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# THE COMPUTATION AND THE EXPANSION OF SOME TRIPLE INTEGRALS ORIGINATING FROM THE THEORY OF COSMIC RAYS

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R 261

# The computation and the expansion of some triple integrals originating from the theory of cosmic rays.

## Introduction.

In the theory of cosmic rays the following integrals occur

$$I(r,B) = r \int_{\Theta=0}^{\pi/2} (1-e^{-B\alpha})(1-e^{-B\beta})(1-e^{-B\gamma})\,d\Theta \qquad (0.0.1)$$

$$J^*(B) = B^{-5/2} \int_{r=0}^{\infty} I(r,B)\,dr \qquad (0.0.2)$$

$$J(r) = \int_{B=0}^{\infty} B^{-5/2}\, I(r,B)\,dB \qquad (0.0.3)$$

$$K = \int_{r=0}^{\infty} J(r)\,dr = \int_{B=0}^{\infty} J^*(B)\,dB \qquad (0.0.4)$$

where $\alpha$, $\beta$ and $\gamma$ are functions of r.

Two cases are considered here:

first case: $$\alpha = \frac{0{,}0245}{r^3} \qquad (0.0.5)$$

$$\beta = \frac{0{,}0245}{r_1^3} \qquad (0.0.6)$$

$$\gamma = \frac{0{,}0245}{r_2^3} \qquad (0.0.7)$$

second case: $$\alpha = \frac{0{,}454}{r}(1+4r)e^{-4r^{2/3}} \qquad (0.0.8)$$

$$\beta = \frac{0{,}454}{r_1}(1+4r_1)e^{-4r_1^{2/3}} \qquad (0.0.9)$$

$$\gamma = \frac{0{,}454}{r_2}(1+4r_2)e^{-4r_2^{2/3}} \qquad (0.0.10)$$

where $r_{1,2} = (a^2+r^2 \pm 2ar\cos\Theta)^{\frac{1}{2}}$ and a denotes a positive constant.

In this report the behaviour of $J^*(B)$ for small positive and for large positive values of B is considered. Furthermore there are given numerical values of the functions $J(r)$, $J^*(B)$ and K for certain values of a in both cases.

# CHAPTER I.

## $1^{st}$ case:

### 1. Expansion of $J^*(B)$ for small positive values of $B$.

We split up the domain D of integration ($0 \leq r \leq \infty$, $0 \leq \theta \leq \frac{\pi}{2}$) into three domains:

$$D_1:\ 0 \leq \theta \leq \frac{\pi}{2}\ ;\ 0 \leq r \leq \frac{a}{2}$$

$$D_2:\ 0 \leq r_2 \leq B^{1/18}$$

and $$D_3:\ D - D_1 - D_2 .$$

Let $j(b)$ be defined as the integral extended over the region D of the integrand $(1-e^{-br^{-3}})(1-e^{-br_1^{-3}})(1-e^{-br_2^{-3}})$ with $r_{1,2} = (a^2+r^2 \pm 2\ ar \cos\theta)^{\frac{1}{2}}$.

The integral over $D_1$ is called $h(b)$, that over $D_2$ is called $i(b)$ and finally that over $D_3$ is called $l(b)$, so that

$$j(b) = h(b) + i(b) + l(b).$$

### 2. Integration over the domain $D_1$.

On $D_1$ we can expand the function $(1-e^{-br_1^{-3}})(1-e^{-br_2^{-3}})$ in a power-series of b, since on $D_1$ holds $r_1 \geq a$ and $r_2 \geq a/2$.

The function $h(b)$ appears to be equal to

$$b^2 \int_{r=0}^{\infty} r(1-e^{-br^{-3}})dr \int_{\theta=0}^{\frac{\pi}{2}} \left\{(a^2+r^2)^2 - 4a^2r^2\cos^2\theta\right\}^{-\frac{3}{2}} d\theta + \dots \qquad (1.2.1)$$

We know already:

$$\int_{\theta=0}^{\pi/2} \left\{(a^2+r^2)^2 - 4a^2r^2\cos^2\theta\right\}^{-\frac{3}{2}} d\theta = (a^2+r^2)^{-3}(1-k^2)^{-1}\,E(k) \qquad (1.2.2)$$

where $k = 2\ ar(a^2+r^2)^{-1}$ and $E(k)$ denotes the elliptic function of the second kind;

if we substitute this result into (1.2.1) we find

$$h(b) = b^2 \int_{r=0}^{a/2} r(1-e^{-br^{-3}})(a^2+r^2)^{-1}(a^2+r^2)^{-2}.E(k)dr + \dots \qquad (1.2.3)$$

Substituting $u = b.r^{-3}$, it follows that

$$h(b) = \frac{1}{3} \; b^{8/3} \int_{b(\frac{a}{2})^{-3}}^{\infty} \frac{u^{-5/3}\,(1-e^{-u}).\,E(k)}{(a^2+u^{-2/3}b^{2/3})(a^2-u^{-2/3}b^{2/3})^2}\,du + \ldots \tag{1.2.4}$$

with $k = \dfrac{2au^{-1/3}.b^{1/3}}{a^2+u^{-2/3}.b^{2/3}}$ .

