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SEQUENTIAL TEST WITH THREE POSSIBLE DECISIONS FOR THE COMPARISON OF TWO UNKNOWN PROBABILITIES

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We consider two series of independent trials, e.g. two processes, each trial resulting in a success or a failure with probabilities p, $1-p$ and p', $1-p'$ respectively for the two processes.

Executing the trials in pairs consisting of one trial for each process, a sequential test with two possible decisions, developed by Wald [1] (p. 106), may be used for the comparison of p and p'.

For groups of trials of both processes a sequential test with two possible decisions has been described in [2]. This test is carried out as follows:

Suppose the group of trials constituting the i^{th} step of the test consists of n_i trials for the first process and m_i trials for the second one. If the number of successes for the two processes are $\boldsymbol{a}_i$ and $\boldsymbol{b}_i$ respectively, then

$$\boldsymbol{x}_i = 2\left\{\text{arc sin}\sqrt{\frac{\boldsymbol{a}_i}{n_i}} - \text{arc sin}\sqrt{\frac{\boldsymbol{b}_i}{m_i}}\right\}$$

is for large n_i and m_i approximately normally distributed with mean

$$(1)\quad \mu = \mathscr{E}\boldsymbol{x}_i = 2\{\text{arc sin}\sqrt{p} - \text{arc sin}\sqrt{p'}\} = 2\,\text{arc sin}\{\sqrt{pq'} - \sqrt{p'q}\} \qquad \begin{matrix} q' = 1-p' \\ q = 1-p \end{matrix}$$

and variance (2) $\sigma_i^2 = \sigma^2(\boldsymbol{x}_i) = (n_i+1)n_i^{-2} + (m_i+1)m_i^{-2}$ (see e.g. [3]).

The ordinary sequential test with two possible decisions for the mean of a normal distribution with known variance, given by Wald [1] (p. 117), may then be applied to the $\boldsymbol{x}_i$.

Both abovementioned tests for comparing two unknown probabilities can be generalized to tests with three possible decisions. For the first one this generalization may be based directly on a method described by de Boer [4].

The test for groups of trials can be generalized by applying the test with three possible decisions for the mean μ of a normal distribution with known variance, developed by Sobel and Wald [5] to the abovementioned random variables $\boldsymbol{x}_i$.

For this test two values ξ and η and four values μ_1, μ_2, μ_3 and μ_4 of μ must be chosen, with (3) $\mu_1 < \xi < \mu_2 < \mu_3 < \eta < \mu_4$
the three possible decisions being

$$(4)\qquad 1.\ \ \mu < \xi, \qquad 2.\ \ \mu > \eta, \qquad 3.\ \ \xi \leqq \mu \leqq \eta,$$

the intervals (μ_1, μ_2) and (μ_3, μ_4) being indifference regions.

To translate this into terms of p and p', let

$$\sqrt{pq'} - \sqrt{p'q} = \delta \tag{5}$$

then (see (1)) we have $\mu = 2 \arcsin \delta$.

The functional relation between p and p' for a given value of δ^2 is given by those two parts of the ellipse

$$p^2 + p'^2 - 2pp'(1 - 2\delta^2) - 2\delta^2(p + p') + \delta^4 = 0 \tag{6}$$

for which $\delta^2 \leqq p + p' \leqq 2 - \delta^2$; this set of points (p, p') consists of a part $\mathscr{C}$ for which $p > p'$ and a part $\mathscr{C}'$ for which $p < p'$.

Choosing two values ε and ζ and four values δ_1, δ_2, δ_3 and δ_4 of δ, with

$$\delta_1 < \varepsilon < \delta_2 < 0 < \delta_3 < \zeta < \delta_4, \tag{7}$$

(δ_1, δ_2) and (δ_3, δ_4) constituting the indifference regions, the three decisions (4) are equivalent with:

$$1.\quad \delta_1 < \varepsilon, \qquad 2.\quad \delta_2 > \zeta, \qquad 3.\quad \varepsilon \leqq \delta \leqq \zeta \tag{8}$$

and hence with the following decisions for p and p':

(9) 1. the point (p, p') lies outside the part $\mathscr{C}'$ of the ellipse (6) corresponding to $\delta = \varepsilon$,
2. the point (p, p') lies outside the part $\mathscr{C}$ corresponding to $\delta = \zeta$,
3. the point (p, p') lies between or on the parts $\mathscr{C}'$ and $\mathscr{C}$ corresponding to $\delta = \varepsilon$ and $\delta = \zeta$.

The probabilities of errors are determined as described in [5].

REMARK. It is not necessary that p and p' are constant throughout the experiment. We only need a constant δ.

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