Table of the integral $\int_0^1 \exp(-v^{-2}-xv)v^{-p}\,dv$.

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1. Introduction.

In a certain physical problem a function plays a role that can be written in the form

$$f_p(y)= \int_0^{y^{\frac{1}{2}}} e^{-y(u^{-2}+u)}\, u^{-p}\, du, \tag{1,1}$$

where p takes moderately small nonnegative integer values and y may take any positive value.

Putting $v = y^{-\frac{1}{2}} u$, and $x = y^{\frac{3}{2}}$ it follows that

$$f_p(y)= y^{-\frac{1}{2}(p-1)}\, F_p(x), \tag{1,2}$$

where

$$F_p(x)= \int_0^1 e^{-v^{-2}-xv}\, v^{-p}\, dv. \tag{1,3}$$

The function $F_p(x)$ is more manageable than $f_p(y)$, and is, therefore, tabulated rather than $f_p(y)$ itself. Several methods of computing $F_p(x)$ are presented, and a table is given for $p = 0(1)10$, and $x = 0(0.25)3.5(0.5)14.5$ in five decimals.

2. Computation of $F_p(x)$.

There are many ways to compute $F_p(x)$. First of all, for $x = 0$ one has:

$$F_p(0)= \int_0^1 e^{-v^{-2}}\, v^{-p}\, dv, \tag{2,1}$$

and by putting $v^{-2} = t$ it follows that

$$F_p(0)= \tfrac{1}{2} \int_1^{\infty} e^{-t}\, t^{\frac{1}{2}(p-3)}\, dt, \tag{2,2}$$

so that $F_p(0)$ is directly expressible in terms of the incomplete factorial function. In particular one has e.g.

$$F_1(0)= -\tfrac{1}{2}\,\mathrm{Ei}(-1)= 0.109692$$

$$F_2(0)= 0.139403.$$

For other values of the order p one computes $F_p(0)$ from these startingvalues easily enough by means of the recurrence relation that follows directly from (2,2), viz.

$$2\, F_p(0) + (3-p)\, F_{p-2}(0)= e^{-1}. \tag{2,3}$$

A short table of $F_p(0)$ is given in Table I. Also negative values of p have been included there for reasons that will be obvious furtheron.

Furthermore, it is obvious from (1,3) that

$$F_p(\infty)= 0. \qquad (2,4)$$

Now, differentiation of (1,3) with respect to x yields

$$F_p'(x)= - F_{p-1}(x), \qquad (2,5)$$

or, in general

$$F_p^{(k)}(x)= (-)^k F_{p-k}(x). \qquad (2,6)$$

Hence, using Taylor's theorem, one obtains directly the series expansion

$$F_p(x)= \sum_{k=0} \frac{1}{k!} F_{p-k}(x_o) (x_o-x)^k. \qquad (2,7)$$

Taking $x_o = 0$, one has the $F_{p-k}(0)$ at hand, and (2,7) can be used to compute $F_p(x)$ for not too large values of x. This procedure can then be repeated with a new value of x_o for which $F_p(x)$ has just been calculated. In doing so, one looses values for the large negative values of p. This difficulty can easily be overcome, as by partial integration of (1,3) the more general recurrence formula is seen to hold

$$2 F_p(x) + (3-p) F_{p-2}(x) - x F_{p-3}(x)= e^{-1-x}. \qquad (2,8)$$

This relation can either be used to supply values for large negative values of p, or even to fill directly gaps in the range of p, or again as a check. The direction in which (2,8) is stable with respect to rounding-off errors varies, but proper application yields good results.

Another way is to compute, say, $F_0(x)$ by means of numerical integration for a suitably dense set of values of x. If x = 0 is included, a certain check on the accuracy of the numerical process is obtained. Now, from (2,5) together with (2,4) it follows that

$$F_p(x) = \int_x^{\infty} F_{p-1}(t)\, dt, \qquad (2,9)$$

so that by numerical integration of $F_0(x)$, $F_1(x)$ is obtained, from that again $F_2(x)$, and so on, one single integration providing from now onwards a complete solution for all x. The value $F_p(0)$, so obtained forms each time a valuable overall check on the integration. As $F_p(x)$ is a very smooth function tending rapidly to 0 as x tends to infinity the integration process is rapid and accurate.

