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On the expansion of a function in a series of sperical harmonics.

by

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On the expansion of a function in a series of spherical harmonics

0. In geophysics the problem arises how to find some analytic expression for a function, which is experimentally determined over the surface of the earth. Assuming that the shape of the earth may be approximated by a sphere, this can be achieved by expanding the function in a series of spherical harmonics. Such an expansion has been given by Prey for the topography of the earth [1].

A reconsideration of Prey's work seems necessary for the following reasons.

1. Since 1921 much information has been gathered concerning the depth of the oceans.
2. In Prey's development the highest order of the spherical harmonics is 16 and this is to low for some applications.
3. The smoothing of the data which is necessary for a rapidly fluctuating function, such as the topography of the earth, has been carried out by Prey in an unsatisfactory way.

This question of improving on Prey's work has been raised by Prof. F.A. Vening Meinesz in connection with his theory of the origin of the continents based on convection currents in the earth. Indeed, the investigations which lead to the present report, have been carried out on his request.

In this report the formulae are given which are needed for the computation of the development coefficients in an expansion of an arbitrary function in spherical harmonics. The improvements over Prey's work are

1. A new method for the calculation of the zeros of a Legendre polynomial which, moreover, can be used for the calculation of the zeros of any function, satisfying a second order homogeneous differential equation.
2. A simple method for the determination of the weights w_i occurring in the formulae for the development coefficients A_n^m and B_n^m has been used.
3. An improved method has been developed for smoothing data obtained from a rapidly fluctuating function.

Although the investigations have been carried out with a definite purpose, i.e. to find an improved expansion of the earths topography, the methods developed in this report are general and can be used for the expansion of other functions and for arbitrary highest order of the spherical harmonics.

The results of the application of the formulae, given in this report, to the special case of the earths topography with a highest order of the spherical harmonics equal to $n = 36$, will constitute the second part of this report.

1. Assuming a function $F(\varphi,\vartheta)$ on a sphere sufficiently well-behaved, it can be expanded in the series

$$F(\varphi,\vartheta) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \varepsilon_m (A_n^m \cos m\varphi + B_n^m \sin m\varphi)\, P_n^m (\cos\vartheta) \qquad 1.1$$

where the coefficients follow from

$$A_n^m = \frac{2n+1}{2} \frac{(n-m)!}{(n+m)!} \int_0^{\pi} C_m(\vartheta) \sin\vartheta \, P_n^m(\cos\vartheta) d\vartheta \qquad 1.2$$

$$B_n^m = \frac{2n+1}{2} \frac{(n-m)!}{(n+m)!} \int_0^{\pi} S_m(\vartheta) \sin\vartheta \, P_n^m(\cos\vartheta) d\vartheta \qquad 1.3$$

$$C_m(\vartheta) = \frac{1}{\pi} \int_0^{2\pi} F(\varphi,\vartheta) \cos m\varphi \, d\varphi \qquad 1.4$$

$$S_m(\vartheta) = \frac{1}{\pi} \int_0^{2\pi} F(\varphi,\vartheta) \sin m\varphi \, d\varphi , \qquad 1.5$$

and $\varepsilon_0 = \frac{1}{2}$, $\varepsilon_m = 1$ $(m \neq 0)$.

2. If in the expansion (1.1) only the terms up to $n = p$ are retained, the A_n^m and B_n^m can be found by summations instead of integrations following a method developed by F. Neumann For the derivation of his formulae we refer to the literature [2][3] .
Let the expansion be

$$F(\varphi,\vartheta) = \sum_{n=0}^{p} \sum_{m=0}^{n} \varepsilon_m (A_n^m \cos m\varphi + B_n^m \sin m\varphi) P_n^m (\cos\vartheta) \qquad 2.1$$

$$= \sum_{m=0}^{p} \varepsilon_m \left[C_m(\vartheta) \cos m\varphi + S_m(\vartheta) \sin m\varphi \right] \qquad 2.2$$

where

$$C_m(\vartheta) = \sum_{j=m}^{p} A_j^m P_j^m(\cos\vartheta) \qquad 2.3$$

$$S_m(\vartheta) = \sum_{j=m}^{p} B_j^m P_j^m(\cos\vartheta), \qquad 2.4$$

and

$\varepsilon_0 = \frac{1}{2}$, $\varepsilon_p = \frac{1}{2}$, $\varepsilon_m = 1$ $(m \neq 0,\ m \neq p)$.

