# A CLASS OF ENTIRE FUNCTIONS USED IN ANALYTIC INTERPOLATION.

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# A class of entire functions used in analytic interpolation.

## 1. Introduction.

This report deals with some properties of the functions $F_k(t,u)$ defined by:

$$F_k(t,u)= \sum_{n=-\infty}^{\infty} e^{-t\pi(u+n)^2} \left\{\frac{\sin\pi(u+n)}{\pi(u+n)}\right\}^K , \qquad (1,1)$$

k being a non-negative integer, and consists of nine sections.

Naturally, the introduction is the first of all, the second giving some general properties of this class of functions.

It is the intention to show that for positive values of t the zeros of $F_k(t,u)$ are on the lines Re $u = n + \frac{1}{2}$, with n an integer. The proof of this theorem, however, is very difficult for values of $k > 2$ and so only the number of zeros in the strip $0 \leqslant$ Re $u \leqslant 1$ is given.

This is dealt with the sections three and four.

The next two sections give the productrepresentations of $F_k(t,u)$, and the seventh section is used for recurrence-relations and the differential-equation.

The case $k = 0$ is dealt with in the following section, and the report ends with an asymptotic expansion, connected with the zeros of $F_1(t,u)$, and some numerical values.

Some notations are given here:

Re stands for the real part of,
Im stands for the imaginary part of,
$*$ stands for the complex conjugate of,
[ ] stands for the integer part of.

## 2. Some general properties.

The functions $F_k(t,u)$ are entire functions of the parameter u as long as Re $t > 0$. The case $t = 0$ is treated by I.J. Schoenberg (1).

The functions are all periodic and one finds:

$$F_k(t,u+1)= F_k(t,u) \qquad (2,1)$$

$$F_k(t,-u) = F_k(t,u) \qquad (2,1)$$

$$F_k(t,u^*) = F_k(t,1-u) \qquad (2,1)$$

It also appears that one only needs to consider half the strip $0 \leqslant$ Re $u \leqslant 1$, Im $u \geqslant 0$.

It is easily seen that $F_k(t,u) > 0$ when Im u = 0. Further for every integer $k > 0$ the following expansion holds:

$$2^k F_k(t,u) = \sum_{n=-\infty}^{\infty} e^{-t\pi(u+n)^2} \left\{ \frac{2\sin\pi(u+n)}{\pi(u+n)} \right\}^k =$$

$$= \sum_{n=-\infty}^{\infty} e^{-t\pi(u+n)^2} \int_{-1}^{+1} \cdots \int_{-1}^{+1} e^{\pi i(u+n)(s_1+s_2+\cdots+s_k)} ds_1 ds_2 \cdots ds_k$$

$$= \int_{-k}^{k} \omega(\sigma) d\sigma \sum_{n=-\infty}^{\infty} e^{-t\pi(u+n)^2 + \pi i\sigma(u+n)} \qquad (2,2)$$

Here, $\omega(\sigma)$ is the volume obtained by cutting the k-dimensional cube of edge-length 2 with a (k-1)-dimensional space perpendicular to the principal diagonal of the k-cube in such a way that the distance between the (k-1)-dimensional space and the origin is equal to $\sigma$. One may be acquainted with the properties of $\omega(\sigma)$:

$-k < \sigma < k$, then $\omega(\sigma) > 0$.

outside this interval $\omega(\sigma) = 0$.

Further when $0 < \sigma_1 < \sigma_2 < k$, then $\omega(\sigma_1) > \omega(\sigma_2)$ and $\omega(-\sigma) = \omega(\sigma)$.

Using the identity of Poisson-Fourier, the relation (2,2) is transformed into

$$2^k F_k(t,u) = \frac{1}{\sqrt{t}} \int_{-k}^{k} \omega(\sigma) d\sigma \sum_{n=-\infty}^{\infty} \exp\left\{ 2\pi inu - \frac{\pi}{t}\left(n - \frac{\sigma}{2}\right)^2 \right\}, \qquad (2,3)$$

and assuming that $u = iy$, i.e. Re u = 0, one will find directly $F_k(t,iy) > 0$, and even one can prove that:

