omega(t)}{r} - \frac{1}{\pi r^2}\frac{\partial u_o^2}{\partial t}\int_0^{\pi/2}\frac{P(u^2)}{N(u^2)}\left\{t-H\alpha-(H-Z)\beta\right\}\cos^2\psi\, d\psi$$

$$-\frac{1}{2\pi r^2}\frac{\partial u_o^2}{\partial t}\int_0^{\pi/2}\frac{P(u^2)}{N(u^2)}\left\{t-H\alpha-(H-Z)\beta\right\}\frac{\cos^2\psi}{\sin^2\psi}\, d\psi$$

$$-\frac{(a'^2+u_o^2)}{\pi r^2}\cdot\frac{\partial u_o^2}{\partial t}\int_0^{\pi/2}\frac{\partial}{\partial u^2}\left[P(u^2)\left\{t-H\alpha-(H-Z)\beta\right\}\right]\frac{\cos^4\psi}{N(u^2)}\, d\psi$$

$$+\frac{1}{2\pi r^2}\int_0^{\pi/2}\frac{P(u^2)}{N^3(u^2)}\left\{t-H\alpha-(H-Z)\beta\right\}\cdot$$

$$\left[\frac{-2\left\{t-H\alpha-(H-Z)\beta\right\}-\frac{\partial u_o^2}{\partial t}\left[Q(u^2)-N^2(u^2)\right]\cos^2\psi}{\sin^2\psi}\right]\cos^2\psi\, d\psi . \qquad (4,2)$$

Subtracting (3,19) from (4,2), and using the identity

$$\frac{\partial u_o^2}{\partial r} = +\, i\, u_o\frac{\partial u_o^2}{\partial t} \quad , \qquad (4,3)$$

one gets

$$V = \frac{\Omega(t)}{r} - \frac{1}{\pi r^2}\frac{\partial u_o^2}{\partial t^2}\int_0^{\pi/2}\frac{P(u^2)}{N(u^2)}\left\{t-H\alpha-(H-Z)\beta + iu_o r\right\}\cos^2\psi\, d\psi$$

$$-\frac{1}{\pi r^2}\int_0^{\pi/2}\frac{P(u^2)}{N^3(u^2)}\frac{\left[\left\{t-H\alpha-(H-Z)\beta\right\}^2+u^2r^2\right]}{\sin^2\psi}\cos^2\psi\, d\psi \cdot$$

$$-\frac{1}{2\pi r^2}\frac{\partial u_o^2}{\partial t}\int_0^{\pi/2}\frac{P(u^2)}{N^3(u^2)}\left[Q(u^2)-N^2(u^2)\right]\left[\left\{t-H\alpha-(H-Z)\beta\right\}+i\,u_o\, r\right]\frac{\cos^4\psi}{\sin^2\psi}\, d\psi +$$

$$-\frac{1}{2\pi r^2}\frac{\partial u_o^2}{\partial t}\int_0^{\pi/2}\frac{P(u^2)}{N(u^2)}\frac{\cos^2\psi}{\sin^2\psi}\left\{t-H\alpha-(H-Z)\beta + i\,u_o\, r\right\} d\psi$$

$$-\frac{(a'^2+u_o^2)}{\pi r^2}\frac{\partial u_o^2}{\partial t}\int_0^{\pi/2}\frac{\partial}{\partial u^2}\left[P(u^2)\left\{t-H\alpha-(H-Z)\beta + i\,u_o\, r\right\}\right]\frac{\cos^4\psi}{N(u^2)}\, d\psi . \qquad (4,4)$$

Here V stands for the difference between the results for V_r found by the two methods.

