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# ON THE MOTION OF THE BOUNDARY BETWEEN THE NORMAL AND THE SUPERCONDUCTIVE STATE IN A CYLINDRICAL WIRE CARRYING A STRONG CURRENT

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**Synopsis**

The inward motion of the boundary between the superconductive core and the normally conducting sheeth in a cylindrical wire through which a large current is passed has been calculated by computing the solution of Maxwell's equations in the normally conductive region. If superconductivity is reestablished immediately when the local field drops below the critical value, F. London's result for the resistance is found. If supercooling retards the reestablishment of superconductivity, a periodic motion of the boundary may result accompanied by a slight increase of the average resistance. Orders of magnitude of periods and upper limits of resistance variations are estimated.

1. In a recent paper [1]) the behaviour of the intermediate superconductive state when it carries an electric current was analysed. An alternative model was suggested to that of F. London who supposed that the boundaries between the normal and the superconductive regions would place themselves perpendicular to the current [2]). One of the cases considered in the paper mentioned was a cylindrical wire carrying a current larger than Silsbee's critical value, which means that the magnetic field at the surface of the wire exceeds the critical field at the temperature of the experiment.

It was supposed that the current could be kept constant in spite of a varying voltage drop along the wire and it was argued that superconductivity would be disturbed near the surface of the wire, the cylindrical boundary between the superconductive and the normal region moving inward. Furthermore making the plausible assumption that the tangential field at the moving boundary retains the critical value, and that the curl of the magnetic field due to Maxwell's displacement current can be neglected, it was easy to write down Maxwell's equations in the normal shell as well as the boundary conditions at its inner and outer circumference.

No rigorous solution of the resulting mathematical problem was given though it was made plausible that, in accordance with a remark of Pippard, who had considered this matter earlier [3]) the magnetic field would fall in some regions below the threshold value. In those regions superconductivity could be restored though a certain supercooling should be expected. According to the model proposed for the structure of the intermediate state core, this would lead to creation of hedged-in normally conducting rings which contract themselves in the direction of the axis of the wire, eventually disappearing there. According to F. London's model, however, the intermediate state core would have a static structure and be composed of flat superconductive double cones with their tops touching each other and of normal matter in between.

In the present paper we shall not at all be concerned with the structure of the intermediate state core. We shall present and discuss the result of numerical computations carried out in the "Mathematisch Centrum", Amsterdam, about the fields in the normal sheeth and the resulting motion of its internal boundary. The inward motion of the boundary has been computed. Supposing that superconductivity will partially but instantaneously be restored in all regions where the magnetic field has fallen below the threshold field, an evaluation will be given of the average voltage drop along the wire.

Also an estimate has been made in two cases of the oscillatory components of the voltage drop. Since there is no reason to suppose that in all parts of a long wire the oscillations are in phase this estimate is an upper limit.

2. The field in the normal shell is ruled by Maxwell's equations for the quasistationary case which read here

$$r^{-1}\,\partial\,(rH)/\partial r = 4\pi\,c^{-1}\,j, \tag{1}$$

$$\partial E/\partial r = c^{-1}\,\partial H/\partial t. \tag{2}$$

Because of the cylindrical symmetry $E$ and $H$ are functions of the radius and of time only, $E$ being parallel to the axis, $H$ perpendicular to the axis as well as to the radius. By Ohm's law one has further

$$j = \sigma E. \tag{3}$$

The boundary conditions are $H = q_1 H_c$ for $r = a$, $H = H_c$ for $r = r_b(t)$ where $a$ = radius of the wire, $r_b(t)$ = radius of the moving boundary between the normal and superconducting phases, $H_c$ = critical magnetic field. The motion of this boundary is determined by the induction law

$$E(r_b(t),\,t) = -\,c^{-1}\,H_c\,\mathrm{d}r_b/\mathrm{d}t. \tag{4}$$

Equations (1)–(4) determine the electric and magnetic fields if the initial values $r_b(0)$ and $H(r, 0)$, $r_b(0) \leqslant r \leqslant a$, are known. The voltage drop over a distance $z$ along the wire is $V(t) = E(a, t)z$. Instead of this instantaneous value one will measure a mean value over a certain time

$$V = E(a)z = z(t_2 - t_1)^{-1} \int_{t_1}^{t_2} E(a, t)\, \mathrm{d}t. \tag{5}$$