There is a constant $K_0 > 0$ independent of y so that

$$F_k(t,iy) > K_0. \qquad (2,4)$$

In order to consider the functions on the line Re u = $\frac{1}{2}$ one will find from (2.2) and (2.3) assuming that $u = \frac{1}{2} + i\frac{\lambda}{t}$:

$$2^k F_k\left(t, \tfrac{1}{2} + \tfrac{i\lambda}{t}\right) = \exp\left(\frac{\lambda^2\pi}{t}\right) \int_{-k}^{k} \omega(\sigma) d\sigma \sum_{n=-\infty}^{\infty} \exp\left\{ -(n+\tfrac{1}{2})^2 - \pi i(2n+1)\left(\lambda - \tfrac{\sigma}{2}\right) - \frac{\pi\lambda\sigma}{t} \right\}, \qquad (2,5)$$

and

$$2^k F_k\left(t, \tfrac{1}{2} + \tfrac{i\lambda}{t}\right) = \frac{1}{\sqrt{t}} \int_{-k}^{k} \omega(\sigma) d\sigma \sum_{n=-\infty}^{\infty} (-1)^n \exp\left\{ \frac{2\pi n\lambda}{t} - \frac{\pi}{t}\left(n - \frac{\sigma}{2}\right)^2 \right\}. \qquad (2,6)$$

From (2,6) it is apparent that $F_k(t,u)$ is real for Re u = $\frac{1}{2}$ and every integer $k \geq 0$.

3. The number of zeros.

Let N(m) be defined as the number of zeros contained in the rectangle $(0,\ 1,\ 1 + i\ \frac{2m+1}{t},\ i\ \frac{2m+1}{t})$, m being a natural number.

According to a well-known theorem in function-theory one will find the identity:

$$N(m) = \frac{1}{2\pi} \underset{c}{\Delta} \arg F_k(t,u),$$

where $\underset{c}{\Delta}$ arg means the increase of the argument by moving round the rectangle in the positive direction.

The condition that there are no zeros on the boundary can be satisfied here: from section 2 one knows already that there are no zeros on three sides of the rectangle, and it is possible to choose a number $\mathcal{E}$ very small so that there is no zero on the side $(i\ \frac{2m+1+\mathcal{E}}{2t}\ ,\ 1 + i\ \frac{2m+1+\mathcal{E}}{2t})$ either. Knowing that $F_k(t,u) > 0$ on three sides of the rectangle, - so the argument does not change there -, one can also use the symbol $\underset{c}{\Delta}$ arg for the increase of the argument when only moving along the upper side.

Rewriting (1,1), one will find:

$$F_k(t,u) = \left\{\frac{\sin \pi u}{\pi}\right\}^k \sum_{n=-\infty}^{\infty} (-1)^{nk} \frac{e^{-t\pi(u+n)^2}}{(u+n)^k}\ , \tag{3,1}$$

and regarding the loop described by $\sin \pi u$, u moving along the upperside of the rectangle, it appears that the increase of the argument of $\left\{\frac{\sin \pi u}{\pi}\right\}^k$ is equal to $k\pi$ .

In order to determine the increase in argument of the sum of formula (3.1) one has to define the argument of the terms of this sum in the point $u = 1 + i\ \frac{2m+1+\mathcal{E}}{2t}$ , and in such a way that these arguments can be fixed between $-(k-2)\frac{\pi}{2}$ and $-(k+2)\frac{\pi}{2}$. The largest term of the sum (the one with index $n = -1$) then has the argument $-k\ \frac{\pi}{2}$ .

Now a large number $m_0$ exists, so, that for all numbers $m > m_0$

$$\left|\arg \left\{ 1 + i\ \frac{2m+1+\mathcal{E}}{2t}\right\} - \frac{\pi}{2}\right| < \frac{\delta}{k}\pi \tag{3,2}$$

where $\delta$ is a small positive number.

From (3.2) it follows that also

$$\left|\arg(n+i\ \frac{2m+1+\mathcal{E}}{2t}) - \arg\ (n-1+i\ \frac{2m+1+\mathcal{E}}{2t}\ )\right| < \frac{\delta}{k}\pi$$

and this shows that the increase of the argument of each term is bounded by $(2m+\mathcal{E}-\delta)\pi$ and $(2m+2+\mathcal{E}+\delta)\pi$.

Now consider separately the cases:

1. k even: $\delta > 0$ and by choosing $-\mathcal{E} > \delta$ one can deduce that the increase of the argument of the sum is equal to $2m\pi$ .

Indeed, it has to be an even multiple of $\pi$. Or:

$$N(m+\frac{\varepsilon}{2}) = \left[m + \frac{k+1}{2}\right] \quad , \text{ when } -\varepsilon > \delta .$$

But choosing $\varepsilon > \delta$, one will find as above:

$$N(m+\frac{\varepsilon}{2}) = \left[m + 1 + \frac{k+1}{2}\right] \quad , \quad \text{when } +\varepsilon > \delta .$$

It is evident that for $m > m_0$ and $\varepsilon > \delta$, there is one and only one zero in the strip $0 \leqslant \text{Re } u \leqslant 1$, $\frac{2m+1-\varepsilon}{2t} \leqslant \text{Im } u \leqslant \frac{2m+1+\varepsilon}{2t}$ and $\lim_{m_0 \to \infty} \varepsilon = 0$.

