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On the Coefficients of the Modular Invariant $J(\tau)$

A. van Wijngaarden

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BY

A. VAN WIJNGAARDEN

(REPORT R 122 OF THE COMPUTATION DEPARTMENT OF THE
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MATHEMATICS

ON THE COEFFICIENTS OF THE MODULAR INVARIANT $J(\tau)$

BY

A. VAN WIJNGAARDEN

(Report R 122 of the Computation Department of the Mathematical Centre, Amsterdam)

(Communicated by Prof. B. VAN DER POL at the meeting of September 26, 1953)

1. *Introduction*

In recent years several authors paid attention to properties and the computation of the Fourier coefficients of the modular invariant $J(\tau)$. H. S. ZUCKERMAN [9], showed a relationship with the partition function, which enabled him to compute the first 24 coefficients with relative ease, thus extending considerably the first tabulation by W. E. H. BERWICK [1], who gave the first 7 coefficients. In order to proceed in this way one needs, however, first tables of the partition function that go further than the existing ones of H. GUPTA [2], [3]. D. H. LEHMER [4] proved several congruences and relations between the coefficients themselves and between the coefficients, the divisor functions and Ramanujan's tau function, and computed, moreover, the 25-th coefficient. B. VAN DER POL [7] gave also many relations of that last type, and relations between $J(\tau)$ and the theta functions. In his paper also values of the next three coefficients are given that were computed by the author of this paper, the last of which being erroneous due to a regrettable coincidence. In the present paper a table of the first 100 coefficients is given, together with some new properties, found empirically and proved afterwards.

The author is indebted to professor VAN DER POL for many substantial contributions and to Miss G. BOTTERWEG, Miss H. C. HAGENAAR and Mr W. KLEIN for their share in the numerical work.

2. *Relations between theta functions and $J(\tau)$*

Let M_k be a function of the integer k and the three numbers x , y and z defined by

$$(2.1) \quad M_k = x^k + y^k + z^k,$$

so that always $M_0 = 3$, and let, moreover, x , y and z be such that

$$(2.2) \quad M_1 = 0.$$

Then we have

$$\begin{aligned} M_2 &= x^2 + y^2 + z^2 = -2(xy + yz + zx), \\ M_3 &= x^3 + y^3 + z^3 = 3xyz \end{aligned}$$

so that

$$\begin{aligned} {}^{1/3}M_3 M_k &= xyz(x^k + y^k + z^k), \\ {}^{1/2}M_2 M_{k+1} &= -(xy + yz + zx)(x^{k+1} + y^{k+1} + z^{k+1}), \end{aligned}$$

from which it follows by addition, and taking into account (2.1) and (2.2)

$$(2.3) \quad M_{k+3} = {}^{1/2}M_2 M_{k+1} + {}^{1/3}M_3 M_k.$$

The solution of this recurrence relation that satisfies (2.2) can be written down in explicit form.

Introducing

$$(2.4) \quad I = ({}^{1/2}M_2)^3 ({}^{1/3}M_3)^{-2}$$

one finds after some tedious arithmetic

$$(2.5) \quad \left\{ \begin{aligned} M_0 &= 3, & M_1 &= 0, \\ M_{2k} &= ({}^{1/2}M_2)^k \sum_{h=0}^{[k/3]} \frac{2k}{k-h} \binom{k-h}{2h} I^{-h}, & k > 0, \\ M_{2k+1} &= ({}^{1/2}M_2)^{k+1/2} I^{-1/2} \sum_{h=0}^{[k-1/3]} \frac{2k+1}{k-h} \binom{k-h}{2h+1} I^{-h}, & k > 0, \\ M_{-k} &= ({}^{1/2}M_2)^{-1/2k} \sum_{h=0}^{[k/3]} (-)^h \frac{k}{k-2h} \binom{k-2h}{h} I^{1/2k-h}, & k > 0. \end{aligned} \right.$$