Then according to Neumann one has

$$A_n^m = \frac{2n+1}{2}\frac{(n-m)!}{(n+m)!}\sum_{i=0}^{p} W_i\, P_n^m(\cos\vartheta_i)C_m(\vartheta_i) \qquad 2.5$$

$$B_n^m = \frac{2n+1}{2}\frac{(n-m)!}{(n+m)!}\sum_{i=0}^{p} W_i\, P_n^m(\cos\vartheta_i)S_m(\vartheta_i) \qquad 2.6$$

$$C_m(\vartheta) = \frac{1}{p}\sum_{k=0}^{2p-1} F(\vartheta, k\varphi_0)\cos m k\varphi_0 \qquad 2.7$$

$$S_m(\vartheta) = \frac{1}{p}\sum_{k=0}^{2p-1} F(\vartheta, k\varphi_0)\sin m k\varphi_0 \qquad 2.8$$

where $\varphi_0 = \pi/p$, 2.9

The weights W_i follow from the set of equations

$$\sum_{i=1}^{p+1} W_i\,\mu_i^{\,s} = \int_{-1}^{+1} x^s dx, \qquad s = 0, \ldots .\ 2p+1, \qquad 2.10$$

where $\mu_i = \cos\vartheta_i$
and it is proved, that the μ_i are the zeros of $P_{p+1}(x)$.

3. Consider the formulae for Legendre-Gauss quadrature, [4].
One has

$$\int_{-1}^{+1} f(x)dx = \sum_{i=1}^{n} W_i\, f(x_i) \qquad 3.1$$

$$W_i = \frac{2}{nP_{n-1}(x_i)P_n'(x_i)} \qquad 3.2$$

where the x_i are the zeros of $P_n(x)$.
Furthermore

$$(1-x^2)P_n'(x) = n\left[P_{n-1}(x) - xP_n(x)\right]$$

or

$$(1-x_i^2)P_n'(x_i) = n\,P_{n-1}(x_i).$$

Hence

$$W_i = \frac{2(1-x_i^2)}{n^2\left[P_{n-1}(x_i)\right]^2}\ . \qquad 3.3$$

In particular, if $f(x) = x^s$, $n = p+1$, we have

$$\int_{-1}^{+1} x^s dx = \sum_{i=1}^{p+1} W_i x_i^s$$

which corresponds to (2.10).

One finds that the weights W_i are given by

$$W_i = \frac{2(1-x_i^2)}{(p+1)^2\left[P_p(x_i)\right]^2} = 2\left[\frac{\sin\vartheta_i}{(p+1)P_p(\cos\vartheta_i)}\right]^2. \qquad 3.4$$

4. The calculation of the zeros of $P_n(\cos\vartheta)$ can be effected in the following way.

Legendre's differential equation reads

$$\frac{d}{dx}\left[(1 - x^2)\frac{dP_n}{dx}\right] + n(n+1)P_n = 0 \qquad 4.1$$

Substitute $x = \sin\psi$, $\psi = \frac{\pi}{2} - \vartheta$, then

$$\frac{d^2P_n}{d\psi^2} - \operatorname{tg}\psi \frac{dP_n}{d\psi} + n(n+1)P_n = 0.$$

Now put $P_n(\sin\psi) = u(\psi)\cos^{-\frac{1}{2}}\psi$. The zeros of $P_n(\sin\psi)$ then coincide with the zeros of $u(\psi)$ One finds

$$\frac{d^2u_n}{d\psi^2} + \left[(n+\tfrac{1}{2})^2 + \frac{1}{4\cos^2\psi}\right] u_n = 0. \qquad 4.2$$

