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A GENERALIZATION OF THE METHOD OF m RANKINGS

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The method of m rankings due to M. Friedman [5] is a test for the hypothesis that m rankings of equal length are independent and that for each of these all permutations of the ranks have equal probabilities. If the ranks are arranged in the rows of a rectangular scheme, the test statistic used is the variance of the column-totals. This method is a non-parametric analogon of the test for the columneffect in the two-way analysis of variance with one observation per cell. The test has been generalized by J. Durbin [4] to incomplete block designs. The further generalization proposed here is applicable to schemes with an arbitrary number of observations in each cell.

We denote the number of rankings by m, the number of columns by n, the number of observations in cell (μ, ν) (μ^{th} ranking, ν^{th} column) by $k_{\mu\nu}$ and $\sum_{\nu=1}^{n} k_{\mu\nu}$ by k_μ; then the arithmetic mean of the ranks in the μ^{th} ranking is $\frac{1}{2}(k_\mu+1)$. Subtracting this from these ranks, we get the reduced ranks. The sum of the reduced ranks in cell (μ, ν) is denoted by $\tilde{\boldsymbol{u}}_{\mu\nu}$ and the column-total $\sum_{\mu=1}^{m} \tilde{\boldsymbol{u}}_{\mu\nu}$ by $\tilde{\boldsymbol{u}}_\nu$.

If there are no ties we have under the assumption that the hypothesis tested (which is the same as the hypothesis of Friedman's original test) is true:

$$\sigma_{\nu\nu'} = \mathscr{E}\boldsymbol{u}_\nu \boldsymbol{u}_{\nu'} = \tfrac{1}{12} \sum_{\mu=1}^{m} k_{\mu\nu}(\delta_\nu^{\nu'} k_\mu - k_{\mu\nu'})\,(k_\mu + 1)$$

where
$$\delta_\nu^{\nu'} = \begin{cases} 0 \text{ if } \nu \neq \nu' \\ 1 \text{ if } \nu = \nu'. \end{cases}$$

The statistic of the test is a homogeneous quadratic function of $n-1$ columntotals $\tilde{\boldsymbol{u}}_\nu$; its matrix is the inverse of the covariance matrix of the same $n-1$ column-totals. Except for degenerate cases, this statistic can be written in the form:

$$\chi_r^2 = - \begin{vmatrix} \sigma_{11} & \cdots & \sigma_{1,n-1} \\ \vdots & & \vdots \\ \sigma_{n-1,1} & \cdots & \sigma_{n-1,n-1} \end{vmatrix}^{-1} \cdot \begin{vmatrix} \sigma_{11} & \cdots & \sigma_{1,n-1} & \tilde{\boldsymbol{u}}_1 \\ \vdots & & \vdots & \vdots \\ \sigma_{n-1,1} & \cdots & \sigma_{n-1,n-1} & \tilde{\boldsymbol{u}}_{n-1} \\ \tilde{\boldsymbol{u}}_1 & \cdots & \tilde{\boldsymbol{u}}_{n-1} & 0 \end{vmatrix}$$

and it has, under the hypothesis tested, for large values of m asymptotically a χ^2-distribution, with $n-1$ degrees of freedom. The method can be adjusted for the case that ties are present.

The general form of this statistic is difficult to compute but necessary and sufficient conditions can be given for the statistic being a linear combination of the squared column-totals. A number of wellknown non-parametric tests can be considered as special cases of the generalized method of m rankings, in particular Friedman's original method, Durbin's generalization, the k-sample test of Kruskal [6] and Terpstra [7] and the distribution-free test for one-and for two-factor experiments of Brown and Mood [2], [3].

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CONFIDENCE LIMITS FOR THE RATIO OF TWO MEANS

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The method to determine confidence limits for the ratio of the means ξ and η of two variates with a two-dimensional normal distribution, usual in biological assays (E. C. Fieller, 1944), can be brought into the following more general form.

General formulation.

To determine confidence limits for $\alpha = \frac{\xi}{\eta}$ we try to find five functions $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a}_{11}, \boldsymbol{a}_{12}$ and $\boldsymbol{a}_{22}$ of the observations $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k$, such that:

1. $\boldsymbol{x}$ and $\boldsymbol{y}$ are $N(\xi, \eta; \sigma_{11}, \sigma_{12}, \sigma_{22})$ distributed with unknown parameters;

2. $$\boldsymbol{z} \stackrel{\text{def}}{=} \eta \boldsymbol{x} - \xi \boldsymbol{y} \tag{1}$$

and

$$\boldsymbol{s_z} \stackrel{\text{def}}{=} + \sqrt{\eta^2 \boldsymbol{a}_{11} - 2\eta\xi \boldsymbol{a}_{12} + \xi^2 \boldsymbol{a}_{22}} \tag{2}$$

are independently distributed;

3. for some known integer f

$$\frac{f \boldsymbol{s}_{\boldsymbol{z}}^2}{\sigma_{\boldsymbol{z}}^2}, \text{ with } \sigma_{\boldsymbol{z}}^2 = \eta^2\sigma_{11} - 2\eta\xi\sigma_{12} + \xi^2\sigma_{22}$$

has a χ_f^2 -distribution.