Inspecting the exponents of M_2 and I in these formulae it appears that the product $M_k M_{-k}$ takes a particularly simple form. In effect

$$(2.6) \quad \left\{ \begin{aligned} M_0 M_0 &= 9, & M_1 M_{-1} &= 0, \\ M_{2k} M_{-2k} &= I^{k-[k/3]-[2k/3]} \sum_{h=0}^{[k/3]} \frac{2k}{k-h} \binom{k-h}{2h} I^{[k/3]-h} \\ &\quad \cdot \sum_{h=0}^{[2k/3]} (-)^h \frac{2k}{2k-2h} \binom{2k-2h}{h} I^{[2k/3]-h}, & k > 0. \\ M_{2k+1} M_{-(2k+1)} &= I^{k-[k-1/3]-[2k+1/3]} \sum_{h=0}^{[k-1/3]} \frac{2k+1}{k-h} \binom{k-h}{2h+1} I^{[k-1/3]-h} \\ &\quad \cdot \sum_{h=0}^{[2k+1/3]} (-)^{h+1} \frac{2k+1}{2k+1-2h} \binom{2k+1-2h}{h} I^{[2k+1/3]-h}, & k > 0. \end{aligned} \right.$$

Hence, $M_k M_{-k}$ is a polynomial in I of degree $[k/2]$ with integer coefficients, that for $I = 0$ takes the value 9 if $k \equiv 0 \pmod{3}$ and vanishes if $k \not\equiv 0 \pmod{3}$. More specifically

$$(2.7) \quad M_k M_{-k} = \sum_{n=0}^{[k/2]} m_k(n) I^n,$$

where the coefficients $m_k(n)$ for $k = 0(1)13$ are given in the following table:

k	$m_k(0)$	$m_k(1)$	$m_k(2)$	$m_k(3)$	$m_k(4)$	$m_k(5)$	$m_k(6)$
0	9						
1	0						
2	0	2					
3	9	— 3					
4	0	— 8	2				
5	0	25	— 5				
6	9	— 12	— 9	2			
7	0	— 49	49	— 7			
8	0	96	— 40	— 8	2		
9	9	— 27	— 135	78	— 9		
10	0	— 150	355	— 100	— 5	2	
11	0	242	— 121	— 242	110	— 11	
12	9	— 48	— 828	892	— 201	— 0	2
13	0	— 338	1521	— 507	— 338	143	— 13

Now, introducing the Jacobian theta functions ¹⁾

$$\theta'_1 = \theta'_1(q) = \{\partial/\partial z \theta_1(z, q)\}_{z=0}, \quad \theta_j = \theta_j(q) = \theta_j(0, q) \quad (j = 2, 3, 4),$$

it is seen from the first of the two fundamental relations

$$(2.8) \quad \theta_2^4 - \theta_3^4 + \theta_4^4 = 0, \quad \theta'_1 = \theta_2 \theta_3 \theta_4,$$

that a set of x , y and z satisfying (2.2) is given by $x = \theta_2^4$, $y = -\theta_3^4$, $z = \theta_4^4$.

Accordingly

$$(2.9) \quad M_k = \theta_2^{4k} + (-\theta_3^4)^k + \theta_4^{4k},$$

and, (confer [7]),

$$(2.10) \quad I = 2^{-2} 3^3 J(\tau).$$

Hence, the equations (2.6) or (2.7) are as many relations between theta functions and the absolute modular invariant $J(\tau)$. Most of these results can be found in [7]. They are given here because of the explicit form into which they have been brought, and, moreover, because slight errors in the values of some coefficients occur in formula (28) of [7], however without any further consequences.

3. Relations between theta functions and the generating functions of the partition functions

Some classical results are needed for our purpose. Be $p(n)$ the number of partitions of the natural number n into the sum of natural numbers without restrictions. If restrictions are imposed it is denoted by an index.

¹⁾ Here, and in what follows, q is considered as argument rather than τ ($q = e^{\pi i \tau}$), since only arithmetical properties are examined. The argument is, moreover, suppressed everywhere when it is q , but only used in cases like $\theta_4(q^4)$, where others might prefer $\theta_4(0, 4\tau)$. It seemed too hard to be consistent and to denote by J or $J(q)$ rather than by $J(\tau)$ a function that owes its name to its modular properties.