But from the last equation, (2,1), it follows that when $u = x+iy$ is a zero of $F_k(t,u)$ then also $u=(1-x)+iy$ is a zero. $(0 < x \leqslant 1)$. So this zero, mentioned above, can only be on Re $u = \frac{1}{2}$.

2. k odd. The only possibility for the increase becomes here $(2m+1)\pi$, for it is an odd multiple of $\pi$, and

$$N(m+\frac{\varepsilon}{2}) = \left[m + \frac{k+1}{2}\right] .$$

When it has been shown that there is no zero on the line Im $u = \frac{2m+1}{2t}$, the $\frac{\varepsilon}{2}$ can be omitted, and

$$N(m) = \left[m + \frac{k+1}{2}\right] .$$

In a similar way as for even k, it is possible to deduce:

For odd k and $m > m_0$, there exists such an $\varepsilon > \delta$ that there is one zero $u_{0,m}$ of $F_k(t,u)$ on the line Re $u = \frac{1}{2}$, and its imaginary part is bounded by $\frac{m-\varepsilon}{t} < \text{Im } u < \frac{m+\varepsilon}{t}$, and $\lim_{m \to \infty} \varepsilon = 0$.

Also:

For even k and $m > m_0$ the number of zeros contained in the rectangle $(0,\ 1,\ 1 + i\frac{m}{t},\ i\,\frac{m}{t})$ is equal to $(m + \frac{k}{2} + 1)$ when it has been shown first that there is no zero on Im $u = \frac{m}{t}$.

4. <u>The line Re $u = \frac{1}{2}$.</u>

Defining

$$\vartheta(z) = \sum_{n=-\infty}^{\infty} \exp\left\{-(n+\tfrac{1}{2})^2 \pi t - 2\pi i (n+\tfrac{1}{2}) z\right\} ,$$

one knows that $\vartheta(z)$ is an even entire function of z, and

$$\vartheta(z+2) = \vartheta(z),$$
$$\vartheta(z+1) = -\vartheta(z).$$

Further for real z this function takes on only real values.

Remembering the properties of $\omega(\sigma)$ mentioned in section 2, and using those of $\vartheta(z)$ one can transform (2,4) into:

$$2^{k-1} F_k(t,\tfrac{1}{2} + i\,\frac{2m+1}{2t}) = (-1)^{m+1}\, e^{\frac{\pi}{4t}(2m+1)^2} \int_0^k \omega(\sigma) \sinh\left\{\frac{\pi(2m+1)\sigma}{2t}\right\} \vartheta(\frac{1-\sigma}{2})\, d\sigma , \qquad (4,1)$$

and

$$2^{k-1}\ F_k(t,\tfrac{1}{2} + i\ \tfrac{m}{t}) = (-1)^m\ e^{\frac{m^2\pi}{t}} \int_0^k \omega\ (\sigma)\cosh\left\{\frac{\pi m\sigma}{t}\right\} \vartheta\left(\frac{\sigma}{2}\right) d\sigma. \qquad (4,2)$$

First, k = 1: $\vartheta(\frac{1-\sigma}{2})$ remains positive for $0 \leqslant \sigma \leqslant 1$ and the integral of (4,1) has a positive integrand. $F_1(t,\frac{1}{2} + i\ \frac{2m+1}{t})$ has the same sign as $(-1)^m$ and together with the results of section 3 one gets the theorem for k = 1:

The zeros of $F_k(t,u)$ are represented by:

$$u_{n,m} = (n + \tfrac{1}{2}) + y_m\ i \qquad (4,3)$$

in which $y_m$ is real.

But also $\vartheta(\frac{\sigma}{2})$ remains positive and from (4,2) it follows that the sign of $F_1(t,\frac{1}{2} + i\ \frac{m}{t})$ is equal to the sign of $(-1)^m$, so $\frac{m}{t} < y_m < \frac{m+\frac{1}{2}}{t}$.

Using this for $m > m_0$, $\frac{m-\varepsilon}{t} < y_m < \frac{m+\varepsilon}{t}$ it appears:

$$\frac{m}{t} < y_m < \frac{m+\varepsilon}{t} \quad \text{for } m > m_0 \text{ and } \lim_{m_0 \to \infty} \varepsilon = 0.$$

Second, k = 2: Again $\vartheta(\frac{1-\sigma}{2})$ remains positive, for $0 \leqslant \sigma \leqslant 2$ and one will find no more than (m+1) zeros in the rectangle $(0,1,1+i\ \frac{2m+1}{2t},\ i\ \frac{2m+1}{2t})$ on the line $\mathrm{Re}(u) = \frac{1}{2}$.