For large values of n an approximate solution can be found by solving

$$\frac{d^2u_n}{d\psi^2} + (n+\tfrac{1}{2})^2 u_n = 0 \qquad 4.3$$

or $u_n \approx A\cos(n+\frac{1}{2})\psi + B\sin(n+\frac{1}{2})\psi$. $\qquad$ 4.4

From

$$P_{2m}(x) = (-1)^m \frac{(2m)!}{4^m m!^2} F(-m, m+\tfrac{1}{2}; \tfrac{1}{2}; x^2)$$

$$P_{2m+1}(x) = (-1)^m \frac{(2m+1)!}{4^m m!^2} x\, F(-m, m+\tfrac{3}{2}; \tfrac{3}{2}; x^2)$$

$$u_n(\ ) = \cos^{\frac{1}{2}}\ P_n(\sin\psi)$$

one sees that $u_n(-\psi) = u_n(\psi)$ n even

$u_n(-\psi) = -u_n(\psi)$ n odd

or,

$u_n \approx \cos(n+\frac{1}{2})\psi$, n odd, which vanishes for

$(n+\frac{1}{2})\psi \approx (k+\frac{1}{2})\pi$, $\psi \approx \frac{2k+1}{2n+1}\pi$

$u_n \approx \sin(n+\frac{1}{2})\psi$, n even, which vanishes for

$(n+\frac{1}{2})\psi \approx k\pi$, $\psi \approx \frac{2k}{2n+1}\pi$.

As a first approximation for the zeros of $P_n(\cos\vartheta)$ one can use therefore

$$\left.\begin{aligned} \vartheta_k &= \left(\tfrac{1}{2} - \frac{2k+1}{2n+1}\right)\pi \quad \text{n odd} \\ \vartheta_k &= \left(\tfrac{1}{2} - \frac{2k}{2n+1}\right)\pi \quad \text{n even.} \end{aligned}\right\} \qquad 4.5$$

An improved value may be found in the following way. Let

$$y = f(x) \qquad 4.6$$

be a given function and let $x = \alpha$ be one of its roots. Then if x is approximately equal to that root,

$$\alpha = x - \frac{y}{y'} - \frac{1}{2}\frac{y''}{y'}\left(\frac{y}{y'}\right)^2 - \frac{1}{6}\left[3\left(\frac{y''}{y'}\right)^2 - \frac{y'''}{y'}\right]\left(\frac{y}{y'}\right)^3 + \ldots \qquad 4.7$$

according to Schröder [5].

Moreover, if f(x) satisfies a differential equation

$$y'' = 2Py' + Qy, \qquad 4.8$$

one finds the third order approximation

$$\alpha \approx x - \frac{y}{y'} - P\left(\frac{y}{y'}\right)^2 - \frac{1}{3}\left(4P^2 - P' + Q\right)\left(\frac{y}{y'}\right)^3. \qquad 4.9$$

In the case of Legendre's differential equation

$$\frac{d^2P_n}{d\varphi^2} + \cot\varphi \frac{dP_n}{d\varphi} + n(n+1)P_n = 0$$

one has $P = -\frac{1}{2}\cot\varphi$, $Q = -n(n+1)$. Hence

$$\alpha \approx \varphi - \frac{P_n}{P_n'} + \frac{1}{2}\cot\varphi\left(\frac{P_n}{P_n'}\right)^2 - \frac{1}{6}\left[\cot^2\varphi - 1 - 2n(n+1)\right]\left(\frac{P_n}{P_n'}\right)^3 \qquad 4.10$$

A related method first transforms the differential equation. Put

$$y = v \exp. \int P dx$$

then

$$v'' = (P^2 - P' + Q)v$$

Furthermore

$$\frac{v'}{v} = \frac{y'}{y} - P$$

And the approximation becomes

$$\alpha \approx \varphi - \frac{v}{v'} - \frac{1}{3}R\left(\frac{v}{v'}\right)^3 \qquad 4.11$$

where $R = P^2 - P' + Q$ and $\frac{v'}{v} = \frac{y'}{y} - P$. 4.12

In the case of Legendre's differential equation one has

$$R = \frac{1}{4}\cot^2\varphi - \frac{1}{2\sin^2\varphi} - n(n+1) = -\left(n+\frac{1}{2}\right)^2 - \frac{1}{4\sin^2\varphi}.$$