The following scheme is proposed. Restrictions on the character of the terms are denoted by a lower index, restrictions on the number of terms by an upper index. Moreover, *e* stands for *even*, *o* for *odd*, *d* for *different*, *s* for *same*, *p* for *prime*. A number, if need be together with algebraical symbols, has an obvious meaning, and other symbols may be introduced by additional conventions. Hence, $p_d(n)$ is the number of partitions of n into different (unequal) terms; $p_{do}(n)$ is the number of partitions of n into unequal odd terms; $p_d^o(n)$ is the number of partitions of n into an odd number of unequal terms; $p_{d>6}^3(n)$ is the number of partitions of n into three unequal terms each greater than six; $p_p^2(2n) > 0$, $n > 1$, expresses the conjecture of GOLDBACH, and so on. By definition, a value is attributed to a partition function of the argument 0, viz. the one that suits its generating function.

Some generating functions are in classical notation

$$(3.1) \quad \left\{ \begin{array}{l} q_0 = q_0(q) = \prod_{n=1}^{\infty} (1 - q^{2n}) = \sum_{n=-\infty}^{\infty} (-)^n q^{n(3n+1)}, \\ q_1 = q_1(q) = \prod_{n=1}^{\infty} (1 + q^{2n}) = \sum_{n=0}^{\infty} p_d(n) q^{2n}, \\ q_2 = q_2(q) = \prod_{n=1}^{\infty} (1 + q^{2n-1}) = \sum_{n=0}^{\infty} p_{do}(n) q^n, \\ q_3 = q_3(q) = \prod_{n=1}^{\infty} (1 - q^{2n-1}) = \sum_{n=0}^{\infty} (-)^n p_{do}(n) q^n, \end{array} \right.$$

where apparently by definition $p_d(0) = p_{do}(0) = 1$.

From the product expansions of the theta functions it follows that

$$(3.2) \quad \left\{ \begin{array}{l} \theta_1' = 2q^{1/4} q_0^3, \quad \theta_2 = 2q^{1/4} q_0 q_1^2, \\ \theta_3 = q_0 q_2^2, \quad \theta_4 = q_0 q_3^2. \end{array} \right.$$

Again, the functions of section 2 take the following form

$$(3.3) \quad \left\{ \begin{array}{l} M_2 = 2q_0^8 q_1^{-8} (2^8 q^2 q_1^{24} + 1), \quad M_3 = -2^4 3 q q_0^{12}, \\ I = 2^{-8} q^{-2} q_1^{-24} (2^8 q^2 q_1^{24} + 1)^3, \end{array} \right.$$

whereas the relations (2.8) read

$$(3.4) \quad 2^4 q q_1^8 = q_2^8 - q_3^8, \quad q_1 q_2 q_3 = 1.$$

From (3.1) and (3.4) one derives easily

$$(3.5) \quad \left\{ \begin{array}{l} q_0(q^2) = q_0(q) q_1(q) = q_0(iq) q_1(iq), \\ q_2(q^2) = q_1^{-1}(iq), \quad q_3(q^2) = q_1^{-1}(q). \end{array} \right.$$

Hence, from (3.2) and (3.5) it follows

$$(3.6) \quad \theta_3(q^2) = q_0(iq) q_1^{-1}(iq), \quad \theta_4(q^2) = q_0(q) q_1^{-1}(q).$$

At last a special combination will be examined, viz.

$$\begin{aligned}\theta_2^{-8}(q) + \theta_2^{-8}(iq) &= 2^{-8} q^{-2} q_0^{-8}(q^2) \{q_1^{-8}(q) - q_1^{-8}(iq)\} \\ &= 2^{-8} q^{-2} q_0^{-8}(q^2) \{q_3^8(q^2) - q_2^8(q^2)\} \\ &= -2^{-4} q_0^{-8}(q^2) q_1^8(q^2) = -2^{-4} \theta_4^{-8}(q^4),\end{aligned}$$

whence

$$(3.7) \quad \theta_4^{-8}(q^4) = -2^4 \{\theta_2^{-8}(q) + \theta_2^{-8}(iq)\}.$$