But section 2 gives

$$\left.\begin{array}{l} N(m-\varepsilon) = m+1 \\ \text{and } N(m+\varepsilon) = m+2 \end{array}\right\} \text{ when } m > m_0 .$$

Because of the fact that if there is one zero in the strip $0 \leqslant \mathrm{Re}\ u \leqslant 1$ outside the line $\mathrm{Re}\ u = \frac{1}{2}$, then there is a second, one will find directly:

$$\left.\begin{array}{l} N(m) = m+1 \\ N(m+\varepsilon) = m+2 \end{array}\right\} \quad \begin{array}{l} \text{when } m > m_0 \\ \text{and } \varepsilon > \delta > 0. \end{array}$$

The zeros of $F_2(t,u)$ can also be represented by (4,3).

Third, k = 3: One can prove, that for large values of $\lambda \gg 0$,

$$\int_0^3 \omega\ (\sigma) \vartheta\left(\frac{1-\sigma}{2}\right) \sinh\left(\frac{\lambda\sigma}{t}\right) d\sigma \qquad (4,4)$$

takes on a negative sign.

The only function which becomes negative is $\vartheta(\frac{1-\sigma}{2})$, the interval being $2 < \sigma \leqslant 3$.

Splitting up (4,4) into two parts

$$\int_0^3 = \int_0^2 + \int_2^3 ,$$

one can majorize the first part by putting

$$0 \leqslant \omega(\sigma)\ \sinh\left(\frac{\lambda\sigma}{t}\right) \vartheta\left(\frac{1-\sigma}{2}\right) \leqslant \omega\ (0) \vartheta(0)\ \sinh\ \frac{2\lambda}{t},$$

and, therefore,

$$\tfrac{1}{2} \int_0^2 \omega\ (\sigma) \vartheta\left(\frac{1-\sigma}{2}\right) \sinh\left(\frac{\lambda\sigma}{t}\right) d\sigma \leqslant \omega\ (0) \vartheta(0)\ \sinh\ \frac{2\lambda}{t} .$$

To minorize $-\int_2^3 \omega(\sigma)\vartheta\left(\frac{1-\sigma}{2}\right)\sinh\frac{\lambda\sigma}{t}\,d\sigma \geqslant -\int_{3/4}^{9/4} \omega(\sigma)\vartheta\left(\frac{1-\sigma}{2}\right)\sinh\frac{\lambda\sigma}{t}\,d\sigma$

$\geqslant \frac{1}{2}\left|\omega\left(\frac{9}{4}\right)\vartheta\left(\frac{7}{8}\right)\sinh\frac{9\lambda}{4t}\right|.$

From these two inequalities the statement (4,4) is proved by increasing $\lambda$ to infinity. Now combining this with (4,2) and considering the sign of $F_3(t,u)$ in the points $u = \frac{1}{2}$, $u_m = \frac{1}{2} + i\,\frac{2m+1}{2t}$, one can deduce directly that there are at least m zeros of $F_3(t,u)$ on the line $\mathrm{Re}\, u = \frac{1}{2}$.

There are also at most 2 zeros which are not on that axis.

Similar treatment for other values of k gives the theorem: The zeros of $F_k(t,u)$ in the strip $0 \leqslant \mathrm{Re}\, u \leqslant 1$ are situated on the axis $\mathrm{Re}\, u = \frac{1}{2}$ with the exception of $2\left[\frac{k+1}{4}\right]$ zeros, which have by twos the same imaginary part.

## 5. The product-representation.

For k = 1 and 2 and $t > 0$ it has been shown in the preceding section that the zeros of $F_k(t,u)$ can be written in the form :

$$u_{n,m} = n + \tfrac{1}{2} + i\, y_m \qquad n,m \text{ integers}$$

so that for large m

$$|y_m - y_{m+1}| \leqslant \frac{1}{t},$$

but also $|y_m - y_{m+1}| > \frac{1}{2t}$.

This means $Y_m \sim O(m)$ (5,1)

For values of $k > 2$ there may be exceptions to this rule, but suppose the same had been proved, the following considerations also hold for $k > 2$.

Now only $t > 0$ is dealt with.

Write down the product-representation:

$$F_k(t,u) = \exp\left\{Q(u,t)\right\} \prod_{m=0}^{\infty} \left(1 + e^{2\pi i u - 2\pi y_m}\right)\left(1 + e^{-2\pi i u - 2\pi y_m}\right) \qquad (5,2)$$

in which $Q(u,t)$ is an entire function of u. Of course it is also a function of t. According to the theorem of Weierstrasz the product (5,2) is absolutely convergent, (owing to (5,1)).

