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Computation of the Goldstein-factor $\varkappa$ for a three-bladed propeller in a homogeneous flow.

by

The Staff of the Computation Department

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1. Statement of the problem

On behalf of the Netherlands Ship Model Basin values of the Goldstein-factor $x^{(z)}$ were computed for a three bladed propeller in a homogenous flow. Following Goldstein [1] and Kramer [2] we write:

$$x^{(z)} = \frac{1+\mu^2}{\mu^2} \, \frac{z}{\pi} \Phi(\mu).$$

where z is the number of blades and $\Phi(\mu)$ the solution for $\xi = \frac{\pi}{z}$ of the partial differential equation:

$$\mu^2 \frac{\partial^2 \Phi}{\partial \mu^2} + \mu \frac{\partial \Phi}{\partial \mu} + (1+\mu^2) \frac{\partial^2 \Phi}{\partial \xi^2} = 0,$$

with boundary conditions:

$$\frac{\partial \Phi}{\partial \xi} = -\frac{\mu}{1+\mu^2} \quad \text{for} \quad 0 \leq \mu \leq \mu_0 \qquad \text{and} \quad \Phi = 0 \quad \text{for} \quad \mu > \mu_0 \text{. for}$$

$$\xi = \frac{\pi}{z}, \frac{3\pi}{z} \ldots \frac{2z-1}{z}\pi$$

so, that $\frac{\partial \Phi}{\partial \mu}$ and $\frac{\partial \Phi}{\partial \xi}$ tend to zero as μ approaches infinity.

2. Method of solution

The differential equation can be solved analytically for $0 \leqslant \mu \leqslant \mu_0$ and for $\mu \geqslant \mu_0$. This was done by Goldstein and Kramer and results in solving an infinite set of linear equations with infinite unknowns. We solved the equation by a numerical method after mapping of the sector $0 \leqslant \xi \leqslant \frac{\pi}{z}$ into a strip. This may be done since $\Phi(\mu, \xi)$ is a periodic function of ξ . Moreover $\Phi(\xi) = -\Phi(-\xi)$.

3. The mapping

As in our problem z = 3, we have

$$\frac{1}{\mu} \frac{\partial}{\partial \mu} \left(\mu \frac{\partial \Phi}{\partial \mu} \right) + \left(1 + \frac{1}{\mu^2}\right) \frac{\partial^2 \Phi}{\partial \xi^2} = 0$$

with

$$\frac{\partial \Phi}{\partial \xi} = -\frac{\mu^2}{1+\mu^2} \qquad \text{for } \xi = \frac{\pi}{3}, \ 0 \leqslant \mu \leqslant \mu_0$$

$$\frac{\partial \Phi}{\partial \xi} = 0 \qquad \text{for } \xi = \frac{\pi}{3}, \ \mu \geqslant \mu_0$$

$$\Phi = 0 \qquad \text{for } \xi = 0$$

Put $u = \sqrt{1+\mu^2} - \frac{1}{2} \log \frac{\sqrt{1+\mu^2}+1}{\sqrt{1+\mu^2}-1}$

$$v = \xi$$

$$f = (1+\mu^2)^{1/4} \Phi$$

By this transformation the sector $0 \leqslant \xi \leqslant \frac{\pi}{3}$ is mapped into the strip $0 \leqslant v \leqslant \frac{\pi}{3}$.

The equation is transformed into

$$\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = \frac{\mu^2(4 - \mu^2)}{4(1 + \mu^2)^3} f.$$

with $\frac{\partial f}{\partial v} = -\mu^2(1 + \mu^2)^{-3/4}$ for $-\infty \leqslant u \leqslant u_0$, $v = \frac{\pi}{3}$
and $f = 0$ for $u > u_0$, $v = \frac{\pi}{3}$ and for $-\infty < u < \infty$, $v = 0$,

where $u_0 = \sqrt{1 + \mu_0^2} - \frac{1}{2} \log \frac{\sqrt{1 + \mu_0^2} + 1}{\sqrt{1 + \mu_0^2} - 1}$.