The last integral is partially integrated:

$$\frac{1}{2\pi r^2}\frac{\partial u_o^2}{\partial t}\int_o^{\pi/2}\frac{\partial}{\partial\varphi}\left[P(u^2)\left\{t-H\alpha-(H-Z)\beta+i\,u_o\,r\right\}\right]\frac{\cos^3\psi}{N(u^2)\sin\psi}\,d\psi =$$

$$= \frac{1}{2\pi r^2}\frac{\partial u_o^2}{\partial t}\int_o^{\pi/2}\frac{P(u^2)}{N^2(u^2)}\left\{t-H\alpha-(H-Z)\beta+i\,u_o\,r\right\}\frac{\partial N(u^2)}{\partial\psi}\frac{\cos^3\psi}{\sin\psi}\,d\psi +$$

$$+ \frac{1}{2\pi r^2}\frac{\partial u_o^2}{\partial t}\int_o^{\pi/2}\frac{P(u^2)}{N(u^2)}\left\{t-H\alpha-(H-Z)\beta+i\,u_o\,r\right\}\frac{(1+2\sin^2\psi)\cos^2\psi}{\sin^2\psi}\,d\psi. \qquad (4.5)$$

Again $\frac{\partial N(u^2)}{\partial\psi}$ is computed from (3,12). Inserting (4,5) into (4.4) it is found eventually that V = 0.

In the same way U_z can be checked:

$$\omega(p) = -\frac{2pi}{\pi}\int_o^\infty\int_o^\pi e^{-p\{H\alpha+(H-Z)\beta+i\,u\,r\cos\varphi\}}\,u^2 f(u)\cos\varphi\,d\,u\,d\varphi,$$

$$\frac{\partial\omega(p)}{\partial Z} = -\frac{2p^2 i}{\pi}\int_o^\infty\int_o^\pi e^{-p\{H\alpha+(H-Z)\beta+iur\cos\varphi\}}\beta(u^2)f(u)\,du\,\cos\varphi\,d\varphi$$

$$= -\frac{2p^2 i}{\pi}\int_\theta^\infty\int_o^\pi e^{-pt}\beta(u^2)\,f(u)\,u^2\frac{\partial u}{\partial t}\,dt\,\cos\varphi\,d\varphi$$

$$= \frac{2pi}{\pi}\int_{e^{-p\theta}}^\infty de^{-pt}\int_o^\pi\beta(u^2)\,f(u)\,u^2\frac{\partial u}{\partial t}\,dt\,\cos\varphi\,d\varphi$$

$$= -\frac{2pi}{\pi}\int_\theta^\infty e^{-pt}\,dt\,\frac{d}{dt}\int_o^\pi\beta(u^2)\,f(u)\,u^2\frac{\partial u}{\partial t}\cos\varphi\,d\varphi.$$

Hence,

$$U_z = 0 \qquad , \text{ when } t<\theta,$$

$$U_z = -\frac{2i}{\pi}\frac{d}{dt}\int_o^\pi\beta(u^2)\,f(u)\,u^2\frac{\partial u}{\partial t}\cos\varphi\,d\varphi, \text{ when } t\geqslant\theta.$$

Or again:

$$U_z = -\frac{2i}{\pi}\frac{d}{dt}\int_{-u_o}^{u_o}\beta(u^2)\,f(u)\,u^2\frac{\partial u}{\partial t}\frac{\partial\varphi}{\partial u}\cos\varphi\,du$$

$$= \frac{-2}{\pi r}\frac{d}{dt}\int_{-u_o}^{u_o}\beta(u^2)\,f(u)u\,\text{cotg}\,\varphi\,du$$

$$= \frac{2i}{\pi r}\frac{d}{dt}\int_{-u_o}^{u_o}\frac{\beta(u^2)f(u)\{t-H\alpha-(H-Z)\beta\}}{\sqrt{u^2r^2+\{t-H\alpha-(H-Z)\beta\}^2}}\,u\,du$$

$$= \frac{+2i}{\pi r}\frac{d}{dt}\int_{ia'}^{u_o}\frac{[f(u)-f(-u)]\,\beta(u^2)\{t-H\alpha-(H-Z)\beta\}}{\sqrt{u^2r^2+\{t-H\alpha-(H-Z)\beta\}^2}}\,u\,du$$

$$= \frac{i}{2\pi r}\frac{d}{dt}\int_{ia'}^{u_0}\frac{P(u^2,t,Z)\beta(u^2)}{N(u^2,t,Z,r)}\sqrt{\frac{u^2+a'^2}{u^2-u_0^2}}\,du^2$$

$$= \frac{-1}{\pi r}\frac{d}{dt}\left\{(a'^2+u_0^2)\int_0^{\pi/2}\frac{P(u^2,t,Z)\beta(u^2)}{N(u^2,t,Z,r)}\cos^2\psi\,d\psi\right\}.$$