The idea is to determine $Q(u)$ first, $Q(u)$ considered as a function of u. Taking the logarithmic derivative of equation (5,2) one will find:

$$Q'(u) = \frac{F_k'(t,u)}{F_k(t,u)} - 2\pi i \sum_{m=0}^{\infty} \frac{e^{2\pi i u}}{e^{2\pi y_m} + e^{2\pi i u}} + 2\pi i \sum_{m=0}^{\infty} \frac{e^{-2\pi i u}}{e^{2\pi y_m} + e^{-2\pi i u}}. \qquad (5,3)$$

The two series in the right-hand side of this equation absolutely converge for every u, except for the zeros of $F_k(t,u)$, but these singularities are annihilated by those of $\frac{F_k'(t,u)}{F_k(t,u)}$. Of course one can restrict oneself to the strip $0 \leqslant Re(u) \leqslant 1$

From (3,1) it may be deduced that

$$\frac{F_k'(t,u)}{F_k(t,u)} = k\pi \operatorname{cotg}\pi u - k \frac{\sum\limits_{n=-\infty}^{\infty}(-1)^{nk} \frac{e^{-t\pi(u+n)^2}}{(u+n)^{k+1}}}{\sum\limits_{n=-\infty}^{\infty}(-1)^{nk} \frac{e^{-t\pi(u+n)^2}}{(u+n)^k}} +$$

$$- 2t\pi \frac{\sum\limits_{n=-\infty}^{\infty}(-1)^{nk} \frac{e^{-t\pi(u+n)^2}}{(u+n)^{k-1}}}{\sum\limits_{n=-\infty}^{\infty}(-1)^{nk} \frac{e^{-t\pi(u+n)^2}}{(u+n)^k}} \qquad (5,4)$$

Again the cases of even k and odd k are dealt with separately:

1. k even: For every $t > 0$ there exists an integer $N > 0$ so, that for $n > N$:

$$e^{-t\pi n^2} < \varepsilon/2k , \qquad (5,5)$$

and a $y_0 > 0$ so, that

$$\frac{N}{y_0} < \delta/k, \qquad (5,5)$$

in which $\varepsilon$ and $\delta$ are small positive numbers.

Assuming that $u = x + iy$, $0 \leqslant x \leqslant 1$, one knows that the denominator $(u+n)^k$ has an argument between $(k\frac{\pi}{2} - \delta)$ and $(\frac{\pi}{2}k + \delta)$ for $|y| > y_0$ and $|n| < N$.

Choosing $y = \frac{m}{t}$, m being so large that $y > y_0$, and for that value of y using:

$$\exp\left\{-t\pi(u+n)^2\right\} = \exp\left\{\frac{m^2\pi}{t} - t\pi(x+n)^2 - 2m\pi i x\right\},$$

it is apparent that the argument of the exponential does not depend on n. That means:

$$\left|\sum_{n=-\infty}^{\infty} \frac{\exp\{-t\pi(u+n)^2\}}{(u+n)^k}\right| > (1-\varepsilon)(1-\delta)\left|\sum_{n=-N}^{N} \frac{\exp\{\frac{m^2\pi}{t} - t\pi(x+n)^2\}}{(iy)^k}\right|$$

and

$$\left|\sum_{n=-\infty}^{\infty} \frac{\exp\{-t\pi(u+n)^2\}}{(u+n)^k}\right| < (1+\varepsilon)(1+\delta)\left|\sum_{n=-N}^{N} \frac{\exp\{\frac{m^2\pi}{t} - t\pi(x+n)^2\}}{(iy)^k}\right|.$$

In what follows $K_n$ means a constant independent of x and y. One will find for $y = \frac{m}{t} > y_0$ and fixed k:

$$\left| k\pi \operatorname{cotg} \pi u \right| < K_1,$$

$$\left| \frac{k \sum_{n=-\infty}^{\infty} \frac{e^{-t\pi(u+n)^2}}{(u+n)^{k+1}}}{\sum_{n=-\infty}^{\infty} \frac{e^{-t\pi(u+n)^2}}{(u+n)^k}} \right| \leqslant \frac{k(1+\varepsilon)(1+\delta)}{(1-\varepsilon)(1-\delta).(y)} < K_2,$$

$$\left| \frac{\sum_{n=-\infty}^{\infty} \frac{e^{-t\pi(u+n)^2}}{(u+n)^{k-1}}}{iy \sum_{n=-\infty}^{\infty} \frac{e^{-t\pi(u+n)^2}}{(u+n)^k}} \right| \leqslant \frac{(1+\varepsilon)(1+\delta)}{(1-\varepsilon)(1-\delta)}$$ , and also