Since the resistance of a wire entirely in the normal phase is given by $R_n = z/\pi a^2 \sigma$ and the total current in the wire by $I = cq_1 a H_c/2$ one finds the ratio

$$R/R_n = 2\pi a \sigma E(a)/cq_1 H_c. \tag{6}$$

3. For the actual computation it is convenient to introduce reduced variables by $\rho = r/a$, $\tau = c^2 t/4\pi\sigma a^2$, $q = H/H_c$ and $e = 4\pi\sigma a E/cH_c$. Then, by eliminating $E$ and $j$, one obtains from (1)–(3) a parabolic differential equation for $q$

$$\frac{\partial q}{\partial \tau} = \frac{\partial}{\partial \rho}\left(\frac{\partial q}{\partial \rho} + \frac{q}{\rho}\right) \tag{7}$$

with boundary conditions $q = q_1$ for $\rho = 1$, $q = 1$ for $\rho = \rho_b(\tau)$. The motion of the boundary follows, according to (4), from the ordinary differential equation

$$\mathrm{d}\rho_b/\mathrm{d}\tau = -\,(\partial q/\partial \rho + q/\rho)_{\rho=\rho_b(\tau)}. \tag{8}$$

The resistance ratio turns out to be

$$\frac{R}{R_n} = \frac{e(1)}{2q_1} = \frac{1}{2q_1(\tau_2 - \tau_1)} \int_{\tau_1}^{\tau_2} \left(\frac{\partial q}{\partial \rho} + \frac{q}{\rho}\right)_{\rho=1} \mathrm{d}t. \tag{9}$$

Putting $\rho = 1$ in the relation

$$\int_{\tau_1}^{\tau_2} \left(\frac{\partial q}{\partial \rho} + \frac{q}{\rho}\right) \mathrm{d}\tau = \int_{\rho_b(\tau_2)}^{\rho} q(\rho', \tau_2)\, \mathrm{d}\rho' - \int_{\rho_b(\tau_1)}^{\rho} q(\rho', \tau_1)\, \mathrm{d}\rho', \tag{10}$$

which is easily proved, one can transform (9) so that numerical differentiation is avoided.

The equations (7) and (8) were integrated by means of finite difference approximation. To this end the range $0 \leqslant \rho \leqslant 1$ was divided into $N$ equal intervals with length $h = N^{-1}$ whereas the intervals in the $\tau$-direction were chosen so that the successive values of $\rho_b(\tau)$ are successive multiples of $h$. Defining $\rho_i$ by $\rho_i = 1 - ih$, $\tau_j$ by $\rho_j = \rho_b(\tau_j)$ and $q_{ij}$ by $q_{ij} = q(\rho_i, \tau_j)$ one can replace (7) by the difference equation

$$\left(1 + \frac{h}{2\rho_i}\right) q_{i-1,j} - \left(2 + \frac{h^2}{\rho_i^2} + \frac{h^2}{\tau_j - \tau_{j-1}}\right) q_{i,j} + \left(1 - \frac{h}{2\rho_i}\right) q_{i+1,j} = -\frac{h^2}{\tau_j - \tau_{j-1}}\, q_{i,j-1}. \tag{11}$$

The values of $q_{i,j-1}$, $i = 0(1)j - 1$, being known for a certain $j$, $\tau_j$ can be calculated according to (8) from

$$\tau_j = \tau_{j-1} + h\{(\partial q/\partial \rho)_{\rho_{j-1},\tau_{j-1}} + 1/\rho_{j-1}\}^{-1} \tag{12}$$

with a suitable difference approximation for the derivative $\partial q/\partial \rho$.

Taking into account that $q_{0,j} = q_1$ and $q_{j,j} = 1$ one can determine $q_{i,j}$, $i = 0(1)j$ with (11) and the procedure may be repeated for the next value of $j$.

A more detailed description of the numerical methods is given in a separate report [4]).