So the result is:

$$U_z = \frac{-(a'^2+u_0^2)}{\pi r}\int_0^{\pi/2}\frac{P(u^2,t,Z)\beta(u^2)}{\{t-H\alpha-(H-Z)\beta\}N(u^2)}\cos^2\psi\,d\psi$$

$$-\frac{1}{\pi r}\frac{\partial u_0^2}{\partial t}\int_0^{\pi/2}\frac{P(u^2,t,Z)\beta(u^2)}{N(u^2)}\cos^2\psi\,d\psi$$

$$-\frac{(a'^2+u_0^2)}{\pi r}\frac{\partial u_0^2}{\partial t}\int_0^{\pi/2}\frac{\partial}{\partial u^2}\left\{P(u^2,t,Z)\beta(u^2)\right\}\frac{\cos^4\psi}{N(u^2)}\,d\psi$$

$$+\frac{(a'^2+u_0^2)}{\pi r}\int_0^{\pi/2}\frac{P(u^2,t,Z)\beta(u^2)}{N^2(u^2)}\frac{\partial N(u^2)}{\partial t}\cos^2\psi\,d\psi. \qquad (4,6)$$

Using the identities:

$$\frac{\partial u_0^2}{\partial Z} = \sqrt{u_0^2+b^2}\,\frac{\partial u_0^2}{\partial t}, \qquad (4,7)$$

$$\frac{\partial N(u^2)}{\partial t} = \frac{-2\{t-H\alpha-(H-Z)\beta\}-\frac{\partial u_0^2}{\partial t}\left[Q(u^2)-N^2(u^2)\right]\cos^2\psi-N^2(u^2)\frac{\partial u_0^2}{\partial t}}{2N(u^2)(a'^2+u_0^2)\sin^2\psi}, \qquad (4,8)$$

$$\frac{\partial N(u^2)}{\partial Z} = \frac{-2\{t-H\alpha-(H-Z)\beta\}\beta-\cos^2\psi\frac{\partial u_0^2}{\partial Z}\left[Q(u^2)-N^2(u^2)\right]-N^2(u^2)\frac{\partial u_0^2}{\partial Z}}{2N(u^2)\left[a'^2+u_0^2\right]\sin^2\psi} \qquad (4,9)$$

and subtracting (3,20) from (4,6) one gets as new difference V'

$$V' = \frac{1}{\pi r}\frac{\partial u_0^2}{\partial t}\int_0^{\pi/2}\frac{P(u^2,t,Z)}{N(u^2)}\cos^2\psi\left[\sqrt{u_0^2+b^2}-\sqrt{u^2+b^2}\right]d\psi$$

$$+\frac{(a'^2+u_0^2)}{\pi r}\frac{\partial u_0^2}{\partial t}\int_0^{\pi/2}\frac{\partial\left\{P(u^2,t,Z)\left[\sqrt{u_0^2+b^2}-\sqrt{u^2+b^2}\right]\right\}}{\partial u^2}\frac{\cos^4\psi}{N(u^2)}\,d\psi$$

$$+\frac{1}{2\pi r}\frac{\partial u_0^2}{\partial t}\int_0^{\pi/2}\frac{P(u^2,t,Z)}{N^3(u^2)}\left[Q(u^2)-N^2(u^2)\right]\left[\sqrt{u_0^2+b^2}-\sqrt{u^2+b^2}\right]\frac{\cos^4\psi}{\sin^2\psi}\,d\psi$$

$$+\frac{1}{2\pi r}\frac{\partial u_0^2}{\partial t}\int_0^{\pi/2}\frac{P(u^2,t,Z)}{N(u^2)}\left[\sqrt{u_0^2+b^2}-\sqrt{u^2+b^2}\right]\frac{\cos^2\psi}{\sin^2\psi}\,d\psi, \qquad (4,10)$$