Next we apply the conformal transformation

$$e^x \cos y = \frac{1}{2} \frac{e^{\frac{\pi}{v_0}(u - u_0)} + \cos\frac{\pi}{v_0} v}{\cosh\frac{\pi}{v_0}(u - u_0) + \cos\frac{\pi}{v_0} v}$$

$$e^x \sin y = \frac{1}{2} \frac{\sin\frac{\pi}{v_0} v}{\cosh\frac{\pi}{v_0}(u - u_0) + \cos\frac{\pi}{v_0} v} \quad \text{with } v_0 = \frac{\pi}{3}$$

The equation becomes

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{v_0^2}{\pi^2} \frac{1}{e^{2x} - 2e^x \cos y + 1} \cdot \frac{\mu^2(4 - \mu^2)}{4(1 + \mu^2)^3} \cdot f = p(x,y) f.$$

with

$$\frac{\partial f}{\partial y} = \frac{v_0}{\pi} \frac{\mu^2(1 + \mu^2)^{-3/4}}{1 + e^x} = q(x) \text{ for } y = \pi$$

$f = 0$ for $y = 0$

The numerical solution of this equation is discussed in the next section.

4. Numerical solution

First we replace derivatives by series of finite differences:

$$\frac{\partial^2}{\partial x^2} = \frac{1}{h^2}(\delta_x^2 - \frac{1}{12}\delta_x^4 + \frac{1}{90}\delta_x^6 \ldots.)$$

$$\frac{\partial^2}{\partial y^2} = \frac{1}{k^2}(\delta_y^2 - \frac{1}{12}\delta_y^4 + \frac{1}{90}\delta_y^6 \ldots.)$$

$$\frac{\partial}{\partial y} = \frac{1}{k}(\mu\delta_y - \frac{1}{6}\mu\delta_y^3 + \frac{1}{30}\mu\delta_y^5 \ldots.)$$

where the δ's are central differences and h en k interval lengths in resp. x- and y-direction. We put $x = mh$, $y = nk$ where $m = 0, \pm 1, \pm 2, \ldots$; $n = 0, 1, 2, \ldots K$.

With these substitutions our equation is replaced by:

$$(\frac{1}{h^2}\delta_x^2 + \frac{1}{k^2}\delta_y^2 - \delta) f = p(x,y) f.$$

$f = 0$ for $y = 0$.

$(\frac{1}{k}\mu\delta_y - \varepsilon)f = q(x)$ for $y = Kk$.

with

$$\delta = \frac{1}{12}\left(\frac{1}{h^2}\delta_x^4 + \frac{1}{k^2}\delta_y^4\right) - \frac{1}{90}\left(\frac{1}{h^2}\delta_x^6 + \frac{1}{k^2}\delta_y^6\right) + \ldots.$$

$$\varepsilon = \frac{1}{k}\left(\frac{1}{6}\mu\delta_y^3 - \frac{1}{30}\mu\delta_y^5 + \ldots.\right)$$

We now try the following iteration:

$$\left(\frac{1}{h^2}\delta_x^2 + \frac{1}{k^2}\delta_y^2\right)f_{s+1} = pf_s + \delta f_s = F_s(x,y), \quad s = 1,2,\ldots \qquad (4.1)$$

with $f_{s+1} = 0$

for $n = 0$ and $\frac{1}{k}\mu\delta_y f_{s+1} = q + \varepsilon f_s = G_s(mh)$ for $n = K$ (4.2)

We put

$$f_s = A_s + B_s \qquad (4.3)$$

Where A_s satisfies

$$\left(\frac{1}{h^2}\delta_x^2 + \frac{1}{k^2}\delta_y^2\right)A_{s+1} = F_s \qquad (4.4)$$

with $A_{s+1} = 0$ for $n = 0$ and $\mu\delta_y A_{s+1}$ arbitrary for $n = K$

Now it is possible to build up A_{s+1} if A_{s+1} is known for $n = 1$. If we put $A_{s+1}(mh,k) = f_s(mh,k)$ we have the following conditions for B_{s+1}:

$$\left(\frac{1}{h^2}\delta_x^2 + \frac{1}{k^2}\delta_y^2\right)B_{s+1} = 0$$

with $B_{s+1} = 0$ for $n = 0$

and $\mu\delta_y B_{s+1} = kG_s - \mu\delta_y A_{s+1} = C_s(mh)$ for $n = K$. (4.5)

We now define a "standard" function U as follows:

$$\left(\frac{1}{h^2}\delta_x^2 + \frac{1}{k^2}\delta_y^2\right)U = 0$$

with

$U = 0$ for $n = 0$

$\mu\delta_y U = 0$ for $n = K$ and $m \neq 0$

$\mu\delta_y U = 1$ for $n = K$ and $m = 0$

Then we have:

$$B_{s+1} = \sum_{m'=-\infty}^{\infty} C_s(m'h)U(mh - m'h, nk) \qquad (4.6)$$

If the "standard" function U is known the procedure is as follows: let be available an estimate f_s. Then using (4.1) we calculate F_s and using (4.4) A_{s+1}. From (4.2) and (4.5)

we calclulate C_s and using (4.6) we find B_{s+1}. Then we have $f_{s+1} = A_{s+1} + B_{s+1}$.
This proces converges rapidly if F_s or what is the same p.f, is small. In our problem this happened to be so.