4. The interesting case concerns a field varying periodically with time. Then the time average in (5) and (9) is taken over a period. To introduce periodicity in this dynamic model one can use the phenomenon of "undercooling" of the normal phase. The assumption is made that the normal phase persists when the magnetic field falls below the critical value $H_c$ and that an "intermediate state" mixture is set up wherever $H < H_c$ if the field has fallen to a value $q_r H_c$ anywhere in the normal region ($q_r$ being a parameter lower than 1).

The outer boundary of this region may be supposed to be superconductive and its subsequent motion is then determined again by (4) and (8).

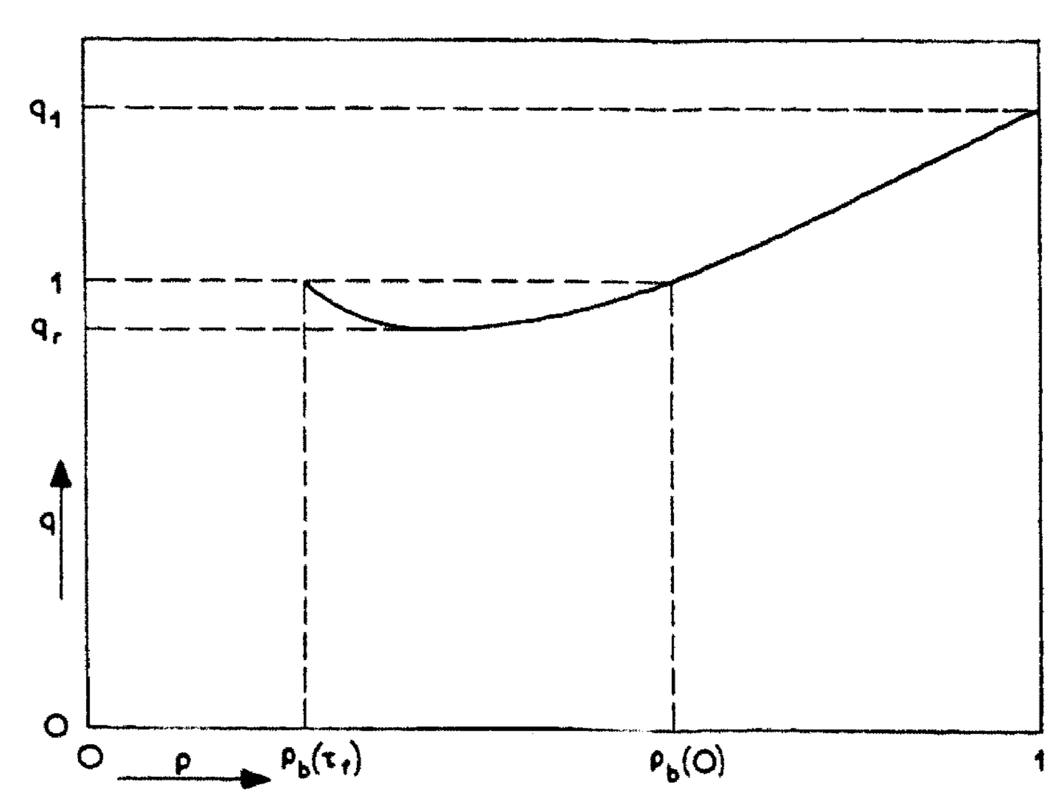


Fig. 1. Initial and final field in periodical case.

Mathematically formulated, such initial conditions at $\tau = 0$ are sought that, after a certain time $\tau_f$, one finds $q(\rho, \tau_f) = q(\rho, 0)$ in the range $\rho_b(0) \leqslant \leqslant \rho \leqslant 1$, while in the range $\rho_b(\tau_f) \leqslant \rho \leqslant \rho_b(0)$ a minimum value $q_r$ occurs (see fig. 1).

