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R 110

Deformation of Bearing Surfaces, II

1952

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1. Introduction.

The following computations were performed on request of the Royal Shell Laboratory at Delft. (See your introductory commission of March 15th, 1951).

The subject is concerned with the elastic deformation of bearing surfaces, embodied in the following reports:

(1) The elastic deformation of the surface of a half space under the hydrodynamic pressure distribution, going with a parabolic lubrication-film (IM 524) by H. Blok and J.W. Cohen.

(2) On the elastic hydrodynamic problem of the lubrication of spur gear teeth. By H. Blok.

(3) Deformation of bearing surfaces, R 51 by the Computation Department of the Mathematical Centre.

2. Method and range of the computations.

The specified task was:

1 . To determine x_1 in the neighbourhood of $x = k$ in such a way that

$$y''(x_1) = 0, \qquad (2,1)$$

to state the values

$$y(x_1),\ y'(x_1) \text{ and } y''(x_1),$$

and the function $F(x,x_1)$ defined by (see (2) formula (64)):

$$F(x,x_1) = (x-x_1)(x+x_1-2\, z_\infty) + \frac{2}{y''(x_1)}\Big[\, y(x_1) - y(x)\Big], \qquad (2,2)$$

for the area $x = -2.0(0.1)7.0$.

2 . To determine the function $\gamma(G) = \frac{1+k^2}{2G}\int_{-k}^{\infty} \ln\left\{1-G(t)\right\}^{-1} dt, \qquad (2,3)$

in which

$$\varphi(t) = A.t(1+t^2)^{-2} - B+(1+t^2)^{-1} + B(\tfrac{\pi}{2} - \arctan\, t), \qquad (2,4)$$

$$A = 2(1+k^2) \qquad (2,5)$$

$$B = 1 - 3\,k^2 \qquad (2,6)$$

$$1 + k^2 = 1.225748441 \qquad (2,7)$$

and this for a number of further specified values of the parameter G and at the same time for N = 0(0.2)12.2, the relation between G and N being given by:

$$N = \sqrt{24}\,.\,(1 + k^2)^{-\frac{3}{2}}\,.\, G\left[\gamma (G)\right]^{\frac{3}{2}} \qquad (2,8)$$

It was necessary to compute separately the values of N for

$$G = \left[\pi\ (1 - 3\,k^2)\right]^{-1}. \qquad \text{(For the formulas see (2)).}$$

3 . To determine the function y(x,G), defined by

$$y(x,G) = \frac{1}{G}\int_{-k}^{\infty} \ln\left[1 - G\,\varphi(t)\right]^{-1} \ln(x-t)^2\, dt, \qquad (2,9)$$

for x = k and x = 0,
and G having the values that correspond with N = 2(2)12.

Finally the first and second derivative with respect to x of the function y(x,G) was computed, x and G having the values mentioned above.

For the computations the improper integrals (2,3) and (2,9) were transformed by means of the transformation

$$\operatorname{arccot}\ t = s. \qquad (2,10)$$

$\varphi(t)$ changes into the function p(s)

$$p(s) = -\frac{1+k^2}{4}\sin 4\,s + 2\,k^2 \sin 2\,s + (1-3\,k^2)\,s. \qquad (2,11)$$

Now:

$$\gamma(G) = -\frac{1+k^2}{2G}\int_0^{\frac{\pi}{2}+\arctan\ k} \log\left[1 - G\,p(s)\right]\frac{ds}{\sin^2 s}, \qquad (2,12)$$

and

$$y(x,G) = -\frac{1}{G}\int_0^{\frac{\pi}{2}+\arctan\ k} \log\left[1 - G\,p(s)\right]\log\left|\,x - \cot\ s\right|^2 \frac{ds}{\sin^2 s} \qquad (2,13)$$

The integral $\gamma(G)$ is not improper in the point s = 0, for

$$\frac{\log\left\{1 - G\,p(s)\right\}^{-1}}{\sin^2\ s} = \frac{8}{3}\ G\,s\left\{1 - \frac{7+9k^2}{15}\ s^2 \ldots\right\} \qquad (2,14)$$

For s_0 = arccot k,the integrand of (2,12) contains a logarithmic singularity, amounting to

$$-\frac{2\,\log(s-s_0)}{\sin^2\ s_0}\ .$$

This difficulty has been surmounted by the method described in (3).

The integrand in (2,13) also contains a logarithmic singularity, in cot s_0 = x, in fact.

This logarithmic character of the singularity disappears in evaluating the derivatives with respect to x.

It is easily derived that:

$$y'(x,G) = -\frac{2}{G}\int_0^{\frac{\pi}{2}+\arctan k} \frac{\log\left[1-G\,p(s)\right]}{(x-\cot\ s)\sin^2 s}\,ds \qquad (2,15)$$

which is a principal value in the sense of Cauchy.