$$\left| \frac{\sum_{n=-\infty}^{\infty} \frac{e^{-t\pi(u+n)^2}}{(k+n)^{k-1}}}{iy \sum_{n=-\infty}^{\infty} \frac{e^{-t\pi(u+n)^2}}{(u+n)^k}} \right| \geqslant \frac{(1-\varepsilon)(1-\delta)}{(1+\varepsilon)(1+\delta)} .$$

in which $\varepsilon$ and $\delta$ may be given extremely small values. The argument of the left quotiënt of these two sums lies between $k\,.2\pi - \delta$ and $k.2\pi + \delta$, so this quotiënt is in the neighbourhood of +1.

From (5,5) it follows that

$$\frac{F_k'(t,u)}{F_k(t,u)} = -2\, t\pi\, iy + R, \qquad (5,6)$$

with $R < K_3$ for $y > y_0$.

For x = 0 it is evident that:

$$\left| \frac{F_k'(t,u)}{F_k(t,u)} \right| < K_4\, e^{t\pi\, y^2}. \qquad (5,7)$$

The two sums of (5,3) can be dealt with as follows:

Put $q = \exp(-2\pi iu)$ so that $|q|$ is large.

In order to avoid the zeros of $F_k(t,u)$, one chooses a number r; it may be small. Let $M_1$ be that entire number so that

$$\left| \frac{\exp(2\pi y_{M_1}) + q}{q} \right| = \eta_1 \text{ is a minimum.}$$

By avoiding the circles with radius r and centre in the zeros, one has a $\eta_1 > 0$ and for every m is now holds:

$$\left| \frac{\exp(2\pi y_m)+q}{q} \right| \geqslant \eta_1,$$

inwhich $\eta_1$ does not depend on q.

The first sum is always of the order $q^{-1}$. The second is split up into three parts:

$$\sum_{m=0}^{\infty} = \sum_{m=0}^{M_1-3} + \sum_{m=M_1-2}^{M_1+1} + \sum_{m=M_1+1}^{\infty}$$

$$\left| \sum_{m=M_1-2}^{M_1+1} \right| < \frac{3}{\eta_1}$$

$\displaystyle\sum_{m=M_1+1}^{\infty} \frac{1}{1+\frac{e^{2\pi y_m}}{q}}$ is a convergent sum remaining below a constant $K_5$

$$\sum_{m=0}^{M_1-3} \frac{1}{1+\frac{e^{2\pi y_m}}{q}} = $$

$$=(M_1-2) - \frac{1}{q} \sum_{m=0}^{M_1-3} e^{2\pi y_m} + \frac{1}{q^2} \sum_{m=0}^{M_1-3} e^{4\pi T_m} - \dots$$

$$= M_1 + S'(q).$$

One can divide the zeros of $F_k(t,u)$ in two groups:

$1^\circ$ so, that in every y interval $\frac{m-\frac{1}{2}}{t}$ and $\frac{m+\frac{1}{2}}{t}$ there is only one zero.

$2^\circ$ the remaining ones, at most $\left[\frac{k-1}{2}\right]$ zeros.

Now:

$$|S(q)| \leqslant \left[\frac{k-1}{2}\right] \sum_{n=1}^{\infty} \frac{1}{|q|^n} + \sum_{n=1}^{\infty} \frac{1}{|q|^n} \left| \frac{1-f^{n(M_1-2)}}{1-f^n} \right| =$$

$$= \left[\frac{k-1}{2}\right] \sum_{n=1}^{\infty} \frac{1}{|q|^n} + \sum_{n=1}^{\infty} \frac{1}{f^n(1-f^n)},$$

in which $f = e^{\pi/t} > 1$.

Also $|S(q)| < K_6$ (5,8)

And one has proved:

$$\left| \sum_{m=0}^{\infty} \frac{q}{e^{2\pi y_m}+q} - t\, y \right| < K_7 \qquad (5,9)$$

for $y > y_0$ and avoiding the neighbourhood of the zeros.

From (5,3), (5,6) and (5,9) it appears that

$$|Q'(u)| < K_8 \qquad (5,10)$$

for $y = \frac{m}{t}$.

For $x = 0$, (5,7) holds, but this is also valid for $x = n$.