From (9) follows

$$R/R_n = (1/2q_1\tau_f) \int_{\rho_b(0)}^{\rho_b(\tau_f)} q(\rho', \tau_f)\, d\rho'. \tag{13}$$

In the special case of $q_r \to 1$ this expression approaches to $(2q_1\rho^*)^{-1}$ where $\rho^*$ is the limiting value of both $\rho_b(0)$ and $\rho_b(\tau_f)$. From (7) one can then deduce

$$\rho^* = q_1 - \sqrt{q_1^2 - 1} \tag{14}$$

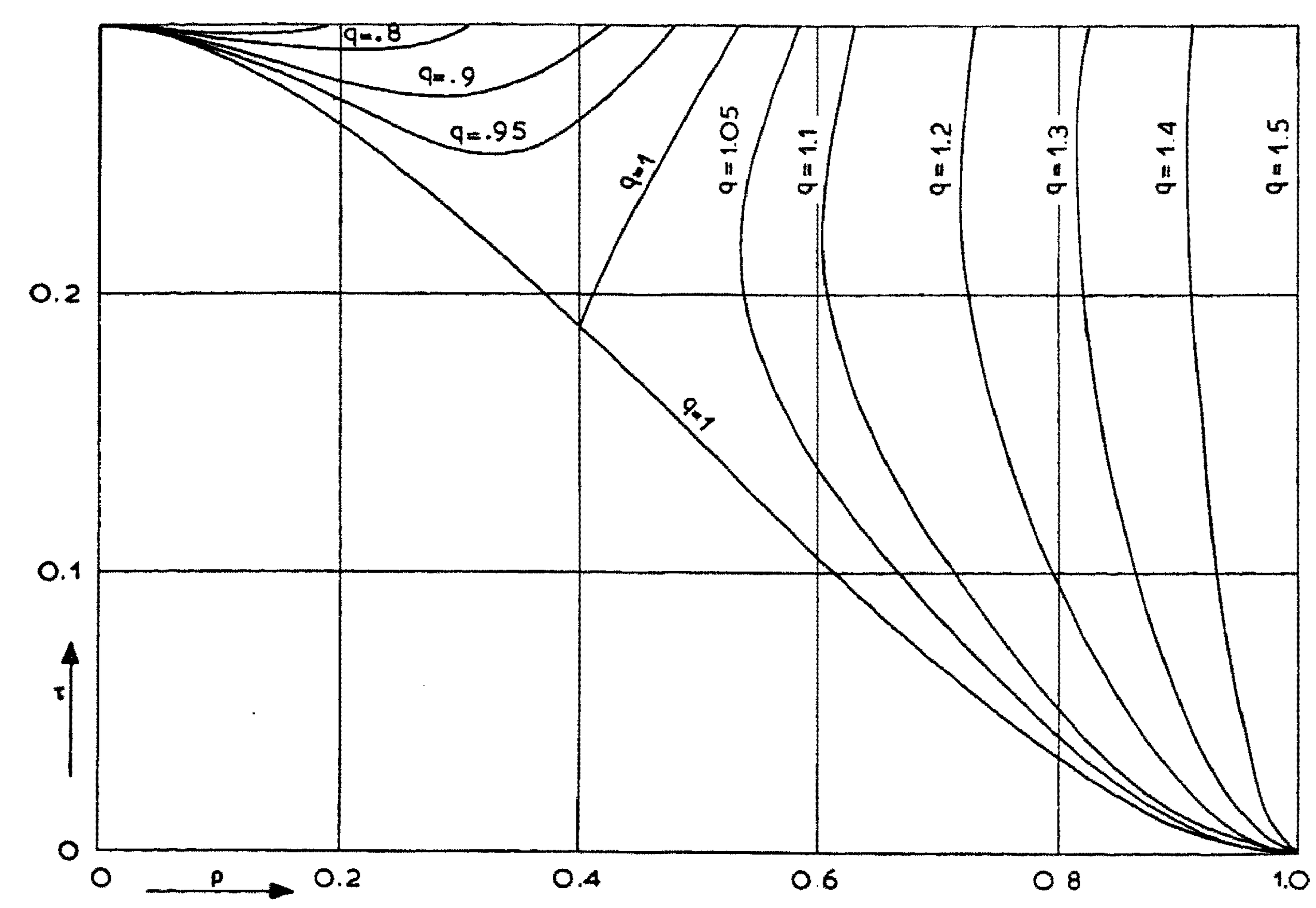


Fig. 2*a*. Solution for $q_1 = 1.1$.