Hence

$$\left.\begin{aligned} \alpha &= \varphi - \beta + \tfrac{1}{3}\left[(n+\tfrac{1}{2})^2 + \frac{1}{4\sin^2\varphi}\right]\beta^3 \\ 1/\beta &= \frac{P_n'}{P_n} - \tfrac{1}{2}\cot\varphi = \left[(n-\tfrac{1}{2})\cos\varphi - n\frac{P_n}{P_{n-1}}\right]/\sin\varphi \end{aligned}\right\} \quad 4.13$$

5. If the function, which has to be expanded, changes rapidly over the sections in which the sphere is divided, it becomes difficult to find a representative value for these sections and some smoothing process must be applied. We propose the following method: First the function is expanded and from the expansioncoefficients we derive the coefficients of the expansion of the original function.

Let $f(\varphi, \vartheta)$ be the original function and let

$$F(\varphi,\vartheta) = \frac{1}{4\varepsilon\delta}\int_{\varphi-\delta}^{\varphi+\delta}\int_{\vartheta-\varepsilon}^{\vartheta+\varepsilon} f(\varphi', \vartheta')d\vartheta' d\varphi' \qquad 5.1$$

be the integral function. Then symbolically

$$F(\varphi,\vartheta) = \frac{\sinh \delta D\varphi}{\delta D\varphi}\,\frac{\sinh \varepsilon D\vartheta}{\varepsilon D\vartheta}\, f(\varphi,\vartheta) \qquad 5.2$$

$$f(\varphi,\vartheta) = \frac{\delta D\varphi}{\sinh \delta D\varphi}\,\frac{\varepsilon D\vartheta}{\sinh \varepsilon D\vartheta}\, F(\varphi,\vartheta) \qquad 5.3$$

Now expand the integral function in spherical harmonics

$$F(\varphi,\vartheta) = \sum_{m=0}^{p}\varepsilon_m\left[C_m(\vartheta)\cos m\varphi + S_m(\vartheta)\sin m\varphi\right] \qquad 2.2$$

where

$$C_m(\vartheta) = \sum_{j=m}^{p} A_j^m P_j^m(\cos\vartheta) \qquad 2.3$$

$$S_m(\vartheta) = \sum_{j=m}^{p} B_j^m P_j^m(\cos\vartheta) \qquad 2.4$$

By aid of 5.3. an expansion of $f(\varphi, \vartheta)$ follows, which however is not an expansion in terms of $\cos m\varphi$, $\sin m\varphi$ and $P_n^m(\vartheta)$ but in terms of $\cos m\varphi$, $\sin m\varphi$ and functions defined by

$$p_n^m(\cos\vartheta) = \frac{\varepsilon D}{\sinh \varepsilon D} P_n^m(\cos\vartheta)\,. \qquad 5.4$$

Consider

$$\frac{\sinh \delta D}{\delta D} \cos m\varphi = \frac{1}{2\delta} \int_{\varphi-\delta}^{\varphi+\delta} \cos m\varphi' d\varphi' = \frac{\sin \delta m}{\delta m} \cos m\varphi$$

or,

$$\frac{\delta D}{\sinh \delta D} \cos m\varphi = \frac{\delta m}{\sin \delta m} \cos m\varphi \qquad 5.5$$

and in the same way

$$\frac{\delta D}{\sinh \delta D} \sin m\varphi = \frac{\delta m}{\sin \delta m} \sin m\varphi. \qquad 5.6$$