This integral has to be expanded for small positive values of b. To this end we use the method described in Theorem 5 of MR 11 of the Computation Department of the Mathematical Centre. Then we find:

$$h(b) = \frac{\pi}{6a^6}\, b^{8/3} \int_{u=0}^{\infty} u^{-5/3}\,(1-e^{-u})du - \frac{\pi}{6a^6}\, b^{8/3} \int_{u=0}^{b(\frac{a}{2})^{-3}} u^{-5/3}(1-e^{-u})du +$$

$$+ \frac{b^{8/3}}{3} \int_{u=b(\frac{a}{2})^{-3}}^{\infty} u^{-2/3} \left\{ \frac{E(k)}{(a^2-u^{-2/3}b^{2/3})^2(a^2+u^{-2/3}b^{2/3})} - \frac{\pi}{2a^6} \right\} du +$$

$$+ \frac{b^{8/3}}{3} \int_{u=b(\frac{a}{2})^{-3}}^{\infty} u^{-5/3} \left\{1-e^{-u}-u\right\} \left\{ \frac{E(k)}{(a^2-u^{-2/3}b^{2/3})^2(a^2+u^{-2/3}b^{2/3})} - \frac{\pi}{2a^6} \right\} du +$$

$$\ldots \tag{1.2.5}$$

We will now compute the integrals occurring in (1.2.5).

The first and second term give the following results:

$$\frac{\pi}{6a^6}\; b^{8/3} \int_0^{\infty} u^{-5/3}\,(1-e^{-u})du = -\frac{\pi}{6a^6}\; b^{8/3}.\,\Gamma(-2/3) \tag{1.2.6}$$

and

$$-\frac{\pi}{6a^6}\; b^{8/3} \int_0^{b(\frac{a}{2})^{-3}} u^{-5/3}(1-e^{-u})du = -\frac{\pi}{2a^6}\; b^3\; (\tfrac{a}{2})^{-1} + \ldots \tag{1.2.7}$$

The third term can be computed by writing $r = (bu^{-1})^{1/3}$. This integral is then equal to:

$$b^3 \int_0^{a/2} r^{-2} \left\{ \frac{E(k)}{(a^2-r^2)^2(a^2+r^2)} - \frac{\pi}{2a^6} \right\} dr \tag{1.2.8}$$

The last integral does not depend on b, so the third term is of the order $b^3$.

Using the expansions:

$$E(k) = \frac{\pi}{2}\left(1 - \frac{k^2}{2} + \dots\right)$$

and

$$(a^2-r^2)^{-2}(a^2+r^2) = a^{-6}\left(1 + \frac{r^2}{a^2} + \dots\right)$$

in the integrand of (1.2.8) one can derive that the third term is approximately equal to

$$0{,}03417\pi\, a^{-7}.b^3 \qquad (1.2.9)$$

From the relations (1.2.6), (1.2.7) and (1.2.9) we deduce:

$$h(b) = -\frac{\pi}{6a^6}\,\Gamma(-2/3)\,b^{8/3}+\dots \qquad (1.2.10)$$

3. Integration over the domain $D_2$.

In order to compute the contribution i(b) we substitute $u=br_2^{-3}$ and find:

$$i(b) = \frac{2}{3}\, b^{8/3} \int_{u=b^{5/6}}^{\infty} Z(u)\,u^{-5/3}(1-e^{-u})\,du$$

with

$$Z(u) = \int_{r=R_1}^{r=R_2} \frac{r^{-2}\,dr}{\left\{2(a^2+r^2)-u^{-2/3}b^{2/3}\right\}^{3/2}\left\{4a^2r^2-(a^2+r^2-u^{-2/3}b^{2/3})^2\right\}^{\frac{1}{2}}}$$

with $R_1 = a - u^{-1/3}\, b^{1/3}$ and $R_2 = a + u^{-1/3}\, b^{1/3}$. (1.3.1.)

Using the substitutions $v = u^{-1/3}\, b^{1/3}$ and

$$r = a + pu^{-1/3}\, b^{1/3}$$

the equation (1.3.1) changes into the expression

$$i(b) = \frac{2}{3}\, b^{8/3} \int_{v=b^{5/6}}^{\infty} (1-e^{-u})u^{-5/3}\, f(v).dv \qquad (1.3.2)$$

with

$$f(v) = v\int_{-1}^{+1} \frac{(a+pv)^{-2}\,dp}{\left\{2(2a^2+2apv+pv^2)-v^2\right\}^{3/2}\left\{4a^2(a+pv)^2-\left[2a^2+2apv-(1-p^2)v^2\right]^2\right\}^{\frac{1}{2}}}$$

we have only to derive the first term of the power series of f(v). This term is found to be equal to

$$f(v) = 8^{-1}\, v.a^{-5} \int_{-1}^{+1} \left\{ 4a^2(1-p^2)v^2 + 4p(1-p^2)av^3 - (1-p^2)^2\, v^4 \right\}^{-\frac{1}{2}} dp + \ldots =$$

$$= 16^{-1}\, a^{-6} \int_{-1}^{+1} (1-p^2)^{-\frac{1}{2}}\, dp + \ldots = \frac{\pi}{16}\, a^{-6} + \ldots \qquad (1.3.3)$$

By using the results of (1.2.6) and (1.3.3) we have

$$i(b) = -\frac{\pi}{24a^6}\, \Gamma(-2/3) b^{8/3} + \ldots \qquad (1.3.4)$$

Integration over $D_3$:

In this domain we find

$$l(b) = b^3 \iint_{D_3} r_1^{-3}.r_2^{-3}\, r^{-2}.dr\, d\Theta + \ldots \qquad (1.4.1)$$

We have to remark that the remainder term is of the order $b^3$, so that we need not to calculate terms of higher order occurring in the expansions for h(b) and i(b).