Describing the contour formed by the lines $y = \pm Y$ and $x = \pm X(Y)$ in which $X(Y)$ is an natural number larger than $e^{(\alpha+t\pi)Y^2}$, $\alpha$ being a positive number, one can deduce from (5,7), (5,10) and the integral-expressions of Cauchy for $y \to \infty$ that all derivatives of $Q'(u)$ are zero and that $Q'(u)$ must be a constant. This constant can be a function of t. Knowing that $Q(u)$ is an even functions one now has:

$Q(u,t) = Q(t)$, a function of t only

and

$$F_k(t,u) = P(t) \prod_{m=0}^{\infty} (1+e^{2\pi iu-2\pi y_m})(1+e^{-2\pi iu-2\pi y_m}) \qquad (5,11)$$

## 6. Further determination of the product-representation.

Most of the preceding chapter also holds for odd K. One needs only to alter $y = \frac{m}{t}$ into $y = \frac{2m+1}{2t}$, the symbol $(-1)^{nk}$ becomes $(-1)^n$ here, and in a similar way one can deduce again (5,11).

Now use the transformation

$$\mathfrak{z} = (2 \sin \pi u)^2. \qquad (6,1)$$

The zeros of $F_k(t,u)$ are now transformed into

$$\mathfrak{z}_m = 2 + e^{-2\pi y_m} + e^{2\pi y_m} \qquad (6,2)$$

so that $\mathfrak{z}_m > 2$ for every m,

and $\mathfrak{z}_m = O(e^{\frac{2\pi}{t} m})$ (6,3)

The expression (5,11) is transformed into

$$F_k(t,u) = G_k(t,\mathfrak{z}) = C(t) \prod_{m=1}^{\infty} (1 - \frac{\mathfrak{z}}{\mathfrak{z}_m})$$

in which $C(t)$ may be calculated by putting $u = n$ or $\mathfrak{z} = 0$, and one has finally

$$F_k(t,u) = G_k(t,\mathfrak{z}) = \prod_{m=1}^{\infty} (1 - \frac{\mathfrak{z}}{\mathfrak{z}_m}) \qquad (6,4)$$

Hence it follows:

For $t > 0$ and $k = 1$ and $2$ the function $\frac{1}{F_k(t,u)}$ admits of an expansion $\sum_{n=0}^{\infty} c_n(t)\mathfrak{z}^n$, so, that $c_n(t) > 0$ and this expansion converges in the region $|\mathfrak{z}| \leqslant 2$.

7. The recurrence-relations and the differential equation.

From (1,1) one can derive by differentiation:

$$\frac{\partial F_k(t,u)}{\partial t} = -\frac{\sin^2\pi u}{\pi} F_{k-2}(t,u) \qquad K \geqslant 2 \qquad (7,1)$$

Now define the two sorts of functions,

$$\rho_k(t,u)= \left\{\left(\frac{\sin\pi u}{\pi}\right)^k\right\} \sum_{n=-\infty}^{\infty} \frac{e^{-t\pi(u+n)^2}}{(u+n)^k}, \qquad (7,2)$$

and

$$\sigma_k(t,u)= \left(\frac{\sin\pi u}{\pi}\right)^k \sum_{n=-\infty}^{\infty} (-1)^n \frac{e^{-t\pi(u+n)^2}}{(u+n)^k}. \qquad (7,3)$$

Also $\rho_k(t,u)= F_k(t,u)$ when k even,

$\sigma_k(t,u)= F_k(t,u)$ when k odd.

From (7,2) one will find:

$$\rho_{k+1}(t,u)= \cos\pi u\, \rho_k(t,u) - \frac{\sin\pi u}{\pi} \frac{\partial}{\partial u} \rho_k(t,u)+ - \frac{2t \sin^2\pi u}{k\pi} \rho_{k-1}(t,u) \qquad (7,4)$$

and $\sigma_u(t,u)$ also satisfies this recurrence relation.

Considering the fact that the zeros of $F_k(t,u)$ are not due to the factor $(\frac{\sin\pi u}{\pi})^k$ one can omit this factor, and defining:

$$\beta_k(t,u)= i^k \sum_{n=-\infty}^{\infty} \frac{e^{-t\pi(u+n)^2}}{(u+n)^k}, \qquad (7,5)$$

$$\gamma_k(t,u)= i^{k-2} \sum_{n=-\infty}^{\infty} (-1)^n \frac{e^{-t\pi(u+n)^2}}{(u+n)^k} \qquad (7,6)$$

one will obtain $\frac{\partial}{\partial t} \beta_k(t,u)= \beta_{k-2}(t,u)$ (7,7)

and

$$\beta_{k+1}(t,u)= \frac{i}{k}\frac{\partial}{\partial u} \beta_k(t,u)+ \frac{2t\pi}{k} \beta_{k-1}(t,u) \qquad (7,8)$$

$\gamma_k(t,u)$ also satisfies the relations (7,6) and (7,7).