Substitution of 2.2 in 5.3 yields, by aid of 5.4, 5.5, 5.6,

$$f(\varphi, \vartheta) = \sum_{m=0}^{p} \varepsilon_m \frac{\delta m}{\sin \delta m} (\bar{c}_m \cos m\varphi + \bar{s}_m \sin m\varphi) \qquad 5.7$$

$$\bar{c}_m(\vartheta) = \sum_{j=m}^{p} A_j^m P_j^m(\cos\vartheta) \qquad 5.8$$

$$s_m(\vartheta) = \sum_{j=m}^{p} B_j^m P_j^m(\cos\vartheta). \qquad 5.9$$

Let

$$f(\varphi, \vartheta) = \sum_{m=0}^{p} \varepsilon_m \left[c_m(\vartheta) \cos m\varphi + s_m(\vartheta) \sin m\varphi\right] \qquad 5.10$$

$$c_m(\vartheta) = \sum_{j=m}^{p} a_j^m P_j^m(\cos\vartheta) \qquad 5.11$$

$$s_m(\vartheta) = \sum_{j=m}^{p} b_j^m P_j^m (\cos\vartheta) \qquad 5.12$$

be the wanted expansion. For this expansion the coefficients a_n^m and b_n^m follow from

$$a_n^m = \frac{2n+1}{2} \frac{(n-m)!}{(n+m)!} \int_0^{\pi} P_n^m(\cos\vartheta) c_m(\vartheta) \sin\vartheta \, d\vartheta \qquad 5.13$$

$$b_n^m = \frac{2n+1}{2} \frac{(n-m)!}{(n+m)!} \int_0^{\pi} P_n^m(\cos\vartheta) s_m(\vartheta) \sin\vartheta \, d\vartheta \qquad 5.14$$

$$c_m(\vartheta) = \frac{1}{\pi} \int_0^{2} f(\varphi, \vartheta) \cos m\varphi \, d\varphi \qquad 5.15$$

$$s_m(\vartheta) = \frac{1}{\pi} \int_0^{2} f(\varphi, \vartheta) \sin m\varphi \, d\varphi \qquad 5.16$$

or, alternatively

$$a_n^m = \frac{2n+1}{2} \frac{(n-m)!}{(n+m)!} \sum_{i=0}^{p} W_i P_n^m(\vartheta_i) c_m(\vartheta_i) \qquad 5.13'$$

$$b_n^m = \frac{2n+1}{2} \frac{(n-m)!}{(n+m)!} \sum_{i=0}^{p} W_i P_n^m(\vartheta_i) s_m(\vartheta_i) \qquad 5.14'$$

$$c_m(\vartheta) = \frac{1}{p} \sum_{k=0}^{2p-1} f(k\varphi_0, \vartheta) \cos mk\varphi_0 \qquad 5.15'$$

$$s_m(\vartheta) = \frac{1}{p} \sum_{k=0}^{2p-1} f(k\varphi_0, \vartheta) \sin mk\varphi_0 \ . \qquad 5.16'$$

Substitution of the auxiliary expansion for $f(\varphi, \vartheta)$, given by 5.7, 5.8, 5.9 in the set 5.13, 5.14, 5.15, 5.16 or 5.13', 5.14', 5.15 , 5.16 yields.

$$c_m(\vartheta) = \frac{1}{\pi}\int_0^{2\pi} \left[\sum_{m'=0}^{p} \varepsilon_{m'} \frac{\delta m'}{\sin \delta m'} (\bar{c}_{m'} \cos m'\varphi + \bar{s}_{m'} \sin m'\varphi) \right] \cos m\varphi d\varphi$$

$$= \varepsilon_m \frac{\delta m}{\sin \delta m} \bar{c}_m = \varepsilon_m \frac{\delta m}{\sin \delta m} \sum_{j=m}^{p} A_j^m P_j^m(\cos \vartheta) \qquad 5.17$$

$$s_m(\vartheta) = \frac{1}{\pi}\int_0^{2\pi} \left[\sum_{m'=0}^{p} \varepsilon_{m'} \frac{\delta m'}{\sin \delta m'} (\bar{c}_{m'} \cos m'\varphi + \bar{s}_{m'} \sin m'\varphi) \right] \sin m\varphi d\varphi$$

$$= \varepsilon_m \frac{\delta m}{\sin \delta m} \bar{s}_m = \varepsilon_m \frac{\delta m}{\sin \delta m} \sum_{j=m}^{p} B_j^m P_j^m(\cos \vartheta) \qquad 5.18$$