As soon as $F_p(x)$ has been computed in this way for three consecutive values of p, the solution can be extended in the p-direction by means of (2,8) for each value of x separately.

At last, an asymptotic expansion for large x can be constructed from (1,3) by means of the method of steepest descent. Little use of it has been made.

Following several of the methods, described above, five decimal values of $F_p(x)$ have been computed for $p = 0(1)10$, and $x = 0(0.25)3.5(0.5)14.5$. The results are given in Table II. The last decimal is not absolutely reliable.

3. Acknowledgement.

The author is indebted to Miss T. Hurts, who carried out the numerical computations required for the tabulation.

TABLE I

Values of $F_p(0) = \int_o^1 \exp(-v^{-2}) v^{-p} dv.$

p	$J_p(0)$
-25	0.013068
-24	0.013546
-23	0.014060
-22	0.014615
-21	0.015215
-20	0.015866
-19	0.016574
-18	0.017348
-17	0.018197
-16	0.019132
-15	0.020167
-14	0.021318
-13	0.022606
-12	0.024056
-11	0.025700
-10	0.027578
- 9	0.029742
- 8	0.032262
- 7	0.035227
- 6	0.038760
- 5	0.043031
- 4	0.048278
- 3	0.054846
- 2	0.063244
- 1	0.074248
0	0.089073
1	0.109692
2	0.139403
3	0.183940
4	0.253641
5	0.367879
6	0.564401
7	0.919699
8	1.594942
9	2.943036
10	5.766237
11	9.748805

TABLE II.

Values of $F_p(x)=\int_0^1 \exp(-v^{-2}-xv)v^{-p}dv$.

x	p = 0	p = 1	p = 2	p = 3	p = 4	p = 5
0	0.08907	0.10969	0.13940	0.18394	0.25364	0.36788
0.25	0.07235	0.08959	0.11458	0.15229	0.21174	0.30986
0.50	0.05883	0.07325	0.09430	0.12627	0.17702	0.26140
0.75	0.04788	0.05996	0.07770	0.10484	0.14822	0.22086
1.00	0.03901	0.04914	0.06411	0.08717	0.12429	0.18689
1.25	0.03181	0.04032	0.05297	0.07258	0.10438	0.15838
1.50	0.02597	0.03312	0.04382	0.06052	0.08779	0.13442
1.75	0.02124	0.02724	0.03630	0.05054	0.07395	0.11426
2.00	0.01737	0.02243	0.03011	0.04227	0.06238	0.09727
2.25	0.01423	0.01850	0.02501	0.03540	0.05270	0.08292
2.50	0.01167	0.01527	0.02080	0.02969	0.04459	0.07078
2.75	0.00958	0.01262	0.01732	0.02493	0.03778	0.06051
3.00	0.00787	0.01045	0.01445	0.02096	0.03206	0.05181
3.25	0.00648	0.00866	0.01207	0.01766	0.02724	0.04442
3.50	0.00534	0.00719	0.01009	0.01490	0.02318	0.03811
4.00	0.00364	0.00497	0.00709	0.01065	0.01685	0.02821
4.50	0.00249	0.00345	0.00501	0.00765	0.01232	0.02097
5.00	0.00172	0.00241	0.00356	0.00553	0.00906	0.01567
5.50	0.00119	0.00170	0.00254	0.00402	0.00669	0.01177
6.00	0.00083	0.00120	0.00183	0.00293	0.00497	0.00888
6.50	0.00058	0.00085	0.00132	0.00215	0.00371	0.00672
7.00	0.00041	0.00061	0.00096	0.00159	0.00278	0.00511
7.50	0.00029	0.00044	0.00070	0.00118	0.00209	0.00390
8.00	0.00021	0.00032	0.00051	0.00088	0.00158	0.00299
8.50	0.00015	0.00023	0.00038	0.00066	0.00120	0.00230
9.00	0.00010	0.00017	0.00028	0.00049	0.00091	0.00178
9.50	0.00008	0.00012	0.00021	0.00037	0.00070	0.00138
10.00	0.00005	0.00009	0.00016	0.00028	0.00054	0.00107
10.50	0.00004	0.00007	0.00012	0.00021	0.00041	0.00083
11.00	0.00003	0.00005	0.00009	0.00016	0.00032	0.00065
11.50	0.00002	0.00004	0.00007	0.00013	0.00025	0.00051
12.00	0.00002	0.00003	0.00005	0.00010	0.00019	0.00040
12.50	0.00001	0.00002	0.00004	0.00007	0.00015	0.00032
13.00	0.00001	0.00002	0.00003	0.00005	0.00012	0.00036
13.50	0.00001	0.00001	0.00002	0.00004	0.00009	0.00020
14.00	0.00000	0.00001	0.00002	0.00003	0.00007	0.00016
14.50	0.00000	0.00001	0.00001	0.00003	0.00006	0.00012