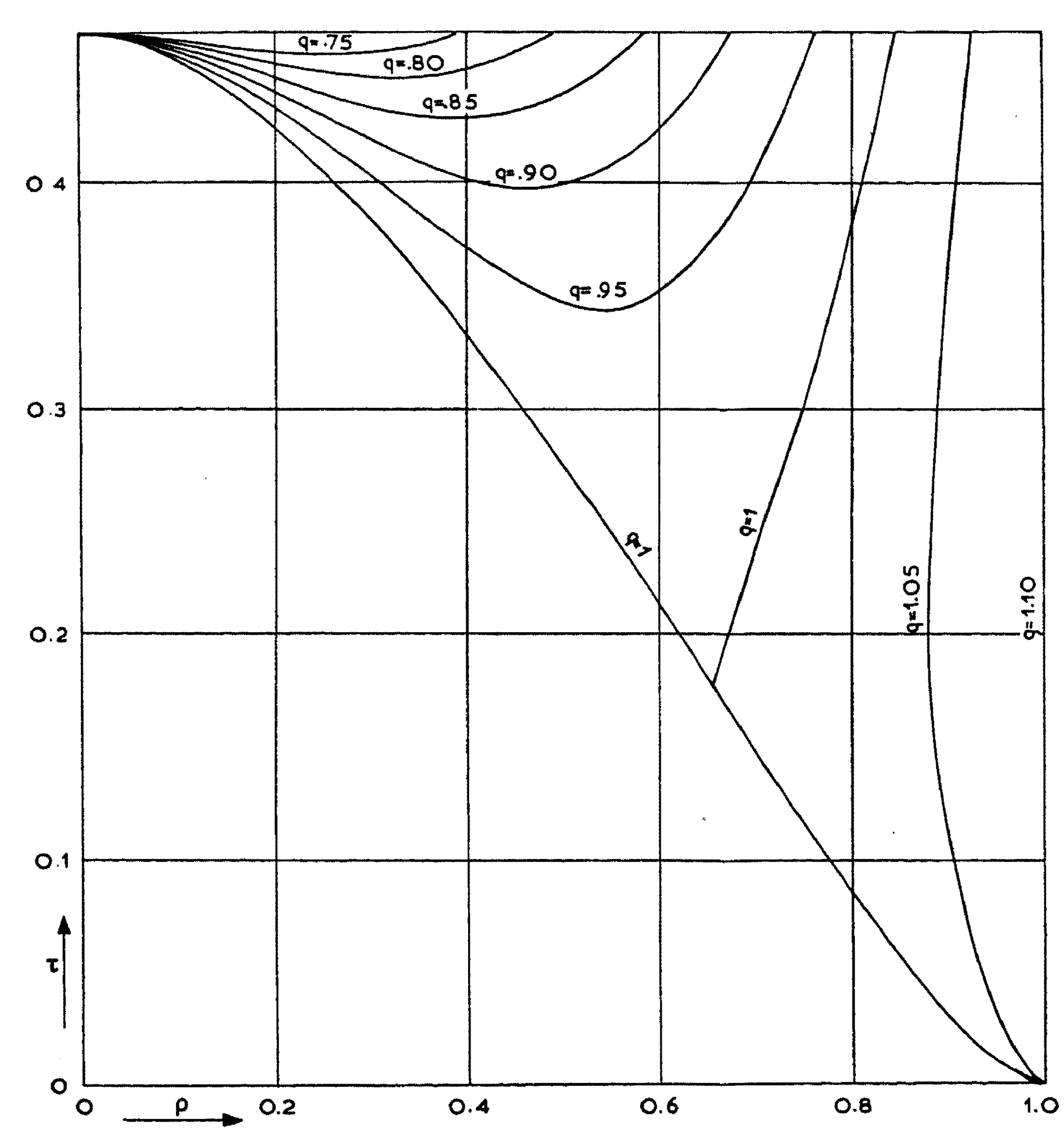


Fig. 2*b*. Solution for $q_1 = 1.5$.

and thus

$$R/R_n = \tfrac{1}{2}(1 + \sqrt{1 - q_1^{-2}}) \tag{15}$$

which are exactly the values obtained by London [2]) with his stationary model!