4. The coefficients of $J(\tau)$

As well known, the coefficients $c(n)$ in the expansion

$$(4.1) \quad j(\tau) = 2^8 I = 12^3 J(\tau) = \sum_{n=-1}^{\infty} c(n) q^{2n}$$

are integers. There are many ways to compute the coefficients, each of one resulting in rather lengthy calculations if n is not very small. This is rather natural since the coefficients grow very rapidly with n , and the digits have to come from somewhere. LEHMER [4] has derived multiplicative relations that for special values of n involve a moderate amount of arithmetical operations but all on large numbers. LEHMER [4] and VAN DER POL [7] have given recurrence relations implying divisor functions and Ramanujan's tau-function. Typical examples are the following ones ([7], formula 54a)

$$(4.2) \quad \begin{cases} \sum_{k=-1}^{n-1} c(k) \tau(n-k) = \frac{720}{691} \{91 \sigma_{11}(n) + 600 \tau(n)\}, \\ \sum_{k=-1}^{n-1} kc(k) \tau(n-k) = 24 \sigma_{13}(n). \end{cases}$$

Here $\tau(n)$ is Ramanujan's function and $\sigma_{2m+1}(n)$ is the sum of the $(2m+1)$ th powers of the divisors of n , if $m = 1, 2, 3, \dots$, whereas by definition $\sigma_{2m-1}(0) = \frac{1}{2} \zeta(1-2m) = -B_{2m}/(4m)$ and B_{2m} are the Bernoullian numbers, $B_0 = 1$, $B_2 = 1/6$, $B_4 = -1/30 \dots$. The τ -function is well tabulated and the σ -functions are easily computed. Moreover, they are not very large. From both formulae $c(n)$ is found by repeating the same procedure. The second formula has moreover an advantage over the first one in that it yields $(n-1)c(n-1)$, whereas the first one yields $c(n-1)$ directly ($\tau(1) = 1$), so that in the second case one has an important numerical check in the remainderless division by $n-1$.

Yet, another type of formulae exists, one of which will be investigated now. The third formula (2.7) together with (4.1) yields

$$(4.3) \quad j(\tau) = 2^7 M_2 M_{-2}.$$

Now, M_2 is a very simple function. In effect (confer [7])

$$(4.4) \quad M_2 = 480 \sum_{n=1}^{\infty} \sigma_3(n) q^{2n},$$

where it should be remembered that $\sigma_3(0) = 1/240$. Be analogously

$$(4.5) \quad M_{-2} = \theta_2^{-8} + \theta_3^{-8} + \theta_4^{-8} = 2^{-8} \sum_{n=-1}^{\infty} d(n) q^{2n},$$

where all $d(n)$ are positive integers ($n \geq -1$). Also,

$$(4.6) \quad \theta_4^{-8} = 2^4 \sum_{n=0}^{\infty} s(n) q^n,$$

where $s(0) = 2^{-4}$ and $s(n)$ is a positive integer for $n > 0$. Since $\theta_4(q) = \theta_3(-q)$, also

$$(4.7) \quad \theta_3^{-8} = 2^4 \sum_{n=0}^{\infty} (-)^n s(n) q^n,$$

Similarly

$$(4.8) \quad \theta_2^{-8} = -2^{-8} \sum_{n=-1}^{\infty} t(n) q^{2n},$$

where all $t(n)$ are integers ($n \geq -1$) with alternating signs. Hence

$$(4.8) \quad d(n) = 2^{13} s(2n) - t(n), \quad n \geq -1,$$

where, of course, $s(-2) = 0$.

The coefficients $s(n)$ and $t(n)$ can be found from the expansions

$$(4.9) \quad \theta_2 = 2q^{1/4} \sum_{n=0}^{\infty} q^{n(n+1)}, \quad \theta_4 = \sum_{n=-\infty}^{\infty} (-)^n q^{n^2},$$

by applying the following well known artifice. Be $a(n)$ given coefficients for $n \geq 0$, and $a(0) \neq 0$. Then the coefficients $b(n)$ satisfying

$$\sum_{n=0}^{\infty} b(n) x^n = \left\{ \sum_{n=0}^{\infty} a(n) x^n \right\}^{m-1}$$

can be found by differentiating logarithmically, multiplying by the arising denominators and equating the coefficients of equal powers of x . Then one gets the recurrence relation

$$n a(0) b(n) = \sum_{k=1}^n (mk - n) a(k) b(n-k), \quad n > 0, \quad b(0) = a(0)^{m-1}.$$