TABLE II (Continued).

Values of $F_p(x)= \int_0^1 \exp(-v^{-2}-xv)v^{-p}dv$.

X	p = 5	p = 6	p = 7	p = 8	p = 9	p = 10
0	0.36788	0.56440	0.91970	1.59494	2.94304	5.76624
0.25	0.30986	0.47990	0.78944	1.38173	2.57156	5.07802
0.50	0.26140	0.40867	0.67863	1.19861	2.24965	4.47631
0.75	0.22086	0.34854	0.58420	1.04105	1.97021	3.94965
1.00	0.18689	0.29769	0.50360	0.90534	1.72732	3.48817
1.25	0.15838	0.25463	0.43471	0.78828	1.51598	3.08336
1.50	0.13442	0.21812	0.37575	0.68716	1.33186	2.72790
1.75	0.11426	0.18710	0.32520	0.59971	1.17126	2.41547
2.00	0.09727	0.16072	0.28181	0.52397	1.03102	2.14058
2.25	0.08292	0.13826	0.24451	0.45829	0.90844	1.89849
2.50	0.07078	0.11909	0.21240	0.40128	0.80116	1.68508
2.75	0.06051	0.10271	0.18473	0.35172	0.70718	1.49680
3.00	0.05181	0.08870	0.16085	0.30860	0.62476	1.33053
3.25	0.04442	0.07670	0.14022	0.27103	0.55242	1.18358
3.50	0.03811	0.06641	0.12237	0.23826	0.48885	1.05359
4.00	0.02821	0.04994	0.09349	0.18464	0.38372	0.83658
4.50	0.02097	0.03774	0.07172	0.14359	0.30211	0.66598
5.00	0.01567	0.02865	0.05523	0.11205	0.23854	0.53147
5.50	0.01177	0.02184	0.04269	0.08771	0.18887	0.42512
6.00	0.00888	0.01671	0.03311	0.06886	0.14993	0.34081
6.50	0.00672	0.01283	0.02577	0.05422	0.11931	0.27381
7.00	0.00511	0.00989	0.02012	0.04281	0.09517	0.22043
7.50	0.00390	0.00766	0.01575	0.03389	0.07609	0.17780
8.00	0.00299	0.00595	0.01237	0.02690	0.06096	0.14368
8.50	0.00230	0.00463	0.00974	0.02140	0.04894	0.11632
9.00	0.00178	0.00362	0.00769	0.01706	0.03937	0.09434
9.50	0.00138	0.00283	0.00609	0.01363	0.03173	0.07663
10.00	0.00107	0.00222	0.00483	0.01091	0.02562	0.06235
10.50	0.00083	0.00175	0.00384	0.00876	0.02072	0.05081
11.00	0.00065	0.00138	0.00306	0.00704	0.01679	0.04147
11.50	0.00051	0.00109	0.00244	0.00567	0.01363	0.03389
12.00	0.00040	0.00087	0.00196	0.00457	0.01108	0.02774
12.50	0.00032	0.00069	0.00157	0.00370	0.00902	0.02274
13.00	0.00036	0.00055	0.00125	0.00300	0.00735	0.01865
13.50	0.00020	0.00044	0.00101	0.00243	0.00600	0.01533
14.00	0.00016	0.00035	0.00083	0.00196	0.00495	0.01272
14.50	0.00012	0.00028	0.00072	0.00160	0.00420	0.01084