So the final result is:

$$j(b) = -\frac{5}{24}\, \frac{\pi}{a^6}\, \Gamma(-\tfrac{2}{3})\, b^{8/3} + O(b^3) \qquad (1.4.2)$$

6. Behaviour for large positive b:

The integration domain D is splitted up into $D_4$ $(0 \leqq r \leqq b^{1/3-\delta};\ 0 \leqq \Theta \leqq \frac{\pi}{2})$ and $D_5 (b^{1/3-\delta} \leqq r \leqq \infty,\ 0 \leqq \Theta \leqq \frac{\pi}{2})$ with $\delta > 0$ but small with respect to 1/3. Let $I_4$ be the integral extended over the region $D_4$ and $I_5$ be the integral extended over $D_5$.

In the domain $D_4$ we have $1-e^{-br^{-3}} \sim 1 + O(e^{-b^{3\delta}})$ and $1-e^{-br_{1,2}^{-3}} \sim 1 + O(e^{-b^{3\delta}})$ for large positive values of b and consequently

$$I_4 \sim \frac{\pi}{4}\, b^{2/3-2\delta} \left\{ 1 + O(e^{-b^{3\delta}}) \right\} \qquad (1.6.1)$$

In the domain $D_5$ we have $r_1 = r + O(b^{-1/3+\delta})$; $r_2 = r + O(b^{-1/3+\delta})$. Hence

$$I_5 \sim \frac{\pi}{2} \int_{b^{1/3-\delta}}^{\infty} r(1-e^{-br^{-3}})^3\, dr + O(b^{-1+3\delta}\, e^{-b^{3\delta}}) \qquad (1.6.2)$$

If we write now $u = br^{-3}$ we deduce from (1.6.2)

$$\frac{\pi}{2} \int_{r=b^{1/3-\delta}}^{\infty} r(1-e^{-br^{-3}})^3 \, dr = -\frac{\pi}{6} b^{2/3} \Gamma(-2/3)\left[1+3^{1/3}(1-2^{2/3})\right] 3^{2/3} - $$

$$- \frac{\pi}{6} b^{2/3} \int_{b^{3\delta}}^{\infty} u^{-5/3}(1-e^{-u})^3 \, du \qquad (1.6.3)$$

Furthermore $\frac{\pi}{6} b^{2/3} \int_{b^{3\delta}}^{\infty} u^{-5/3} (1-e^{-u})^3 \, du = -$

$$- \frac{\pi}{4} b^{2/3-2\delta} \left\{1 + O(e^{-B^{3\delta}})\right\}:$$

By addition of $I_4$ and $I_5$ we find

$$j(b) \sim \frac{\pi}{6} 3^{2/3} \left\{1 + 3^{1/3} (1-2^{2/3})\right\} \Gamma(-2/3) b^{2/3} + O(e^{-b^{3\delta}} b^{2/3-2\delta}) \qquad (1.6.4)$$

# CHAPTER II

## 2nd case:

### Introduction:

In this Chapter we will give the order of the integral

$$R(B)= \int_{r=0}^{\infty} rdr \int_{\theta=0}^{\frac{\pi}{2}} (1-e^{-b\alpha})(1-e^{-b\beta})(1-e^{-b\gamma})d\theta \qquad (2.0.1)$$

for small and large positive values of b.

In (2.0.1) the functions $\alpha$ , $\beta$ and $\gamma$ are given by the relations:

$$\alpha = \frac{c}{r} (1 + 4r)e^{-4r^{2/3}} \qquad (2.0.2)$$

$$\beta = \frac{c}{r_1} (1 + 4r_1)e^{-4r_1^{2/3}} \qquad (2.0.3)$$

$$\gamma = \frac{c}{r_2} (1 + 4r_2)e^{-4r_2^{2/3}} \qquad (2.0.4)$$

where c is a positive constant.

The integration domain D will again be splitted up into three domains:

$D_1$: $0 \leq r \leq a/2;\ 0 \leq \theta \leq \pi/2$

$D_2$: $0 \leq r_2 \leq t$ with t a positive constant

$D_3$: $D-D_1-D_2$ .

Now the integral over $D_1$ is called m(b), the integral over $D_2$ is called n(b) and finally the integral over $D_3$ is called p(b).

### 1. Estimation of m (b) for small positive b.

In this domain we expand the function $(1-e^{-b\beta})(1-e^{-b\gamma})$ in a powerseries of b. Furthermore since

$$\left| \int_0^{\pi/2} \beta . \gamma \, d\theta \right| \text{ is bounded} \qquad \text{and } \left|1-e^{-b\alpha}\right| < b\alpha$$

we find

$$|m(b)| < \frac{\pi}{2} c\ b^3 M \left| \int_0^{a/2} (1+4r)e^{-4r^{2/3}} dr \right| < b^3 M_1 \text{ with } M > 0 \text{ and } M_1 > 0. \qquad (2.1.1)$$

2. Treatment of the integral in the domain $D_2$.

In $D_2$ we can estimate the integral by writing $(1-e^{-b\alpha})(1-e^{-b\beta}) = b^2\beta\alpha + \ldots$, it follows then

$$n(b) = b^2c^2 \iint_{D_2} (1+4r)(1+4r_1)r_1^{-1}e^{-4r_1^{2/3}} \quad e^{-4r^{2/3}} (1-e^{-b\gamma})dr\, d\theta + \ldots \qquad (2.2.1)$$

The function $(1+4r)(1+4r_1)r_1^{-1}e^{-4r_1^{2/3}} \quad e^{-4r^{2/3}}$ is positive in $D_2$ and is furthermore bounded in $D_2$.