Now we expand the integrand into a series

$$\frac{a_{-1}}{u} + a_0 + a_1\,u + a_2\,u^2\ \ldots, \qquad (2,16)$$

in which $u = s - s_0$, and

$$s_0 = \operatorname{arccot}\ x \qquad (2,17)$$

and now we evaluate $a_{-1}\int_0^{\frac{\pi}{2}+\arctan k}\frac{1}{s}\,ds$ analytically and numerically. The difference between the two evaluations is the error that is made by determining (2,15) via the way of numerical integration.

The formula for $y''(x,G)$ is:

$$y''(x,G) = \frac{2}{G}\int_0^{\frac{\pi}{2}+\arctan k}\left\{\log\left[1-G\,p(s)\right] - \log\left[1 - G\,p(s_0)\right]\right\}\cdot \frac{ds}{(x-\cot\ s)^2\sin^2\ s}. \qquad (2,18)$$

(2,18) can be proved by writing the integral (2,15) as follows:

$$y'(x,G) = -\frac{2}{G}\left\{\int_0^{s_0-\varepsilon}\frac{\log\left[1-G\,p(s)\right]}{(x-\cot\ s)\sin^2 s}\,ds + \int_{s_0-\varepsilon}^{s_0+\varepsilon}\frac{\log\left[1-G\,p(s)\right]}{(x-\cot\ s)\sin^2 s}\,ds + \int_{s_0+\varepsilon}^{\frac{\pi}{2}+\arctan k}\frac{\log\left[1-G\,p(s)\right]}{(x-\cot\ s)\sin^2 s}\,ds\right\},$$

and then performing the differentiation with respect to x.

$$\frac{d}{dx}\left[\int_{s_0-\varepsilon}^{s_0+\varepsilon}\frac{\log\left[1-G\,p(s)\right]}{(x-\cot\ s)\sin^2 s}\,ds\right] = \frac{d}{dx}\left[\int_{-\varepsilon}^{\varepsilon}\left(\frac{a_{-1}}{u} + a_0 + a_1u + \ldots\right)du\right]$$

$$= \frac{d}{dx}\left[2\,a_0\varepsilon + \frac{2}{3}\,a_2\varepsilon^3 + \ldots\right] = 2\,b_0\varepsilon + \frac{2}{3}\varepsilon^3 + \ldots$$

in which $b_n = \frac{da_n}{dx}$.

Therefore,

$$\lim_{\varepsilon \to 0} \frac{d}{dx}\left[\int_{s_0-\varepsilon}^{s_0+\varepsilon} \frac{\log\left[1-G\,p(s)\right]}{x \cot s \sin^2 s}\, ds \right] = 0 .$$

Utilizing

$$x - \cot s_0 = 0$$

we find $\frac{ds_0}{dx} = -\sin^2 s_0$ and therefore differentiation of the first integral yields:

$$+ \frac{2}{G}\int_0^{s_0-\varepsilon} \frac{\log\left[1-G\,p(s)\right]}{(x-\cot s)^2 \sin^2 s}\, ds + \frac{2}{G}\,\frac{\log\left[1-G\,p(s_0-\varepsilon)\right]}{\left[x-\cot(s_0-\varepsilon)\right]\sin^2(s_0-\varepsilon)}\sin^2 s_0$$

$$= + \frac{2}{G}\int_0^{s_0-\varepsilon} \left\{ \log\left[1-G\,p(s)\right] - \log\left[1-G\,p(s_0-\varepsilon)\right] \frac{\sin^2 s_0}{\sin^2(s_0-\varepsilon)} \right\} \frac{ds}{(x-\cot s)^2 \sin^2 s}$$

When, in a similar manner, we differentiate the last integral and assume that $\lim \varepsilon \to 0$ we find (2,18).

For the case $x = 0$, this new-found integral is again an integral of which the principal value has to be considered, for $x = k$ the integrand is regular.

3. Numerical results.

All results have an error of at most one unit of the last figure that is stated. Here and there it proved necessary to use more figures than was necessary for the final answer, in view of the fact that figures sometimes drop off.

For x_1 was found

$$x_1 = 0.384043$$

and

$$y(x_1) = -2.5714$$

$$y'(x_1) = -2.18004$$

$$y''(x_1) = 10.53252.$$

In the tables I, II and III the functions $F(x,x_1)$, $\gamma(G)$ as functions of G and N, and $y(x,G)$, $y'(x,G)$ and $y''(x,G)$ follow underneath.