5. Computation of the function U

Put $u = \cos ax \ \sinh by$.

$$\left(\frac{1}{h^2}\delta_x^2 + \frac{1}{k^2}\delta_y^2\right)u = \frac{2u}{h^2}(\cos ah - 1) + \frac{2u}{k^2}(\cosh bk - 1)$$

If this is to be zero, we must have:

$$\frac{1}{h^2}(\cos ah - 1) + \frac{1}{k^2}(\cosh bk - 1) = 0$$

Next we have $\mu\delta_y u = \cos ah \cosh bkK \sinh bk$

Now U is constructed as a linear combination of u's in such a way that

$\mu\delta_y U = 0$ for n = K for all $m \neq 0$ and
$\mu\delta_y U = 1$ for m = 0
This gives

$$U = \frac{h}{\pi}\int_0^{\pi/h} \frac{\cos ax \ \sinh by}{\cosh bkK \sinh bk}\, da \text{ with } \frac{1}{h^2}(\cos ah - 1) + \frac{1}{k^2}(\cosh bk - 1) = 0$$

as may easily be verified.
This expression not being suited for computation we proceed as follows: putting A = ah, B = bk we have:

$$U = \frac{1}{\pi}\int_0^{\pi} \frac{\cos Am \sinh Bn}{\cos BK \sinh B}\, dA \text{ with } \frac{1}{h^2}(\cos A - 1) + \frac{1}{k^2}(\cosh B - 1) = 0$$

Now $\dfrac{\sinh Bn}{\sinh B} = Q_{n-1}(\cosh B)$ and $\cosh BK = T_K(\cosh B)$, where

Q_{n-1} and T_K are Chebyshev-polynomials [3].
We have

$$\frac{Q_{n-1}(\cosh B)}{T_K(\cosh B)} = \frac{1}{K}\sum_0^{K-1}(-)^l \frac{\sin \frac{2l+1}{2K}\pi n}{\cosh B - \cos \frac{2l+1}{2K}\pi}.$$

Putting $h/k = R$ we have:

$$U(m,n) = \frac{R^2}{K}\sum_0^{K-1} (-)^l \sin\frac{2l+1}{2K}\pi n \ .\ M_l(m)$$

where

$$M_l(m) = \frac{1}{\pi R^2}\int_0^{\pi}\frac{\cos Am}{\cosh B - \cos\frac{2l+1}{2K}}\, dA \text{ with } R^2(\cosh B - 1) = 1 - \cos A$$

$$M_l(m) = \frac{1}{\pi}\int_0^{\pi}\frac{\cos Am}{\cosh E_m - \cos A}\, dA \text{ if } \cosh E_m = 1 + 2R^2\sin^2\left(\frac{2m+1}{4K}\right)\pi$$

Putting $z = e^{iA}$ we have:

$$M_1(m) = -\frac{1}{2\pi i}\oint \frac{z^m + z^{-m}}{z - 2\cosh E_m + z^{-1}}\,\frac{dz}{z} = -$$

$$-\frac{1}{2\pi i}\oint \frac{z^m + z^{-m}}{(z - e^{E_m})(z - e^{-E_m})}\,dz$$

where the integration is to be performed along a circle around $z = 0$. The residual of the pole $z = e^{-E_m}$ is $\cosh mE_m / \sinh E_m$ and the residual of $z = 0$ is $-\sinh|m|E_m / \sinh E_m$.

So we have:

$$M_1(m) = e^{-|m|E_m} / \sinh E_m,$$

and

$$U(m,n) = \frac{R^2}{K}\sum_{0}^{K-1} \frac{(-)^l}{\sinh E_m}\, e^{-|m|E_m} \sin \frac{2l + 1}{2K}\pi n.$$

with $R = h/k$ and $\sinh \frac{1}{2}E_m = R \sin \frac{2m + 1}{2K}\pi$.