Therefore we have

$$|n(b)| < M_2\, b^2\, c^2 \left| \iint_{D_2} (1-e^{-b\gamma})d\theta\, dr \right| \quad \text{with } M_2 > 0. \qquad (2.2.2)$$

If we now substitute $r = a\cos\theta + (a^2\sin^2\theta + r_2^2)^{\frac{1}{2}}$ we get finally

$$|n(b)| < M_2\, b^2\, c^2 \int_0^t r_2(1-e^{-b\gamma}) \int_0^{bg\,\sin\frac{r_2}{a}} (a^2\sin^2\theta + r_2^2)^{-\frac{1}{2}}\, d\theta <$$

$$< M_4\, b^3\, c^3 \qquad (2.2.3)$$

where $M_4$ is a positive constant.

3. The domain $D_3$.

Now we can expand the integrand and we find easily

$$p(b) = b^3\, c^3 \iint_{D_3} r\alpha\,.\,\beta\gamma \quad dr\, d\theta + \ldots \qquad (2.3.1)$$

According to (2.1.1), (2.2.3) and (2.3.1) it follows therefore:

$$b^{-5/2} \quad R(b) = b^{\frac{1}{2}}\, O(1) + \ldots \qquad (2.3.2)$$

4. Behaviour for large B:

We devide the integration domain D into two parts. In $D_1$ and $D_2$ the variables r and $\theta$ satisfy the following inequalities:

$D_1$: $0 \leqq r \leqq b^{\varepsilon/2}$ ; $0 \leqq \theta \leqq \frac{\pi}{2}$ where $\varepsilon$ is a small positive number.

$D_2$: $D - D_1$.

After writing $1-e^{-b\alpha} < 1$, it follows that the integral over $D_1$ is smaller than $\frac{\pi}{4}\, b^{\varepsilon}$. (2.4.1)

In the domain $D_2$ holds $r \sim r_1 \sim r_2$ for large positive b. Furthermore $\alpha$, $\beta$ and $\gamma$ are exponentially small. So we can expand the exponential powers and we find the integral over $D_2$ to be equal to:

$$32\, b^3\, c^3 \pi \int_{b^{\varepsilon/2}}^{\infty} r e^{-12r^{2/3}}\, dr = \frac{2}{3}\, b^3\, c^3 \pi\, (72 b^{\frac{2\varepsilon}{3}} + 12\, b^{\frac{\varepsilon}{3}} + 1) e^{-12 b^{\varepsilon/3}} \qquad (2.4.2)$$

Combining (2.4.1) and (2.4.2) we find:

$$b^{-5/2} R(b) \sim b^{-5/2}\, O(1)\ . \qquad (2.4.3)$$

5. <u>Summary</u>.

Combining the results of Chapter I and II we get in the first case after having substituted $b^* = 0{,}0245\, b$ into 1.42 and 1.64

$$b^{-5/2} J^*(b) = 0{,}1332\, 10^{-3}\, a^{-6}\, b^{1/6} + O(b^{\frac{1}{2}}). \qquad (2.5.1)$$

for small positive b.

and

$$b^{-5/2} J^*(b) \sim 0{,}0564\, b^{-11/6} + O(b^{-11/6-2\delta}\, e^{-b^{3\delta}}). \qquad (2.5.2)$$

for large positive b and with $\delta$ a small positive number.

For the second case we have according to (2.3.2) and (2.4.3)

$$b^{-5/2} R(b) = b^{\frac{1}{2}}.O(1) \qquad \text{for small positive b.} \qquad (2.5.3)$$

and

$$b^{-5/2} R(b) = b^{-5/2}\, O(1) \qquad \text{for large positive b.} \qquad (2.5.4).$$

## CHAPTER III

### 1. Method of computation.

In order to compute J(r) and K according to (0.0.3) and (0.0.4) we change the triple integral into a double integral.

We define the function:

$$F(\alpha, \beta, \gamma) = \int_0^\infty B^{-5/2} (1-e^{-B\alpha})(1-e^{-B\beta})(1-e^{-B\gamma})dB \qquad (3.1.0)$$

Now we determine the derivatives with respect to $\alpha$, $\beta$ and $\gamma$. It follows

$$\frac{\partial F}{\partial \alpha} = \int_0^\infty B^{-3/2} e^{-B\alpha} (1-e^{-B\beta})(1-e^{-B\gamma})dB \qquad (3.1.1)$$

$$\frac{\partial^2 F}{\partial\alpha\partial\beta} = \int_0^\infty B^{-\frac{1}{2}} e^{-B\alpha} e^{-B\beta}(1-e^{-B\gamma})dB \qquad (3.1.2)$$

and

$$\frac{\partial^3 F}{\partial\alpha\partial\beta\partial\gamma} = \int_0^\infty B^{+\frac{1}{2}} e^{-B(\alpha+\beta+\gamma)} dB = \tfrac{1}{2}\sqrt{\pi} \quad (\alpha+\beta+\gamma)^{-3/2} \qquad (3.1.3)$$

Furthermore holds $F(0, \beta, \gamma) = 0$ (3.1.4)

$$\frac{\partial F}{\partial \alpha}(\alpha, 0, \gamma) = 0 \qquad (3.1.5)$$

$$\frac{\partial^2 F}{\partial\alpha\partial\beta}(\alpha, \beta, 0) = 0 \ . \qquad (3.1.6)$$