This device is particularly useful if the $a(n)$ form a lacunary sequence. In this way one obtains the recurrence relations

$$(4.10) \quad \begin{cases} n s(n) = -2 \sum_{k=1}^n (-)^k (n + 7k^2) s(n-k^2), & n > 0, \quad s(0) = 2^{-4}, \\ (n+1) t(n) = - \sum_{k=1}^n \left\{ n+1 + 7 \frac{k(k+1)}{2} \right\} t \left\{ n - \frac{k(k+1)}{2} \right\}, & n > -1, \quad t(-1) = -1, \end{cases}$$

where k takes such values that $n - k^2 \geq 0$ resp. $n - k(k+1)/2 \geq -1$.

Both formulae allow numerical checks by the division by n , resp. $n+1$.

Having obtained $s(n)$ and $t(n)$, $d(n)$ follows from (4.8) and hence $c(n)$ from (4.3), (4.4) and (4.5)

$$(4.11) \quad c(n) = 240 \sum_{k=0}^{n+1} \sigma_3(k) d(n-k) = d(n) + 240 \sum_{k=1}^{n+1} \sigma_3(k) d(n-k).$$

The final summation is therefore rather simple, all terms being positive and $\sigma_3(k)$ being only a small factor. Of course, first $s(n)$, $t(n)$ and $d(n)$ have to be computed but these functions are of some interest in themselves. Moreover it will be shown that s can be expressed in t . To that end one combines (3.7) with (4.6) and (4.8):

$$\begin{aligned}\theta_4^{-8}(q^4) &= 2^4 \sum_{n=0}^{\infty} s(n) q^{4n}, \\ \theta_2^{-8}(q) + \theta_2^{-8}(iq) &= -2^{-8} \left\{ \sum_{n=-1}^{\infty} t(n) q^{2n} + \sum_{n=-1}^{\infty} (-1)^n t(n) q^{2n} \right\} = \\ &= -2^{-8} \sum_{n=0}^{\infty} t(2n) q^{4n},\end{aligned}$$

whence

$$(4.12) \quad s(n) = 2^{-8} t(2n), \quad n \geq 0,$$

or together with (4.8)

$$(4.13) \quad d(n) = 2^5 t(4n) - t(n),$$

so that the computation of $c(n)$ may be completely based on that of $t(n)$ and $\sigma_3(n)$. In actual practice, if one wants to compute $c(n)$ up to a fixed upper value of n , N say, it is easier to compute $t(n)$ for $-1 \leq n \leq N$ and $s(n)$ for $N/2 \leq n \leq 2N$ directly from (4.10) than to compute $t(n)$ for $-1 \leq n \leq 4N$ and then to apply (4.12).

Actually (4.12) was found empirically. From the conjecture that it should hold followed then (3.7) which was proved afterwards in the given way.

This method of computing $c(n)$ has the disadvantage, however, that after the formation of $s(n)$ and $t(n)$ no internal checks occur in the formulae. It is therefore advisable to have an efficient numerical check on the values of $c(n)$ for all values of n .

5. Congruences satisfied by $c(n)$

One knows many congruences satisfied by the coefficients $c(n)$. LEHMER [4] proved (more than) that

$$(5.1) \quad (n+1) c(n) \equiv 0 \pmod{24},$$

$$(5.2) \quad c(n) \equiv 0 \pmod{5} \text{ if } n \equiv \pm 2 \pmod{5}.$$

J. LEHNER [5], [6] proved that

$$(5.3) \quad \left\{ \begin{array}{l} c(n) \equiv 0 \pmod{2^{3\alpha+8} 3^{2\beta+3} 5^{\gamma+1} 7^\delta 11^\epsilon} \text{ if } 1 < n \equiv 0 \pmod{2^\alpha 3^\beta 5^\gamma 7^\delta 11^\epsilon}; \\ \alpha, \beta, \gamma, \delta \geq 1, \quad \epsilon = 1, 2, 3. \end{array} \right.$$

Moreover he remarked that from the numerical values of $c(n)$ available to him ($n \leq 25$) followed that in the case that n is one of the five primes 2, 3, 5, 7 and 11 or a power of one of these primes (thus if $n = 2, 3, 4, 5,$

7, 8, 9, 11, 16, 25), the congruences (5.3) predict the exact power of that prime dividing $c(n)$, ($n \leq 25$).