After some calculations one obtains the differential equation

$$\frac{\partial^3}{\partial u^2\partial t} \beta_k(t,u)+ 4\pi t^2 \frac{\partial^2\beta_k(t,u)}{\partial t^2} - 2\pi t(2k-5) \frac{\partial\beta_k(t,u)}{\partial t} + + \pi(k-2)(k-1)\beta_k(t,u)= 0 \qquad (7,9)$$

8. <u>k = 0</u>.

k = 0 is very easy:

$$F_0(t,u) = \sum_{n=-\infty}^{\infty} e^{-t\pi(u+n)^2} = e^{-t\pi u^2} \vartheta_{00}(z/\tau)$$

with $\tau = it$

$$z = u\tau .$$

The zeros of $F_0(t,u)$ are at the same time those of $\vartheta_{00}(z/\tau)$

$$u_{n,m} = \frac{2n+1}{2} + \frac{2m+1}{2t} \; i \quad , \qquad n,m \text{ integers} \quad t > 0$$

and one gets the product-representation :

$$F_0(t,u) = f(t) \prod_{m=0}^{\infty} (1 + 2\, q^{2m+1} \cos 2\pi u + q^{4m+2}) \; ,$$

if $q = e^{-\pi/t}$ and f(t) depends on t only. Using again the transformation (6,1) one will find again :

$$\zeta_m = 2 + q^{-(2m+1)} + q^{2m+1} \tag{8,1}$$

and

$$F_0(t,u) = G_0(t,\zeta) = f^*(t) \prod_{m=0}^{\infty} (1 - \frac{\zeta}{\zeta_m}) \; . \tag{8,2}$$

From this last equation it appears that the theorem mentioned at the end of section 6 also holds for k = 0.

9. <u>k = 1</u>.

It is easy to get more information about the zeros $u_{n,m}$ of F(t,u) for large values of m.

For, if $u = \frac{1}{2} + iy$, one can write :

$$F_1(t,u) = \sum_{n=0}^{\infty} \frac{\exp\{-t\pi\{(n+\frac{1}{2})^2 - y^2\}\}}{(n+\frac{1}{2})^2 + y^2} \left\{ (2n+1)\cos\{t\pi(2n+1)y\} + - 2\,y \sin\{t\pi(2n+1)y\} \right\} .$$

The zeros of the equations

$$(2n+1)\cos\{t\pi(2n+1)y\} - 2\,y \sin\{t\pi(2n+1)y\} = 0 \qquad n = 0,1,2,\ldots \tag{9,1}$$

will be also zeros of $F_1(t,u)$. It is clear, however, that there does not exist a value of y which simultaneously satisfies this system of equations. When y is large the solutions of these equations approach each other, that is to say, only those that correspond with the solution of the equation (9,1) with n = 0. By substituting for y the following asymptotic series

$$y = \sum_{k=-1}^{\infty} c_h \, m^{-h} \tag{9,2}$$

one will find that the solutions of different equations deviate only in the term of order $m^{-3}$ only, and one will find

$$y_m \approx \frac{m}{t} + \frac{1}{2\pi m} + O\left(\frac{1}{m^3}\right) \tag{9,3}$$

with m a large integer.

## 10. Computations performed regarding the zeros.

In order to have some more indications about the zeros of $F_k(t,u)$, a number of them are computed. The values of t are $\frac{1}{\pi}$ and $\pi$. It appears that for these values of t all zeros of the functions examined are on the lines Re $u = n + \frac{1}{2}$.

If $u_{n,m} = n + \frac{1}{2} + i\frac{\lambda_m}{t}$, the values of $\lambda_m$ are given in the table below:

$t = 1/\pi$

| k = | 3 | 4 | 5 | 6 |
|---|---|---|---|---|
| n = 0 | 0,0756 | 0,0576 | 0,0466 | 0,0392 |
| 1 | 0,3852 | 0,2422 | 0,1748 | 0,1378 |
| 2 | | | 0,4995 | 0,3395 |

$t = \pi$ :

| | | | | |
|---|---|---|---|---|
| n = 0 | 0,31248 | 0,27742 | 0,24940 | 0,2265 |
| 1 | 0,97120 | 0,86136 | 0,7725 | 0,6996 |
| 2 | 1,7096 | 1,5206 | 1,3626 | 1,2309 |
| 3 | 2,5305 | 2,2706 | 2,0432 | 1,846 |
| 4 | 3,4120 | 3,098 | 2,811 | 2,553 |
| 5 | | | 3,648 | 3,340 |

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## 12. Literature.

(1) I.J. Schoenberg, Contributions to the problem of approximation of equidistant data by analytic functions. Quart.of Appl.Math. IV, 1946, 1, (112-141).