If we integrate the equation (3.1.3) three times and use the conditions (3.1.4), (3.1.5) and (3.1.6) we get finally

$$F(\alpha,\beta,\gamma) = \frac{4}{3}\sqrt{\pi}\left[ -(\alpha+\beta+\gamma)^{3/2} + (\alpha+\beta)^{3/2} + (\alpha+\gamma)^{3/2} + (\beta+\gamma)^{3/2} - \alpha^{3/2} - \beta^{3/2} - \gamma^{3/2} \right] \qquad (3.1.7)$$

According to (3.1.0) we can write J(r) and K as a function of $F(\alpha,\beta,\gamma)$

$$J(r) = \int_0^{\pi/2} r\, F(\alpha,\beta,\gamma) d\theta \qquad (3.1.8)$$

$$K = \int_0^\infty J(r)dr \qquad (3.1.9)$$

The function $rF(\alpha,\beta,\gamma)$ was tabulated as a function of r and $\theta$. The function J(r) was obtained by numerical integration. In the same way the function $J^*(B)$ and the number K was defined for several a's.

## 2. Numerical values.

The functions J(r) and K were computed for the following values of a:

0.005; 0.01; 0.05; 0.1; 0.5; 1.

The function J (B) was computed for a = 0.1 and a = 1.

### 1st case

a=0.005

| r | $10^{-3}$.J(r) | r | $10^{-3}$.J(r) |
|---|---|---|---|
| 0.0000 | 0 | 0.0110 | 24 |
| 0.0005 | 129 | 0.0120 | 18 |
| 0.0010 | 238 | 0.0130 | 14 |
| 0.0015 | 319 | 0.0140 | 11 |
| 0.0020 | 365 | 0.0150 | 9 |
| 0.0025 | 374 | 0.0160 | 7 |
| 0.0030 | 354 | 0.0170 | 6 |
| 0.0035 | 314 | 0.0180 | 5 |
| 0.0040 | 269 | 0.0190 | 4 |
| 0.0045 | 225 | 0.0200 | 4 |
| 0.0050 | 186 | 0.0210 | 3 |
| 0.0055 | 153 | 0.0220 | 3 |
| 0.0060 | 126 | 0.0230 | 2 |
| 0.0065 | 104 | 0.0240 | 2 |
| 0.0070 | 86 | 0.0250 | 2 |
| 0.0075 | 72 | 0.0260 | 1 |
| 0.0080 | 60 | 0.0270 | 1 |
| 0.0085 | 51 | 0.0280 | 1 |
| 0.0090 | 43 | 0.0290 | 1 |
| 0.0095 | 37 | 0.0300 | 0 |
| 0.0100 | 31 | | |

K = 1895

### 1st case

a=0.01

| r | $10^{-1}$.J(r) | r | $10^{-1}$.J(r) |
|---|---|---|---|
| 0 | 0 | 0.0140 | 760 |
| 0.0005 | 585 | 0.0145 | 695 |
| 0.0010 | 1137 | 0.0150 | 632 |
| 0.0015 | 1648 | 0.0155 | 578 |
| 0.0020 | 2107 | 0.0160 | 529 |
| 0.0025 | 2497 | 0.0165 | 488 |
| 0.0030 | 2820 | 0.0170 | 447 |
| 0.0035 | 3062 | 0.0175 | 412 |
| 0.0040 | 3222 | 0.0180 | 379 |
| 0.0045 | 3303 | 0.0190 | 321 |
| 0.0050 | 3303 | 0.0200 | 278 |
| 0.0055 | 3247 | 0.0210 | 240 |
| 0.0060 | 3127 | 0.0220 | 209 |
| 0.0065 | 2971 | 0.0230 | 182 |
| 0.0070 | 2779 | 0.0240 | 159 |
| 0.0075 | 2582 | 0.0250 | 140 |
| 0.0080 | 2381 | 0.0300 | 77 |
| 0.0085 | 2178 | 0.0350 | 48 |
| 0.0090 | 1993 | 0.0400 | 30 |
| 0.0095 | 1816 | 0.0450 | 21 |
| 0.0100 | 1650 | 0.0500 | 15 |
| 0.0105 | 1491 | 0.0600 | 10 |
| 0.0110 | 1354 | 0.0700 | 7 |
| 0.0115 | 1226 | 0.0800 | 5 |
| 0.0120 | 1115 | 0.0900 | 4 |
| 0.0125 | 1006 | 0.1000 | 2 |
| 0.0130 | 916 | 0.1100 | 1 |
| 0.0135 | 834 | 0.1200 | 0 |

K = 338

1st case

a=0.05

| r | J(r) | r | J(r) |
|---|---|---|---|
| 0 | .0 | 0.15 | 2.8 |
| 0.01 | 75.3 | 0.20 | 1.1 |
| 0.02 | 115.3 | 0.25 | 0.5 |
| 0.03 | 111.8 | 0.30 | 0.3 |
| 0.04 | 85.1 | 0.35 | 0.1 |
| 0.05 | 59.2 | 0.40 | 0.1 |
| 0.06 | 39.8 | 0.45 | 0.1 |
| 0.07 | 27.2 | 0.50 | 0 |
| 0.08 | 19.0 | | |
| 0.09 | 13.6 | | |
| 0.10 | 9.9 | | |