We shall now prove some other remarkable congruences. From the formulae of section 2 it follows that

$$J(\tau) = \frac{1}{6} M_2^3 M_3^{-2} = \frac{1}{6} (M_2^3 M_3) M_3^{-3} = 2^2 3^{-2} M_3^{-3} M_9 - 2^2 3^{-4}.$$

Combining this with the formulae of section 3 one gets

$$J(\tau) + 2^2 3^{-4} = 2^2 3^{-2} M_3^{-3} M_9 = 2^{-10} 3^{-5} q^{-3} (q_2^{72} - q_3^{72} - 2^{36} q^9 q_1^{72}),$$

and after introducing (4.1), with $c(-1) = 1$, $c(0) = 744$,

$$24 q^3 \{3q^{-2} + 35.71 + 3 + 3 \sum_{n=1}^{\infty} c(n) q^{2n}\} = \frac{1}{2} (q_2^{72} - q_3^{72}) - 2^{35} q^9 q_1^{72}.$$

Now, in virtue of (3.1) it holds

$$q_h^{72} = q_h(q) q_h^{71}(q) \equiv q_h(q) q_h(q^{71}) \pmod{71}, \quad (h = 1, 2, 3),$$

and moreover $2^{35} \equiv 1 \pmod{71}$, so that

$$q + q^3 + \sum_{n=1}^{\infty} c(n) q^{2n+3} \equiv \frac{1}{2} \{q_2(q) q_2(q^{71}) - q_3(q) q_3(q^{71})\} \\ - q^9 q_1(q) q_1(q^{71}) \pmod{71}.$$

Writing, for a moment, $c'(n) = c(n)$ if $n \neq 0$, $c'(0) = 1$, then together with (3.1) again, one gets

$$\sum_{n=1}^{\infty} c'(n) q^{2n+3} \equiv \frac{1}{2} \left\{ \sum_{n=0}^{\infty} p_{d_0}(n) q^n \sum_{n=0}^{\infty} p_{d_0}(n) q^{71n} \right. \\ \left. - \sum_{n=0}^{\infty} (-)^n p_{d_0}(n) q^n \sum_{n=0}^{\infty} (-)^n p_{d_0}(n) q^{71n} \right\} \\ - q^9 \sum_{n=0}^{\infty} p_d(n) q^{2n} \sum_{n=0}^{\infty} p_d(n) q^{142n} \pmod{71}.$$

By equating the coefficients of the terms containing q^{2n+3} one finds then at last

$$(5.4) \quad \left\{ c(n) \equiv \sum_{k=0}^{[2n+3/71]} p_{d_0}(k) p_{d_0}(2n+3-71k) - \sum_{k=0}^{[n-3/71]} p_d(k) p_d(n-3-71k) \right. \\ \left. \pmod{71}, \quad n \neq 0. \right.$$

The two partition functions $p_d(n)$ and $p_{d_0}(n)$ have been tabulated by G. N. WATSON [8], up to $n = 400$. They are, moreover, easily computable and relatively small numbers. The right hand side of (5.4) contains only very few terms, even less than it appears at first sight since $p_{d_0}(2) = 0$. For instance, for $n \leq 109$, (5.4) runs in full

$$c(n) \equiv p_{d_0}(2n+3) + p_{d_0}(2n+3-71) - p_d(n-3) - p_d(n-3-71) \\ \pmod{71}, \quad 0 \neq n \leq 109.$$

For $n = 100$, for instance, one has

$$c(100) \equiv 354357 + 14157 - 345856 - 165 = 22493 \equiv 57 \pmod{71},$$

which involves really small numbers only in comparison to $c(100)$ itself which is a number of 53 decimal digits.