5. The calculations were carried out on the electronic computer ARMAC of the "Mathematisch Centrum". The integration step was chosen $h = 0.02$. To get an impression of the character of the solution, the equations (7)–(8)

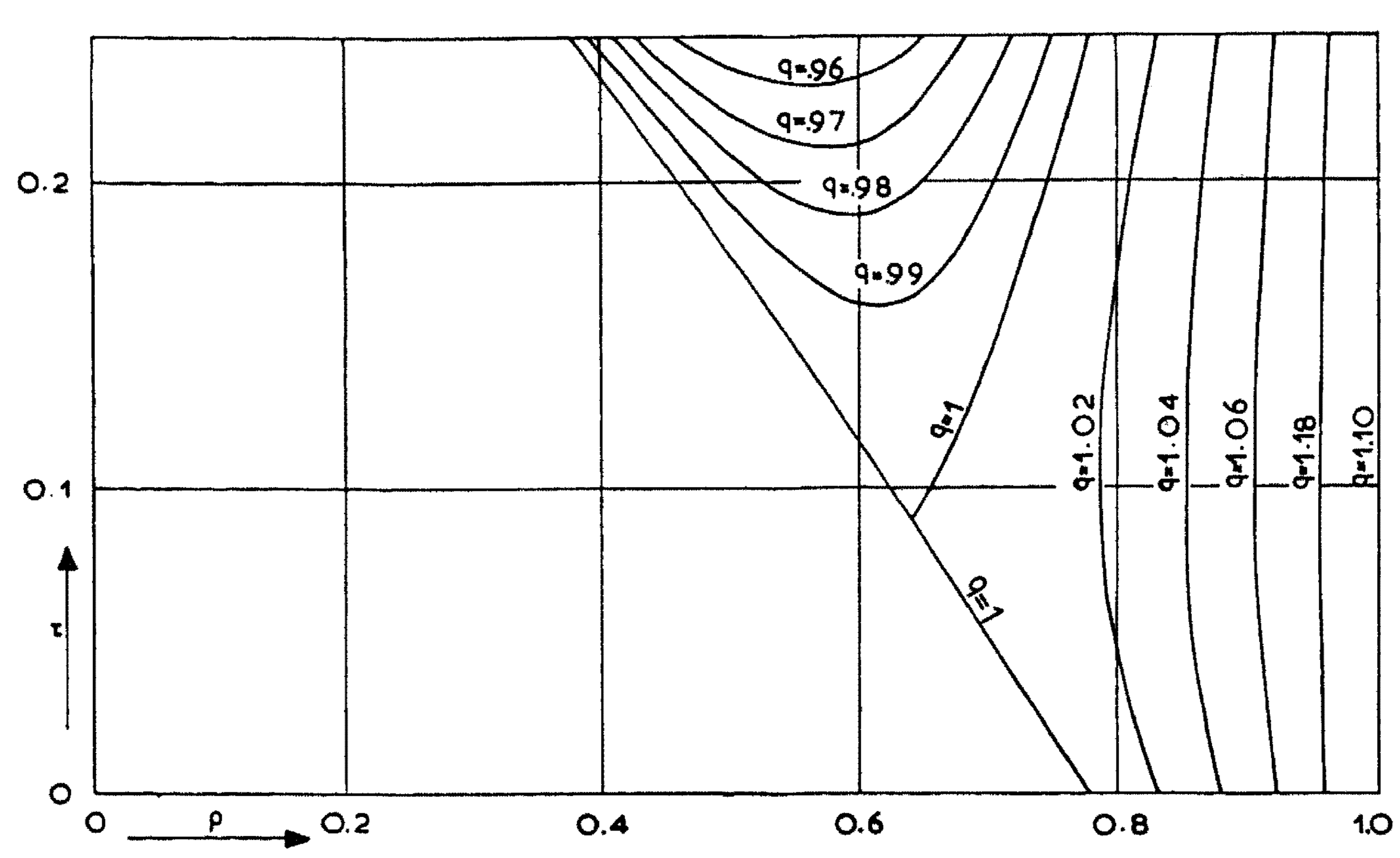


Fig. 3*a*. Periodical solution for $q_1 = 1.1$; $q_r = 0.95$