K = 6.04

1st case

a=0.1

| r | $10^{-1}\cdot J(r)$ | r | $10^{-1}\cdot J(r)$ |
|---|---|---|---|
| 0 | 0 | 0.175 | 0.130 |
| 0.010 | 0.365 | 0.200 | 0.088 |
| 0.020 | 0.668 | 0.225 | 0.061 |
| 0.030 | 0.894 | 0.250 | 0.044 |
| 0.040 | 1.019 | 0.275 | 0.033 |
| 0.050 | 1.046 | 0.300 | 0.025 |
| 0.060 | 0.988 | 0.400 | 0.009 |
| 0.070 | 0.879 | 0.500 | 0.004 |
| 0.080 | 0.753 | 0.600 | 0.002 |
| 0.090 | 0.630 | 0.700 | 0.001 |
| 0.100 | 0.521 | 0.800 | 0.001 |
| 0.125 | 0.319 | 0.900 | 0 |
| 0.150 | 0.200 | | |

K = 1.06

1st case

a=0.5

| r | $10^{2}\cdot J(r)$ | r | $10^{2}\cdot J(r)$ |
|---|---|---|---|
| 0 | 0 | 1.00 | 0.315 |
| 0.05 | 1.287 | 1.10 | 0.285 |
| 0.10 | 2.383 | 1.20 | 0.180 |
| 0.15 | 3.189 | 1.30 | 0.139 |
| 0.20 | 3.645 | 1.40 | 0.110 |
| 0.25 | 3.744 | 1.50 | 0.088 |
| 0.30 | 3.537 | 1.60 | 0.071 |
| 0.35 | 3.146 | 1.70 | 0.059 |
| 0.40 | 2.693 | 1.80 | 0.048 |
| 0.45 | 2.254 | 1.90 | 0.040 |
| 0.50 | 1.864 | 2.00 | 0.034 |
| 0.60 | 1.258 | 2.50 | 0.015 |
| 0.70 | 0.859 | 3.00 | 0.004 |
| 0.80 | 0.600 | 3.50 | 0.002 |
| 0.90 | 0.430 | 4.00 | 0.001 |

K = 0.0190

1st case

a=1

| r | $10^{3}\cdot J(r)$ | r | $10^{3}\cdot J(r)$ |
|---|---|---|---|
| 0 | 0 | 1.4 | 0.759 |
| 0.1 | 1.139 | 1.5 | 0.631 |
| 0.2 | 2.119 | 1.6 | 0.530 |
| 0.3 | 2.820 | 1.7 | 0.455 |
| 0.4 | 3.225 | 1.8 | 0.381 |
| 0.5 | 3.307 | 1.9 | 0.328 |
| 0.6 | 3.126 | 2.0 | 0.282 |
| 0.7 | 2.780 | 2.5 | 0.142 |
| 0.8 | 2.376 | 3.0 | 0.078 |
| 0.9 | 1.996 | 3.5 | 0.045 |
| 1.0 | 1.648 | 4.0 | 0.032 |
| 1.1 | 1.346 | 5.0 | 0.019 |
| 1.2 | 1.096 | 6.0 | 0 |
| 1.3 | 0.915 | | |

K = 0.00336

1st case

a = 0.1

| B | $J^*(B)$ |
|---|---|
| 0 | 0 |
| 0.00005 | 25.24 |
| 0.00010 | 26.47 |
| 0.00025 | 28.17 |
| 0.00050 | 28.98 |
| 0.00100 | 29.41 |
| 0.00500 | 24.15 |
| 0.01000 | 18.10 |
| 0.02000 | 11.42 |
| 0.03000 | 7.955 |
| 0.04000 | 5.906 |
| 0.05000 | 4.597 |
| 0.06000 | 3.708 |
| 0.07000 | 3.070 |
| 0.08000 | 2.580 |
| 0.09000 | 2.226 |
| 0.10000 | 1.907 |
| 0.15000 | 1.066 |
| 0.20000 | 0.688 |
| 0.50000 | 0.157 |
| 1.00000 | 0.0480 |
| 5.00000 | 0.00278 |
| 10.00000 | 0.000797 |
| 20.00000 | 0.000227 |
| 50.00000 | 0.0000434 |

1st case

a = 1 .

| B | $J^*(B)$ |
|---|---|
| 0 | 0 |
| 0.025 | $0.678.10^{-4}$ |
| 0.050 | $0.748.10^{-4}$ |
| 0.075 | $0.785.10^{-4}$ |
| 0.100 | $0.811.10^{-4}$ |
| 0.200 | $0.886.10^{-4}$ |
| 0.300 | $0.928.10^{-4}$ |
| 0.400 | $0.936.10^{-4}$ |
| 0.500 | $0.950.10^{-4}$ |
| 1.000 | $0.973.10^{-4}$ |
| 3.000 | $0.879.10^{-4}$ |
| 5.000 | $0.770.10^{-4}$ |
| 10.000 | $0.576.10^{-4}$ |
| 20.000 | $0.3581.10^{-4}$ |
| 100.000 | $0.6052.10^{-5}$ |
| 1000.000 | $0.1516.10^{-6}$ |
| 10000.000 | $0.2524.10^{-8}$ |