$$a_n^m = \frac{2n+1}{2} \frac{(n-m)!}{(n+m)!} \int_0^{\pi} P_n^m(\cos\vartheta) \, \varepsilon_m \frac{\delta m}{\sin \delta m} \sum_{j=m}^{p} A_j^m P_j^m(\cos\vartheta) \sin\vartheta \, d\vartheta$$

$$= \frac{2n+1}{2} \frac{(n-m)!}{(n+m)!} \varepsilon_m \frac{\delta m}{\sin \delta m} \sum_{j=m}^{p} A_j^m \int_0^{\pi} P_n^m(\cos\vartheta) P_j^m(\cos\vartheta) \sin\vartheta \, d\vartheta$$

$$b_n^m = \frac{2n+1}{2} \frac{(n-m)!}{(n+m)!} \int_0^{\pi} P_n^m(\cos\vartheta) \, \varepsilon_m \frac{\delta m}{\sin \delta m} \sum_{j=m}^{p} B_j^m P_j^m(\cos\vartheta) \sin\vartheta \, d\vartheta$$

$$= \frac{2n+1}{2} \frac{(n-m)!}{(n+m)!} \varepsilon_m \frac{\delta m}{\sin \delta m} \sum_{j=m}^{p} {}^m_j \int_0^{\pi} P_n^m(\cos\vartheta) P_j^m(\cos\vartheta) \sin\vartheta \, d\vartheta$$

or alternatively,

$$a_n^m = \frac{2n+1}{2} \frac{(n-m)!}{(n+m)!} \varepsilon_m \frac{\delta m}{\sin \delta m} \sum_{j=m}^{p} A_j^m \sum_{i=0}^{p} W_i P_n^m(\cos\vartheta_i) P_j^m(\cos\vartheta_i)$$

$$b_n^m = \frac{2n+1}{2} \frac{(n-m)!}{(n+m)!} \varepsilon_m \frac{\delta m}{\sin \delta m} \sum_{j=m}^{p} B_j^m \sum_{i=0}^{p} W_i P_n^m(\cos\vartheta_i) P_j^m(\cos\vartheta_i)$$

which formulae also can be written as

$$a_n^m = \frac{2n+1}{2} \frac{(n-m)!}{(n+m)!} \varepsilon_m \frac{\delta m}{\sin\delta m} \sum_{j=m}^{p} A_j^m D_{n,j}^m \qquad 5.19$$

$$b_n^m = \frac{2n+1}{2} \frac{(n-m)!}{(n+m)!} \varepsilon_m \frac{\delta m}{\sin\delta m} \sum_{j=m}^{p} B_j^m D_{n,j}^m \qquad 5.20$$

where

$$D_{n,j}^m = \int_0^\pi P_n^m (\cos\vartheta)\ P_j^m(\cos\vartheta)\ \sin\vartheta\, d\vartheta \qquad 5.21$$

$$= \sum_{i=0}^{p} W_i P_n^m (\cos\vartheta_i)\ P_j^m (\cos\vartheta_i) \qquad 5.22$$

8. For the actual computation of the P_n^m we derive a useful formula.

From

$$(1-x^2) \frac{d}{dx} P_n^m = \sqrt{1-x^2}\ P_n^{m+1} - mx\ P_n^m$$

and

$$\frac{2mx}{\sqrt{1-x^2}} P_n^m = P_n^{m+1} + (n+m)(n-m+1)\ P_n^{m-1}$$

follows

$$2\sqrt{1-x^2} \frac{d}{dx} P_n^m = P_n^{m+1} - (n+m)(n-m+1)\ P_n^{m-1} .$$

Substitution of $x = \cos\vartheta$ yields

$$P_n^{m+1} = (n+m)(n-m+1)\ P_n^{m-1} - 2\frac{d}{d\vartheta} P_n^m . \qquad 8.1$$