One treats separately:

$$I = \frac{(a'^2+u_0^2)}{\pi r}\frac{\partial u_0^2}{\partial t}\int_0^{\pi/2}\frac{\partial\left\{P(u^2,t,Z)\left[\sqrt{u_0^2+b^2}-\sqrt{u^2+b^2}\right]\right\}}{\partial u^2}\frac{\cos^4\psi}{(u^2)}\,d\psi =$$

$$= \frac{-1}{2\pi r}\frac{\partial u_o^2}{\partial t}\int_0^{\pi/2}\frac{\partial\left\{P(u^2,t,Z)\left[\sqrt{u_o^2+b^2}-\sqrt{u^2+b^2}\right]\right\}}{\partial\psi}\frac{\cos^3\psi}{N(u^2)\sin\psi}\,d\psi$$

$$= \frac{-1}{2\pi r}\frac{\partial u_o^2}{\partial t}\int_0^{\pi/2}\frac{P(u^2,t,Z)\left[\sqrt{u_o^2+b^2}-\sqrt{u^2+b^2}\right]}{N^2(u^2)}\frac{\partial N(u^2)}{\partial\psi}\frac{\cos^3\psi}{\sin\psi}\,d\psi$$

$$- \frac{1}{2\pi r}\frac{\partial u_o^2}{\partial t}\int_0^{\pi/2}\frac{P(u^2,t,Z)\left[\sqrt{u_o^2+b^2}-\sqrt{u^2+b^2}\right]}{N(u^2)}\frac{(1+2\sin^2\psi)\cos^2\psi}{\sin^2\psi}\,d\psi.$$

$$\frac{\partial N(u^2)}{\partial\psi} = \left\{Q(u^2)-N^2(u^2)\right\}\frac{\cos\psi}{\sin\psi}$$

$$I = \frac{-1}{2\pi r}\frac{\partial u_o^2}{\partial t}\int_0^{\pi/2}\frac{P(u^2,t,Z)\left[\sqrt{u_o^2+b^2}-\sqrt{u^2+b^2}\right]}{N^2(u^2)}\left\{Q(u^2)-N^2(u^2)\right\}\frac{\cos^4\psi}{\sin^2\psi}\,d\psi$$

$$\frac{-1}{2\pi r}\frac{\partial u_o^2}{\partial t}\int_0^{\pi/2}\frac{P(u^2,t,Z)\left[\sqrt{u_o^2+b^2}-\sqrt{u^2+b^2}\right]}{N^2(u^2)}\frac{\cos^2\psi}{\sin^2\psi}\,d\psi$$

$$\frac{-1}{\pi r}\frac{\partial u_o^2}{\partial t}\int_0^{\pi/2}\frac{P(u^2,t,Z)\left[\sqrt{u_o^2+b^2}-\sqrt{u^2+b^2}\right]}{N^2(u^2)}\cos^2\psi\,d\psi \qquad (4,11)$$

By means of (4,11) we get from (4,10) again $V' = 0$.

5. Indefinite forms.

In this section the values of some expressions are given for which both denominator and nominator vanish for $u = u_o$.

So, cf. (3,12)

$$N^2(u_o^2) = r^2 + i\,r\,u_o\left\{\frac{H}{\alpha_o}+\frac{H-Z}{\beta_o}\right\}, \qquad (5,1)$$

where $\alpha_o = \sqrt{u_o^2+a^2}$ and $\beta_o = \sqrt{u_o^2+b^2}$.