## 2nd case

a=0.005

| r | $10^{-1}.J(r)$ | r | $10^{-1}.J(r)$ |
|---|---|---|---|
| 0.0000 | 0 | 0.0200 | 0.1653 |
| 0.0005 | 0.0846 | 0.0300 | 0.1253 |
| 0.0010 | 0.1460 | 0.0400 | 0.1013 |
| 0.0015 | 0.1953 | 0.0500 | 0.0852 |
| 0.0020 | 0.2340 | 0.0600 | 0.0737 |
| 0.0025 | 0.2664 | 0.0700 | 0.0648 |
| 0.0030 | 0.2905 | 0.0800 | 0.0[illegible]79 |
| 0.0035 | 0.3103 | 0.0900 | 0.0520 |
| 0.0040 | 0.3267 | 0.1000 | 0.0471 |
| 0.0045 | 0.3381 | 0.2000 | 0.0226 |
| 0.0050 | 0.3561 | 0.3000 | 0.0134 |
| 0.0055 | 0.3346 | 0.4000 | 0.0084 |
| 0.0060 | 0.3264 | 0.5000 | 0.0055 |
| 0.0065 | 0.3125 | 0.6000 | 0.0037 |
| 0.0070 | 0.3008 | 0.7000 | 0.0026 |
| 0.0075 | 0.2934 | 0.8000 | 0.0018 |
| 0.0080 | 0.2835 | 1.0000 | 0.0009 |
| 0.0085 | 0.2756 | 1.2000 | 0.0005 |
| 0.0090 | 0.2678 | 1.4000 | 0.0003 |
| 0.0095 | 0.2601 | 1.6000 | 0.0001 |
| 0.0100 | 0.2529 | 1.8000 | 0.0000 |

K = 0.197

## 2nd case

a=0.01

| r | J(r) | r | J(r) |
|---|---|---|---|
| 0.000 | 0 | 0.0140 | 1.942 |
| 0.001 | 0.559 | 0.0150 | 1.879 |
| 0.002 | 0.967 | 0.0160 | 1.811 |
| 0.003 | 1.287 | 0.0170 | 1.759 |
| 0.004 | 1.545 | 0.0180 | 1.704 |
| 0.005 | 1.753 | 0.0190 | 1.654 |
| 0.006 | 1.915 | 0.0200 | 1.592 |
| 0.007 | 2.040 | 0.0250 | 1.393 |
| 0.008 | 2.132 | 0.0300 | 1.233 |
| 0.0082 | 2.148 | 0.0350 | 1.105 |
| 0.0084 | 2.162 | 0.0400 | 1.004 |
| 0.0086 | 2.176 | 0.0450 | 0.919 |
| 0.0088 | 2.191 | 0.0500 | 0.852 |
| 0.0090 | 2.203 | 0.0600 | 0.738 |
| 0.0092 | 2.215 | 0.0700 | 0.645 |
| 0.0094 | 2.224 | 0.0800 | 0.577 |
| 0.0096 | 2.231 | 0.0900 | 0.518 |
| 0.0098 | 2.241 | 0.1000 | 0.471 |
| 0.0100 | 2.259 | 0.2000 | 0.226 |
| 0.0102 | 2.230 | 0.3000 | 0.134 |
| 0.0104 | 2.216 | 0.4000 | 0.084 |
| 0.0106 | 2.201 | 0.5000 | 0.055 |
| 0.0108 | 2.188 | 0.6000 | 0.037 |
| 0.0110 | 2.170 | 0.7000 | 0.026 |
| 0.0112 | 2.154 | 0.8000 | 0.018 |
| 0.0114 | 2.137 | 1.0000 | 0.009 |
| 0.0116 | 2.122 | 1.2000 | 0.005 |
| 0.0118 | 2.105 | 1.4000 | 0.003 |
| 0.0120 | 2.090 | 1.6000 | 0.001 |
| 0.0130 | 2.017 | 1.8000 | 0.000 |

K = 0.183

## 2nd case

a=0.05

| r | J(r) | r | J(r) |
|---|---|---|---|
| 0 | 0 | 0.150 | 0.305 |
| 0.005 | 0.186 | 0.155 | 0.297 |
| 0.010 | 0.323 | 0.160 | 0.285 |
| 0.015 | 0.429 | 0.165 | 0.276 |
| 0.020 | 0.511 | 0.170 | 0.264 |
| 0.025 | 0.5[illegible]5 | 0.175 | 0.260 |
| 0.030 | 0.623 | 0.180 | 0.251 |
| 0.035 | 0.657 | 0.185 | 0.244 |
| 0.040 | 0.679 | 0.190 | 0.239 |
| 0.045 | 0.692 | 0.195 | 0.226 |
| 0.050 | 0.716 | 0.200 | 0.223 |
| 0.055 | 0.667 | 0.250 | 0.170 |
| 0.060 | 0.634 | 0.300 | 0.130 |
| 0.065 | 0.604 | 0.350 | 0.104 |
| 0.070 | 0.576 | 0.400 | 0.081 |
| 0.075 | 0.554 | 0.450 | 0.066 |
| 0.080 | 0.528 | 0.500 | 0.051 |
| 0.085 | 0.507 | 0.550 | 0.042 |
| 0.090 | 0.480 | 0.600 | 0.035 |
| 0.095 | 0.463 | 0.700 | 0.026 |
| 0.100 | 0.443 | 0.800 | 0.018 |
| 0.105 | 0.426 | 0.900 | 0.013 |
| 0.110 | 0.410 | 1.000 | 0.009 |
| 0.115 | 0.391 | 1.100 | 0.007 |
| 0.120 | 0.376 | 1.200 | 0.005 |
| 0.125 | 0.363 | 1.300 | 0.003 |
| 0.130 | 0.347 | 1.400 | 0.002 |
| 0.135 | 0.338 | 1.500 | 0.001 |
| 0.140 | 0.326 | 1.600 | 0.000 |
| 0.145 | 0.316 | | |