6. Computation of f(x,y)

The iterations described in section 4 were mainly performed by the electronic computer ARRA. As first estimate of $f(x,y)$ we chose $f_c(x,y) = 0$. The interval used was $\frac{1}{2}$ in the x-direction and $\frac{\pi}{8}$ in the y-direction. The calculations were made with five places of decimals. As an illustration of the speed of convergence of the iteration used we give values of f_1, f_2, f_3, f_4 for part of the x,y-region for $\mu_0 = 1$:

f_1	x	$\frac{4}{\pi}y$ 1	2	3	4
	-10	.00295	.00557	.00754	.00860
	- 8	.00694	.01327	.01831	.02146
	- 6	.01500	.02919	.04158	.05092
	- 4	.02785	.05568	.08316	.10921
	- 2	.03988	.08198	.12888	.18395
	0	.03889	.08004	.12621	.18153
	+ 2	.02504	.04977	.07342	.09414
	+ 4	.01198	.02295	.03174	.03693
	+ 6	.00492	.00923	.01234	.01374
	+ 8	.00189	.00352	.00464	.00508
	+10	.00070	.00131	.00172	.00186

	x	$\frac{4}{\pi}y$ = 1	2	3	4
f_2	-10	.00286	.00541	.00733	.00836
	- 8	.00673	.01287	.01778	.02086
	- 6	.01448	.02821	.04028	.04947
	- 4	.02680	.05373	.08056	.10632
	- 2	.03843	.07937	.12553	.18040
	0	.03785	.07830	.12409	.17940
	+ 2	.02464	.04902	.07242	.09297
	+ 4	.01180	.02262	.03128	.03638
	+ 6	.00484	.00909	.01215	.01351
	+ 8	.00186	.00346	.00456	.00499
	+10	.00069	.00129	.00169	.00183
f_3	-10	.00286	.00541	.00733	.00836
	- 8	.00674	.01288	.01780	.02089
	- 6	.01451	.02825	.04032	.04952
	- 4	.02684	.05379	.08064	.10641
	- 2	.03848	.07945	.12563	.18051
	0	.03789	.07835	.12415	.17947
	+ 2	.02465	.04904	.07245	.09300
	+ 4	.01181	.02263	.03129	.03640
	+ 6	.00485	.00909	.01216	.01352
	+ 8	.00186	.00346	.00457	.00500
	+10	.00069	.00129	.00169	.00184
f_4	-10	.00286	.00540	.00732	.00836
	- 8	.00675	.01286	.01777	.02084
	- 6	.01449	.02823	.04029	.04948
	- 4	.02683	.05377	.08063	.10638
	- 2	.03848	.07945	.12563	.18050
	0	.03788	.07835	.12414	.17946
	+ 2	.02464	.04903	.07243	.09298
	+ 4	.01180	.02261	.03127	.03636
	+ 6	.00484	.00908	.01214	.01350
	+ 8	.00186	.00345	.00455	.00499
	+10	.00069	.00128	.00168	.00183

7. <u>Results</u>

As results of the computation we have the following table of $x^{(3)}$ for z= 3 and for various μ_0 and μ/μ_0

$\chi^{(3)}$

μ/μ_0	μ_0: 1	2.5	4	5	10
.1	2.420	1.836	1.456	1.304	1.024
.2	1.5084	1.2250	1.0699	1.0202	.9768
.3	1.1202	1.0154	.9785	.9730	.9878
.4	..8950	.9161	.9504	.9661	.9945
.45	.8121	.8821	.9420	.9647	.9961
.5	.7410	.8523	.9333	.9619	.9970
.6	.6211	.7940	.9079	.9478	.9968
.7	.5158	.7234	.8604	.9123	.9909
.75	.4641	.6786	.8229	.8807	.9822
.8	.4105	.6241	.7715	.8344	.9642
.85	.3526	.5558	.7004	.7663	.9267
.9	.2862	.4665	.5992	.6637	.8483
.925	.2474	.4096	.5310	.5918	.7799
.95	.2016	.3390	.4435	.4974	.6766
.975	.1423	.2429	.3206	.3619	.5082
1.0	0	0	0	0	0

These values were obtained by interpolation from the following table where $\chi^{(3)}$ is given as a function of

$$u = \sqrt{1 - \frac{\mu}{\mu_0}} .$$

u	μ_0: 1	2.5	4	5	10
.0000	0	0	0	0	0
.0125	.0112	.0195	.0259	.0294	.0426
.0250	.0224	.0389	.0518	.0588	.0850
.0375	.0337	.0583	.0776	.0880	.1273
.0500	.0449	.0776	.1034	.1172	.1694
.0625	.0561	.0970	.1290	.1463	.2110
.0750	.0674	.1163	.1546	.1753	.2522
.0875	.0786	.1356	.1801	.2041	.2929
.1000	.0899	.1547	.2054	.2326	.3330
.1125	.1011	.1739	.2306	.2610	.3723
.1250	.1124	.1929	.2556	.2890	.4108