It can be shown by means of (3,18) that

$$Q(u_o^2) = N^2(u_o^2). \qquad (5,2)$$

By means of this relation and

$$2(a'^2+u_o^2)\,N(u^2)\,\frac{\partial N(u^2)}{\partial r} = \frac{2u^2r(a'^2+u_o^2)+(a'^2+u^2)\frac{\partial u_o^2}{\partial r}\,Q(u^2)}{(u^2-u_o^2)} - N^2(u^2)\,\frac{\partial u_o^2}{\partial r},$$

one has

$$2\,N(u_o^2)\,\frac{\partial N(u_o^2)}{\partial r} = 2\,r + \frac{\partial u_o^2}{\partial r}\frac{\partial Q(u_o^2)}{\partial u^2}, \qquad (5,3)$$

where

$$\frac{\partial Q(u_o^2)}{\partial u^2} = \frac{1}{2}\left\{ \frac{H}{\alpha_o} + \frac{H-Z}{\beta_o}\right\}^2 - \frac{iru_o}{2}\left\{\frac{H}{\alpha_o^3} + \frac{H-Z}{\beta_o^3}\right\}. \qquad (5,4)$$

In the same way one has

$$2N(u_o^2)\,\frac{\partial N(u_o^2)}{\partial Z} = -\left\{\frac{H}{\alpha_o} + \frac{H-Z}{\beta_o}\right\}\beta_o - \frac{iru_o}{\beta_o} + \frac{\partial Q(u_o^2)}{\partial u^2}\,\frac{\partial u_o^2}{\partial Z}. \qquad (5,5)$$

and

$$2N(u_o^2)\,\frac{\partial N(u_o^2)}{\partial t} = -\left\{\frac{H}{\alpha_o} + \frac{H-Z}{\beta_o}\right\} + \frac{\partial Q(u_o^2)}{\partial u^2}\,\frac{\partial u_o^2}{\partial t}. \qquad (5,6)$$

These relations serve as a means both for computation and checking.

6. The initial jumps.

To compute the initial jumps, i.e. the values of the displacements for $t = t_o$, one uses the formula's (3,16) and (3,20).

Putting $u_o^2 = -a'^2$, one has

$$P(u^2) = P(-a'^2),$$

and $N(u^2) = N(-a'^2)$,

and one obtains

$$\left[U_r\right]_{t=t_o} = \frac{1}{\pi r}\left\{\frac{\partial u_o^2}{\partial r}\right\}\int_0^{\pi/2}\frac{P(-a'^2)}{N(-a'^2)}\cos^2\psi\,d\psi$$

$$= \frac{1}{4r}\,\frac{P(-a'^2)}{N(-a'^2)}\left\{\frac{\partial u_o^2}{\partial r}\right\}_{u_o=ia}, \qquad (6,1)$$

and

$$\left[U_z\right]_{t=t_o} = -\frac{1}{4r}\,\frac{P(-a'^2)}{N(-a'^2)}\left\{\frac{\partial u_o^2}{\partial Z}\right\}_{u_o=ia}. \qquad (6,2)$$

Apparently

$$\frac{\partial u_o^2}{\partial r} = i\,u_o\,\frac{\partial u_o^2}{\partial t} = \frac{i\,u_o}{\sqrt{u_o^2+b^2}}\,\frac{\partial u_o^2}{\partial Z}.$$

The ratio of the two displacements of the initial jump is therefore found as

$$\frac{U_r}{U_z} = \frac{a'}{\sqrt{b^2-a'^2}}$$

what takes the value 0.4472135 for the assumed values of the parameters.

7. Numerical results.

The computations of the displacements were made by means of both methods.

The discrepancy between the two results was always less than 5.10^{-5}.

t	U_r	U_z
7.238	- 0.01752	- 0.03917
7.3	- 0.01629	- 0.03517
7.45	- 0.01347	- 0.02650
7.6	- 0.01093	- 0.01929
7.75	- 0.00875	- 0.01352
7.9	- 0.00690	- 0.00892
8.05	- 0.00527	- 0.00531
8.2	- 0.00394	- 0.00232
8.4	- 0.00217	+ 0.00127
8.4931	- ∞	- ∞


0·04
0·03
0·02
0·01
0
t
73
75
77
79
81
83
85