K = 0.135

## 2nd case

a=0.1

| r | J(r) |
|---|---|
| 0 | 0 |
| 0.01 | 0.104 |
| 0.02 | 0.181 |
| 0.03 | 0.240 |
| 0.04 | 0.286 |
| 0.05 | 0.320 |
| 0.06 | 0.347 |
| 0.07 | 0.359 |
| 0.08 | 0.370 |
| 0.09 | 0.372 |
| 0.10 | 0.369 |
| 0.125 | 0.318 |
| 0.150 | 0.276 |
| 0.175 | 0.239 |
| 0.200 | 0.208 |
| 0.250 | 0.160 |
| 0.300 | 0.130 |
| 0.350 | 0.100 |
| 0.400 | 0.081 |
| 0.500 | 0.053 |
| 0.600 | 0.036 |
| 0.700 | 0.025 |
| 0.800 | 0.018 |
| 0.900 | 0.013 |
| 1.000 | 0.010 |
| 1.100 | 0.007 |
| 1.200 | 0.005 |
| 1.300 | 0.003 |
| 1.400 | 0.002 |
| 1.500 | 0.001 |
| 1.600 | 0.000 |

K = 0.103

**$2^{nd}$ case**

a=0.5

| r | $10^{+1}.J(r)$ |
|---|---|
| 0.00 | 0 |
| 0.05 | 0.1308 |
| 0.10 | 0.2305 |
| 0.15 | 0.3030 |
| 0.20 | 0.3498 |
| 0.25 | 0.3733 |
| 0.30 | 0.3773 |
| 0.35 | 0.3658 |
| 0.40 | 0.3439 |
| 0.45 | 0.3159 |
| 0.50 | 0.2876 |
| 0.60 | 0.2147 |
| 0.70 | 0.1596 |
| 0.80 | 0.1182 |
| 0.90 | 0.0874 |
| 1.00 | 0.0648 |
| 1.10 | 0.0481 |
| 1.20 | 0.0359 |
| 1.30 | 0.0267 |
| 1.40 | 0.0200 |
| 1.50 | 0.0151 |
| 1.60 | 0.0114 |
| 1.70 | 0.0086 |
| 1.80 | 0.0064 |
| 1.90 | 0.0050 |
| 2.00 | 0.0038 |
| 2.50 | 0.0010 |
| 3.00 | 0.0001 |
| 3.50 | 0.0000 |

K= 0.0245

**$2^{nd}$ case**

a=1

| r | $10^{+2}.J(r)$ |
|---|---|
| 0 | 0 |
| 0.1 | 0.228 |
| 0.2 | 0.406 |
| 0.3 | 0.528 |
| 0.4 | 0.588 |
| 0.5 | 0.595 |
| 0.6 | 0.562 |
| 0.7 | 0.496 |
| 0.8 | 0.423 |
| 0.9 | 0.349 |
| 1.0 | 0.285 |
| 1.1 | 0.222 |
| 1.2 | 0.174 |
| 1.3 | 0.136 |
| 1.4 | 0.106 |
| 1.5 | 0.082 |
| 1.6 | 0.063 |
| 1.7 | 0.050 |
| 1.8 | 0.039 |
| 1.9 | 0.031 |
| 2.0 | 0.024 |
| 2.5 | 0.007 |
| 3.0 | 0.002 |

K = 0.00549

**2nd case**

a = 0.1

| B | $J^*(B)$ |
|---|---|
| 0 | 0 |
| 0.005 | 0.0388 |
| 0.010 | 0.0493 |
| 0.030 | 0.0712 |
| 0.050 | 0.0791 |
| 0.100 | 0.0828 |
| 0.150 | 0.0809 |
| 0.200 | 0.0749 |
| 0.250 | 0.0691 |
| 0.350 | 0.0583 |
| 0.450 | 0.0496 |
| 0.650 | 0.0370 |
| 0.850 | 0.0287 |
| 1.050 | 0.0231 |
| 1.550 | 0.0147 |
| 2.050 | 0.0106 |
| 2.550 | 0.00800 |
| 3.550 | 0.00518 |
| 4.550 | 0.00369 |
| 5.550 | 0.00255 |
| 10.000 | 0.000996 |
| 20.000 | 0.000309 |
| 40.000 | 0.0000886 |

**2nd case**

a = 1

| B | $J^*(B)$ |
|---|---|
| 0 | 0 |
| 0.025 | $0.554.10^{-4}$ |
| 0.050 | $0.756.10^{-4}$ |
| 0.075 | $0.899.10^{-4}$ |
| 0.100 | $0.1013.10^{-3}$ |
| 0.200 | $0.1314.10^{-3}$ |
| 0.300 | $0.1505.10^{-3}$ |
| 0.400 | $0.1639.10^{-3}$ |
| 0.600 | $0.1815.10^{-3}$ |
| 0.800 | $0.1929.10^{-3}$ |
| 1.000 | $0.1998.10^{-3}$ |
| 1.600 | $0.2080.10^{-3}$ |
| 2.200 | $0.2076.10^{-3}$ |
| 2.800 | $0.2039.10^{-3}$ |
| 3.400 | $0.1973.10^{-3}$ |
| 4.000 | $0.1903.10^{-3}$ |
| 5.000 | $0.1778.10^{-3}$ |
| 6.000 | $0.1657.10^{-3}$ |
| 10.000 | $0.1256.10^{-3}$ |
| 20.000 | $0,0705.10^{-3}$ |