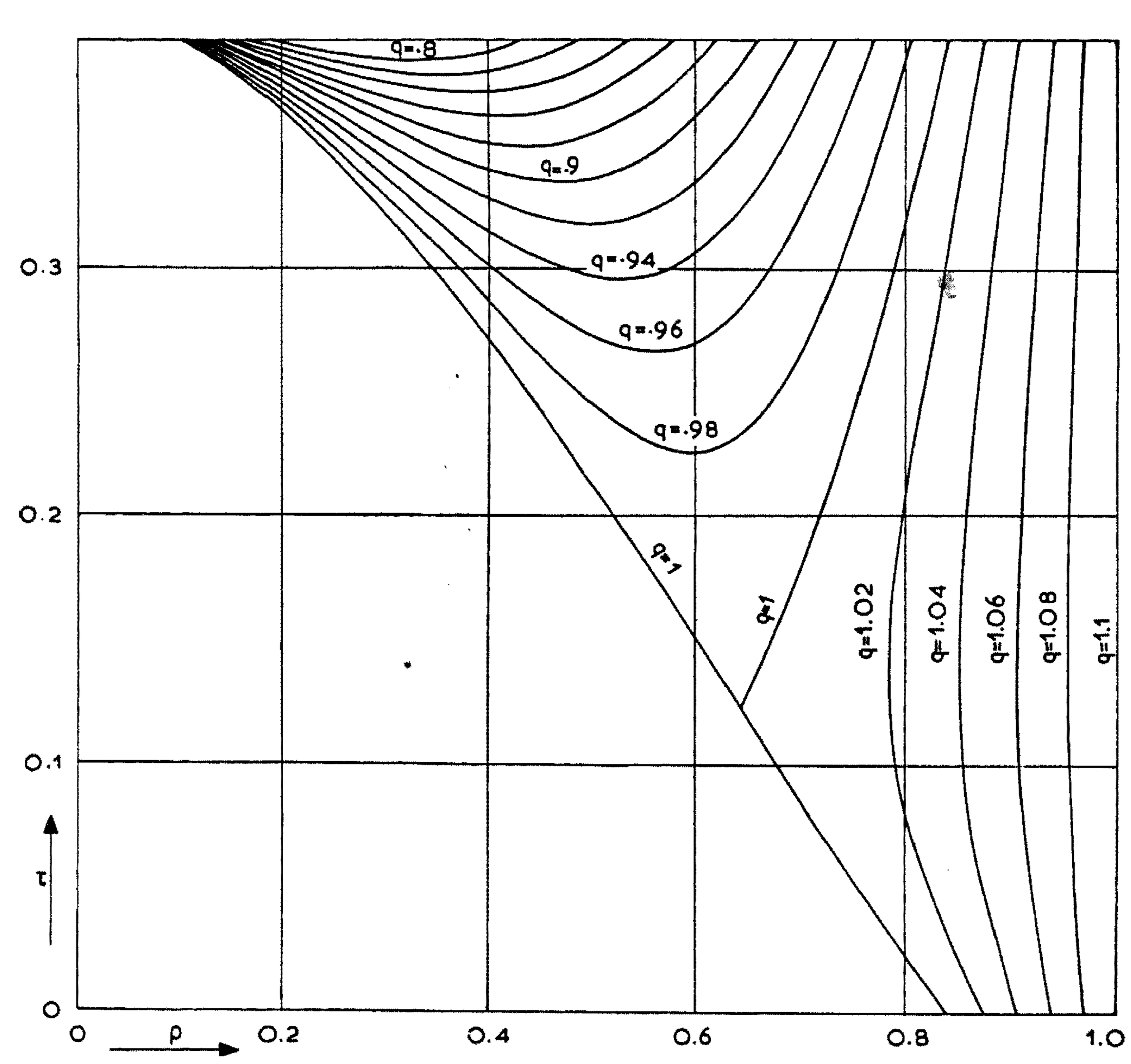


Fig. 3*b*. Periodical solution for $q_1 = 1.1$; $q_r = 0.77$.

were integrated first with initial condition $\rho_b(0) = 1$ for several values of $q_1$. In fig. 2 some lines of $q =$ const. in the $\rho\tau$-plane are plotted for $q_1 = 1.1$ and $q = 1.5$.