1 c

x	$F(x,x_1)$	x	$F(x,x_1)$	x	$F(x,x_1)$
- 2,0	5,49315	2,1	1,68235	6,2	29,95236
- 1,9	5,00686	2,2	1,95412	6,3	31,07174
- 1,8	4,54144	2,3	2,24735	6,4	32,21136
- 1,7	4,09699	2,4	2,56202	6,5	33,37119
- 1,6	3,67358	2,5	2,89806	6,6	34,55125
- 1,5	3,27132	2,6	3,25545	6,7	35,75152
- 1,4	2,89032	2,7	3,63410	6,8	36,97198
- 1,3	2,53075	2,8	4,03398	6,9	38,21265
- 1,2	2,19276	2,9	4,45501	7,0	39,47349
- 1,1	1,87655	3,0	4,89713		
- 1,0	1,58239	3,1	5,36031		
- 0,9	1,31063	3,2	5,84446		
- 0,8	1,06163	3,3	6,34955		
- 0,7	0,83600	3,4	6,87554		
- 0,6	0,63453	3,5	7,42235		
- 0,5	0,45845	3,6	7,98997		
- 0,4	0,31059	3,7	8,57835		
- 0,3	0,19416	3,8	9,18745		
- 0,2	0,10949	3,9	9,81722		
- 0,1	0,05378	4,0	10,46764		
0	0,02173	4,1	11,13868		
0,1	0,00648	4,2	11,83031		
0,2	0,00110	4,3	12,54249		
0,3	0,00004	4,4	13,27520		
0,4	0,00001	4,5	14,02842		
0,5	0,00014	4,6	14,80212		
0,6	0,00164	4,7	15,59627		
0,7	0,00694	4,8	16,41088		
0,8	0,01902	4,9	17,24589		
0,9	0,04091	5,0	18,10132		
1,0	0,07533	5,1	18,97713		
1,1	0,12459	5,2	19,87329		
1,2	0,19057	5,3	20,78982		
1,3	0,27466	5,4	21,72669		
1,4	0,37798	5,5	22,68390		
1,5	0,50130	5,6	23,66140		
1,6	0,64515	5,7	24,65920		
1,7	0,80994	5,8	25,67731		
1,8	0,99594	5,9	26,71568		
1,9	1,20330	6,0	27,77431		
2,0	1,43207	6,1	28,85321		

2 a G	$\gamma(G)$
0,0	1
0,1	1,03263
0,2	1,06935
0,3	1,11112
0,4	1,15944
0,5	1,21655
0,6	1,28609
0,69	1,36458
0,7	1,37456
0,8	1,49587
0,84	1,56041
0,9	1,69062
0,97	1,99381
0,98	2,10312
0,98622858	2,31253

2 b N	$\gamma(G)$
0,0	1
0,2	1,01717
0,4	1,03447
0,6	1,05192
0,8	1,06948
1,0	1,08717
1,2	1,10498
1,4	1,12292
1,6	1,14100
1,8	1,15919
2,0	1,17750
2,2	1,19593
2,4	1,21449
2,6	1,23319
2,8	1,25200
3,0	1,27094
3,2	1,29000
3,4	1,30918
3,6	1,32848
3,8	1,34791
4,0	1,36746
4,2	1,38713
4,4	1,40693
4,6	1,42686
4,8	1,44690
5,0	1,46706
5,2	1,48735
5,4	1,50776
5,6	1,52829
5,8	1,54894
6,0	1,56972
6,2	1,59061

2 b N	$\gamma(G)$
6,4	1,61162
6,6	1,63275
6,8	1,65400
7,0	1,67538
7,2	1,69687
7,4	1,71848
7,6	1,74021
7,8	1,76206
8,0	1,78404
8,2	1,80613
8,4	1,82834
8,6	1,85067
8,8	1,87312
9,0	1,89569
9,2	1,91837
9,4	1,94118
9,6	1,96411
9,8	1,98716
10,0	2,01033
10,2	2,03362
10,4	2,05703
10,6	2,08056
10,8	2,10420
11,0	2,12797
11,2	2,15186
11,4	2,17587
11,6	2,20000
11,8	2,22427
12,0	2,24861
12,2	2,27310

G	N
0,98622858	12,52021

3 <u>a</u>

N	G	y(x,G)	y'(x,G)	y"(x,G)
2	0.43360	- 3.6565	- 1,4716	12,32
4	0.69292	- 4,7485	- 1.7283	19.37
6	0,84514	- 6.052	- 2.0035	31.78
8	0,93001	- 7,623	- 2.2970	57.01
10	0.97191	- 9,584	- 2,6097	127.8
12	0.98580	-12.27	- 2.974	$8,4 \times 10^2$

3 <u>b</u>

	G	y(x,G)	y'(x,G)	y"(x,G)
2	0.43360	- 1,4594	- 6.3674	- 2.471
4	0,69292	- 1.8887	- 7,8194	- 5.364
6	0.84514	- 2.3578	- 9.3633	- 8.920
8	0.93001	- 2.8613	-10.9726	-12.81
10	0,97191	- 3.403	-12.628	-16.76
12	0.98580	- 3.972	-14.33	-20.7