In the derivation of (5.4) M_9 was introduced. If instead one introduces M_6 and proceeds along the same lines one gets the analogous congruence

$$(5.5) \quad \left\{ \begin{aligned} c(n) &\equiv \sum_{k=0}^{\lfloor 2n+2/47 \rfloor} p_{d_0}(k) p_{d_0}(2n+2-47k) + \sum_{k=0}^{\lfloor n-2/47 \rfloor} p_d(k) p_d(n-2-47k) \\ &\pmod{47}, \quad n \neq 0. \end{aligned} \right.$$

The analogous congruence modulo 23 is more complicated and will be omitted. From the numerical data it appears that similar congruences must exist modulo 41 and modulo 59. Actually the existence of (5.4) and (5.5) was also first found empirically.

6. The computation of $c(n)$

The coefficients $c(n)$ were computed for $n = -1(1)100$ from VAN DER POL'S formula [7, (38)]

$$(6.1) \quad J(\tau) = 2/_{27} (\theta_2^8 + \theta_3^8 + \theta_4^8) (\theta_2^{-8} + \theta_3^{-8} + \theta_4^{-8}),$$

What is the explicit form of (4.3), using the methods described in section 4.

The function $s(n)$ was computed for $n = 0(1)200$ and the function $t(n)$ for $n = -1(1)100$ by means of the selfchecking relations (4.10). They were checked moreover by means of (4.12). From (4.8) and (4.11) followed $d(n)$ and $c(n)$ for $n = -1(1)100$. This whole part was checked by duplication. Next $c(n)$ was computed from the second selfchecking equation (4.2) for $n = 1(1)50$ as an independent check. At last all values of $c(n)$ were submitted to the congruence-checks (5.4) and (5.5). For special values of n the congruences (5.2) and (5.3) were used as checks. The congruence (5.1), although valid for all n , was of little use since in both ways of computing the $c(n)$ the factor 24 plays a rôle.

With regard to the remark following (5.3), it is interesting to see whether Lehner's statement also applies to the new powers of primes (< 13) that are available now (viz. $n = 27, 32, 49, 64, 81$). This appears to be the case indeed. This phenomenon is rather interesting. In order to have a further check, $c(128)$ was calculated modulo 2^{30} from Lehmer's duplication formula [4]. It was found that $c(128) \equiv 2^{29} \pmod{2^{30}}$ so that also in this case Lehner's statement holds.

The values of $c(n)$ are given in the following table.

$c(n)$								n
						1	-1	
						744	0	
					1	96884	1	
					214	93760	2	
					8642	99970	3	
				2	02458	56256	4	
				33	32026	40600	5	
				425	20233	00096	6	
				4465	69940	71935	7	
				40149	08866	56000	8	
			3	17644	02297	84420	9	
			22	56739	33095	93600	10	
			146	21191	14995	19294	11	
			874	31371	96857	75360	12	
			4872	01011	17981	42520	13	
			25497	82738	94105	25184	14	
		1	26142	91646	57818	43075	15	
		5	93121	77242	14450	58560	16	
		26	62842	41315	07752	45160	17	
		114	59912	78844	47865	13920	18	
		474	38786	80123	41688	13250	19	
		1894	49976	24889	33900	28800	20	
		7318	11377	31813	75192	45696	21	
		27406	30712	51362	46549	29920	22	
		99710	41659	93718	26935	33820	23	
		3	53074	53186	56142	70998	24	
		12	18832	84330	42251	04333	25	
		41	07899	60190	30790	91576	26	
		135	35635	41518	64687	86750	27	
		436	56892	24858	87663	46104	28	
		1379	83758	34642	99992	55422	29	
		4278	07822	44213	26256	70582	30	
		13023	36938	25770	29512	80448	31	
		38960	80061	70995	91189	43000	32	
		1	14632	93989	00810	63777	33	
		3	31962	77091	39267	16726	34	
		9	46816	61357	02260	43164	35	
		26	61436	58257	53796	26887	36	
		73	77316	99697	25069	76080	37	
		201	76878	99472	28738	64858	38	
		544	76388	17516	16630	12316	39	
		1452	68925	44393	62169	79435	40	
		3827	76775	17393	63485	06559	41	
		9970	41660	02174	43268	73940	42	
		25683	33470	63954	06994	77401	43	
		65452	36773	14992	68312	17028	44	
		1	65078	82156	81861	74782	45	
		4	12189	63080	52167	73489	46	
		10	19253	51589	15767	91938	47	
		24	96774	10595	07166	92603	48	
		60	60574	41541	37209	99542	49	
		145	81598	45321	50199	97540	50	
					39132	61539	50	