These solutions were also used as an estimation for the initial field $q(\rho, 0)$ as described in section 4. Then by one or two steps of an iterative procedure these functions were determined. In fig. 3 the course of the field $q(\rho, \tau)$ within a period is drawn for $q_1 = 1.1$ and two values of $q_r$. In table I the numerical results are collected together with the values calculated from (14)–(15).

From the last column it is clear that the supercooling expressed by $q_r < 1$ leads to an increase in $R$ over the London value of only a few per cent at currents slightly above the critical current. Experimentally the excess is much larger [5] [6] and so this supercooling can at most account for a fraction of the excess.

TABLE I

| Periodical Solutions | | | | | |
|---|---|---|---|---|---|
| $q_1$ | $\tau_f$ | $\varrho_b(0)$ | $\varrho_b(\tau_f)$ | $q_r$ | $R/R_n$ |
| 1.1 | .3917 | .8400 | .1033 | .7683 | .7288 |
| | .3415 | .8200 | .2263 | .8743 | .7249 |
| | .2935 | .8000 | .3090 | .9209 | .7205 |
| | .2481 | .7800 | .3749 | .9488 | .7168 |
| | | $\varrho^* = .6417$ | | 1.0000 | .7083 |
| 1.2 | .2811 | .7200 | .1119 | .7974 | .7842 |
| | .2467 | .7000 | .1957 | .8769 | .7831 |
| | .2126 | .6800 | .2590 | .9196 | .7814 |
| | .1790 | .6600 | .3129 | .9474 | .7797 |
| | | $\varrho^* = .5367$ | | 1.0000 | .7764 |
| 1.3 | .2261 | .6400 | .0701 | .7556 | .8204 |
| | .1972 | .6200 | .1575 | .8636 | .8216 |
| | .1683 | .6000 | .2191 | .9139 | .8211 |
| | .1394 | .5800 | .2714 | .9457 | .8204 |
| | | $\varrho^* = .4693$ | | 1.0000 | .8195 |
| 1.4 | .1634 | .5600 | .1298 | .8505 | .8487 |
| | .1376 | .5400 | .1922 | .9102 | .8493 |
| | .1119 | .5200 | .2439 | .9460 | .8492 |
| | | $\varrho^* = .4204$ | | 1.0000 | .8499 |
| 1.5 | .1471 | .5200 | .0788 | .7940 | .8680 |
| | .1238 | .5000 | .1511 | .8875 | .8700 |
| | .1004 | .4800 | .2058 | .9347 | .8706 |
| | | $\varrho^* = .3820$ | | 1.0000 | .8727 |

6. In order to obtain an impression about the amplitude of the periodic voltage variations a few computations have been made about

$$e(1, \tau) = e(1)\left[1 + \sum_k a_k \cos\left(2\pi \frac{k\tau}{\tau_f} + \varphi_k\right)\right].$$

It is found that for $q_l = 1.1$; $q_r = 0.9488$:

$$a_1 = 0.018, \ a_2 = 0.005, \ a_3 = 0.002,$$

and for $q_l = 1.1$; $q_r = 0.7683$:

$$a_1 = 0.048, \ a_2 = 0.016, \ a_3 = 0.008.$$

It must be remarked, however, that these amplitudes are an upper limit since it might quite well be that the periodic motions are out of phase with each other in different points of a long wire.

7. In order to obtain an order of magnitude of the phenomena concerned we may suppose a rather pure tin wire of a specific resistance of $10^{-8}$ ohm cm and a diameter $2a = 0.02$ cm. Then the characteristic time $t/\tau = 4\pi\sigma a^2/c^2$ becomes $1.3 \times 10^{-4}$ s; the velocity of the boundary is of the order of 200 cm/s and the frequency of the periodicities $10^4 - 10^5$ Hz. If the critical field is 100 Oe the resistance per cm is 30 $\mu\Omega$ and the upper limit of the amplitudes of the voltage variations, in the example given above, $5\mu$V/cm which is quite large indeed.

8. Finally it may be worth while to mention that in the treatment given above the influence of the heat of transformation and of the surface energy between the normal and the superconductive phase [3]) have been left out of consideration. There are indications that the latter might be an important factor [6]).

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