u	μ_0	1	2.5	4	5	10
.1250		.1124	.1929	.2556	.2890	.4108
.1375		.1237	.2118	.2803	.3168	.4484
.1500		.1349	.2307	.3048	.3442	.4850
.1625		.1462	.2494	.3291	.3713	.5205
.1750		.1575	.2680	.3531	.3980	.5549
.1875		.1688	.2865	.3768	.4242	.5881
.2000		.1802	.3048	.4002	.4500	.6200
.2125		.1915	.3230	.4233	.4754	.6506
.2250		.2028	.3410	.4460	.5002	.6798
.2375		.2142	.3588	.4684	.5244	.7075
.2500		.2256	.3764	.4903	.5482	.7338
.2625		.2370	.3939	.5118	.5713	.7586
.2750		.2484	.4112	.5329	.5939	.7820
.2875		.2598	.4282	.5536	.6158	.8038
.3000		.2713	.4450	.5738	.6371	.8241
.3125		.2828	.4616	.5934	.6577	.8429
.3250		.2943	.4780	.6126	.6776	.8603
.3375		.3059	.4941	.6313	.6968	.8763
.3500		.3176	.5100	.6495	.7154	.8908
.3625		.3292	.5256	.6671	.7332	.9041
.3750		.3410	.5409	.6841	.7502	.9161
.3875		.3528	.5560	.7006	.7666	.9269
.4000		.3647	.5708	.7166	.7822	.9366
.4125		.3767	.5853	.7319	.7970	.9452
.4250		.3888	.5995	.7467	.8112	.9528
.4375		.4010	.6135	.7608	.8245	.9596
.4500		.4133	.6271	.7744	.8372	.9655
.4625		.4257	.6404	.7874	.8491	.9706
.4750		.4383	.6535	.7998	.8604	.9751
.4875		.4511	.6662	.8117	.8709	.9790
.5000		.4641	.6786	.8229	.8807	.9822

u	μ_0 1	2.5	4	5	10
.5000	.4641	.6786	.8229	.8807	.9822
.5125	.4772	.6908	.8335	.8899	.9851
.5250	.4907	.7026	.8436	.8984	.9875
.5375	.5044	.7142	.8531	.9063	.9895
.5500	.5184	.7255	.8620	.9136	.9912
.5625	.5327	.7365	.8704	.9203	.9926
.5750	.5474	.7473	.8783	.9264	.9938
.5875	.5625	.7578	.8856	.9320	.9947
.6000	.5781	.7681	.8924	.9370	.9955
.6125	.5942	.7782	.8988	.9415	.9961
.6250	.6108	.7881	.9046	.9456	.9965
.6375	.6282	.7979	.9101	.9492	.9969
.6500	.6462	.8076	.9151	.9523	.9971
.6625	.6651	.8172	.9197	.9551	.9972
.6750	.6849	.8269	.9239	.9574	.9973
.6875	.7058	.8367	.9278	.9594	.9973
.7000	.7278	.8466	.9314	.9611	.9972
.7125	.7512	.8567	.9347	.9625	.9970
.7250	.7762	.8673	.9379	.9636	.9967
.7375	.8028	.8783	.9410	.9644	.9963
.7500	.8316	.8901	.9441	.9651	.9958
.7625	.8625	.9027	.9472	.9656	.9952
.7750	.8962	.9166	.9505	.9662	.9944
.7875	.9330	.9318	.9543	.9667	.9935
.8000	.9734	.9490	.9577	.9675	.9924
.8125	1.0182	.9685	.9640	.9686	.9911
.8250	1.0681	.9910	.9706	.9704	.9896
.8375	1.1242	1.0173	.9792	.9732	.9877
.8500	1.1879	1.0485	.9902	.9775	.9855
.8625	1.2608	1.0860	1.0049	.9841	.9831
.8750	1.3452	1.1316	1.0245	.9940	.9805

u	μ_0	1	2.5	4	5	10
.8750		1.3452	1.1316	1.0245	.9940	.9805
.8875		1.4450	1.1879	1.0511	1.0089	.9780
.9000		1.5642	1.2588	1.0876	1.0312	.9761
.9125		1.7100	1.3497	1.1385	1.0646	.9760
.9250		1.8932	1.4696	1.2107	1.1154	.9804
.9375		2.1325	1.6325	1.3163	1.1940	.9941
.9500		2.4593	1.8653	1.4770	1.3200	1.0287
.9625		2.944	2.224	1.7383	1.5342	1.1107
.9750		3.768	2.849	2.217	1.9426	1.3142
.9875		5.65	4.312	3.376	2.960	1.938
1.0000		∞	∞	∞	∞	∞

8. References

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