$c(n)$										n	
	347	82974	25351	24906	52111	11193	03264	16268		51	
	822	82309	23604	86379	46346	57066	92508	05760		52	
	1930	75525	46782	25741	67329	52965	87752	61720		53	
	4494	97224	12333	74771	55078	53776	07541	22752		54	
	10384	83010	58794	97940	68925	15368	59324	35825		55	
	23814	07585	30992	24134	99951	81283	96335	84128		56	
	54214	49889	87656	47230	00378	95797	97720	88000		57	
1	22553	65475	04082	06615	35516	23305	01657	60000		58	
2	75134	11092	85948	64606	92553	08616	87146	59374		59	
6	13542	89505	30361	36170	69338	27228	48587	77600		60	
13	59250	92428	36550	38097	01809	16661	62894	74168		61	
29	92109	83800	07688	36650	74958	85452	33318	70720		62	
65	45530	43491	65030	30643	85476	04156	99953	65270		63	
142	31976	35972	71606	23108	02114	65424	36536	81152		64	
307	60954	73477	19676	30396	15540	12847	95239	17200		65	
661	00917	73782	87162	74459	09215	08064	15869	54240		66	
1412	35833	72861	18490	82870	80245	89187	32135	44410		67	
3001	00414	97911	12962	58941	10839	46623	40095	18080		68	
6341	98425	35335	41630	77601	14920	60361	94613	13664		69	
13331	26252	93210	23532	85518	96736	23687	92354	81600		70	
27877	50248	90624	32847	67184	93296	34876	93051	98947		71	
57998	94663	06862	70977	78971	24287	02702	89346	56000		72	
1	20064	76859	24154	07996	57067	63561	79539	59481	73320	73	
2	47334	29811	83106	50913	62656	13239	67886	40929	91488	74	
5	07071	19308	98997	08057	00789	06280	84219	65196	46750	75	
10	34690	66408	50426	35622	63168	39259	82257	41159	46496	76	
21	01594	58102	75143	25069	10589	02482	07991	00864	59520	77	
42	49352	00246	86459	96896	93275	41404	17894	12398	69440	78	
85	53998	18184	24975	89405	37694	48098	79634	98086	43878	79	
171	44484	30238	56632	32305	05079	66626	55430	46332	41600	80	
342	15552	55551	89176	73198	38691	23583	94201	19784	93364	81	
679	98684	36672	14052	17195	40980	18582	52260	99449	65120	82	
1345	82384	70689	81684	95259	62168	82155	84589	79008	27370	83	
2652	88632	13847	03560	25223	21296	59440	09217	23815	85408	84	
5208	62134	25202	53933	69315	34883	96012	72044	83857	83600	85	
10186	63549	71409	56830	21681	12072	29975	61148	07976	01792	86	
19845	94685	77153	87241	69587	80804	25504	86362	87388	82125	87	
38518	94383	02834	97365	36939	13362	43138	88225	01457	92000	88	
74484	51892	92890	17811	71998	98327	68142	07693	12594	10120	89	
1	43507	17246	72834	53885	51522	23427	82991	19235	32076	03200	90
2	75501	04261	67891	53749	08061	78938	36796	95113	39297	83496	91
5	27036	05805	32817	64188	08922	00416	29201	19197	55057	56160	92
10	04730	45344	09390	42843	89896	53654	12981	69030	71458	27840	93
19	08864	09832	13103	02488	60473	90986	18405	93893	84773	79584	94
36	14432	17930	44626	81879	67680	91204	64684	97513	08362	05250	95
68	21306	83268	93807	76546	62982	56534	65084	00341	84769	04448	96
128	31568	45093	05662	37049	15719	10171	04861	21743	36342	89960	97
240	60143	44493	76049	97591	58609	03804	73418	08640	16968	39680	98
449	72195	69801	18067	40150	81827	51777	54986	40947	29105	49646	99
837	98831	11070	74769	12751	95038	47574	52703	80191	83390	72000	100

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