If $m = 0$,

$$P_n^1 = n(n+1)\ P_n^{-1} - 2\frac{d}{d\vartheta} P_n \qquad 8.2$$

However,

$$P_n^{-m} = (-)^m \frac{(n-m)!}{(n+m)!} P_n^m$$

and for $m = 1$

$$P_n^{-1} = \frac{1}{n(n+1)} P_n^1 .$$

Hence 8.2 becomes

$$P_n^1 = -\frac{d}{d\vartheta} P_n \quad . \qquad 8.3$$

Because $P_n^m = 0$ $(m>n)$, 8.1 becomes for $m = n$

$$P_n^{n-1} = \frac{1}{n} \frac{d}{d\vartheta} P_n^n . \qquad 8.4$$

If the P_n^m are given in a goniometric series they assume the form

$$P_n^m = \sum_{k=0}^{n} a_{n,k}^m \cos(n - 2k)\vartheta, \quad m \text{ even} \qquad 8.5$$

$$P_n^m = \sum_{k=0}^{n} b_{n,k}^m \sin(n - 2k)\vartheta, \quad m \text{ odd}. \qquad 8.6$$

Substitution in 8.1 yields a relation between the coefficients $a_{n,k}^m$ and $b_{n,k}^m$

$$b_{n,k}^{m+1} = (n + m)(n - m + 1)\, b_{n,k}^{m-1} - 2(n - 2k)a_{n,k}^m, \; m \text{ even}, \qquad 8.7$$

$$a_{n,k}^{m+1} = (n + m)(n - m + 1)\, a_{n,k}^{m-1} + 2(n - 2k)b_{n,k}^m, \; m \text{ odd}. \qquad 8.8$$

The operator $\dfrac{\varepsilon D}{\sinh \varepsilon D}$ can be formally developed

$$\frac{\varepsilon D}{\sinh \varepsilon D} = \sum_{k=0}^{\infty} \alpha_k \varepsilon^{2k} \frac{d^{2k}}{d\vartheta^{2k}}$$

and it is evident, that the operations $\dfrac{d}{d\vartheta}$ and $\dfrac{\varepsilon D}{\sinh \varepsilon D}$ are interchangeable.

Application of the definition

$$p_n^m(\cos\vartheta) = \frac{\varepsilon D}{\sinh \varepsilon D} P_n^m(\cos\vartheta),$$

to 8.1, 8.3, 8.4 then yields

$$p_n^{m+1} = (n + m)(n - m + 1)\, p_n^{m-1} - 2 \frac{d}{d\vartheta} p_n^m, \qquad 8.9$$

$$p_n^1 = - \frac{d}{d\vartheta} p_n , \qquad 8.10$$

$$p_n^{n-1} = \frac{1}{n} \frac{d}{d\vartheta} p_n^n . \qquad 8.11$$

The goniometric series for the p_n^m become, by aid of 5.6,

$$p_n^m = \sum_{k=0}^{n} \alpha_{n,k}^m \cos(n - 2k)\vartheta$$

$$= \sum_{k=0}^{n} \frac{(n - 2k)\varepsilon}{\sin(n - 2k)\varepsilon} a_{n,k}^m \cos(n - 2k)\vartheta, \quad m \text{ even} \qquad 8.12$$

$$p_n^m = \sum_{k=0}^{n} \beta_{n,k}^m \sin(n - 2k)\vartheta$$

$$= \sum_{k=0}^{n} \frac{(n - 2k)\varepsilon}{\sin(n - 2k)\varepsilon} b_{n,k}^m \sin(n - 2k)\vartheta, \quad m \text{ odd} \qquad 8.13$$

Hence multiplication of 8.7, 8.8 by $\dfrac{(n-2k)\varepsilon}{\sin(n - 2k)\varepsilon}$ yields

$$\beta_{n,k}^{m+1} = (n + m)(n - m + 1)\beta_{n,k}^{m-1} - 2(n - 2k)\alpha_{n,k}^m, \quad m \text{ even} \qquad 8.14$$

$$\alpha_{n,k}^{m+1} = (n + m)(n - m + 1)\alpha_{n,k}^{m-1} + 2(n - 2k)\beta_{n,k}^m, \quad m \text{ odd} \qquad 8.15$$

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