From these conditions it follows that $\boldsymbol{t} = \frac{\boldsymbol{z}}{\boldsymbol{s_z}}$ has Student's-distribution with f degrees of freedom. If t_ε is determined by

$$\mathrm{P}\left[\left|\frac{\boldsymbol{z}}{\boldsymbol{s_z}}\right| \leqq t_\varepsilon\right] = 1 - \varepsilon,$$

then it follows from (1) and (2), that the inequality

$$(\boldsymbol{x}^2 - t_\varepsilon^2 \boldsymbol{a}_{11}) - 2\frac{\xi}{\eta}(\boldsymbol{xy} - t_\varepsilon^2 \boldsymbol{a}_{12}) + \frac{\xi^2}{\eta^2}(\boldsymbol{y}^2 - t_\varepsilon^2 \boldsymbol{a}_{22}) \leqq 0 \tag{3}$$

has a probability $1 - \varepsilon$.

All values α' which, substituted for $\frac{\xi}{\eta}$, satisfy (3) (the value ∞ included), form a confidence interval for $\alpha = \frac{\xi}{\eta}$ corresponding to the confidence level $1 - \varepsilon$.

Examples.

The method may e.g. be applied to the following situations:

a. independent pairs of observations $(\boldsymbol{x}_1, \boldsymbol{y}_1), \ldots (\boldsymbol{x}_n, \boldsymbol{y}_n)$, of $\boldsymbol{x}$ and $\boldsymbol{y}$ are given, where $\boldsymbol{x}$ and $\boldsymbol{y}$ have a simultaneous normal distribution with means ξ and η;

b. independent observations $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ and $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m$ of $\boldsymbol{x}$ and $\boldsymbol{y}$ are given, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are independently normally distributed with mean ξ resp. η and variance $\sigma_{\mathbf{x}}^2$ resp. $\sigma_{\mathbf{y}}^2 = k\sigma_{\mathbf{x}}^2$, provided k is a known constant (biological assays).

The more general case, when the ratio of $\sigma_{\mathbf{x}}^2$ to $\sigma_{\mathbf{y}}^2$ is unknown, cannot be solved by this method.

c. The confidence limits for the slope of a line when both variates are subject to errors (with a twodimensional normal distribution), given by A. Wald (1940), is another example of this method.

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AN UPPER BOUND FOR THE DEVIATION FROM NORMALITY OF WILCOXON'S TEST STATISTIC FOR THE TWO-SAMPLE PROBLEM IN THE GENERAL CASE

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Given two independent random samples $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ and $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n$ which are drawn from two populations with (cumulative) distribution functions F and G respectively. Wilcoxon's two-sample test is based on the statistic $\boldsymbol{U}$ defined by the number of pairs (i, j) $(i = 1, \ldots, m; j = 1, \ldots, n)$ with $\boldsymbol{y}_j < \boldsymbol{x}_i$ together with half the number of pairs (i, j) with $\boldsymbol{y}_j = \boldsymbol{x}_i$ (cf. Hemelrijk).

If we define

$$\theta = \mathrm{P}(\boldsymbol{y} < \boldsymbol{x} \mid F, G) + \tfrac{1}{2}\mathrm{P}(\boldsymbol{y} = \boldsymbol{x} \mid F, G) = \tfrac{1}{2}\int_{-\infty}^{\infty} (G(x+0) + G(x-0))\,\mathrm{d}F(x)$$

$$\Gamma_r = \int_{-\infty}^{\infty} \{\tfrac{1}{2}(G(x+0) + G(x-0)) - \theta\}^r\,\mathrm{d}F(x)$$

$$\Lambda_r = \int_{-\infty}^{\infty} \{1 - \tfrac{1}{2}(F(y+0) + F(y-0)) - \theta\}^r\,\mathrm{d}G(y)$$

$$\sigma_1^2 = (m+n)^{-1}\{m\Lambda_2 + n\Gamma_2 + \theta(1-\theta) - (\Lambda_2 + \Gamma_2)\}$$

$$\sigma_2^2 = (m+n)^{-1}(m\Lambda_2 + n\Gamma_2),$$

then we have by means of one of Berry's estimates

$$\sup_{-\infty < \xi < \infty} \left| \mathrm{P}[\boldsymbol{U} - \mathscr{E}(\boldsymbol{U} \mid F, G) \leqq \xi\sigma_{\boldsymbol{U}} \mid F, G] - \Phi(\xi) \right| \leqq \Delta(m, n; F, G)$$

where

$$\Phi(\xi) \stackrel{\text{def}}{=} (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\xi} \mathrm{e}^{-\frac{1}{2}x^2}\,\mathrm{d}x$$

$$\sigma_{\boldsymbol{U}^2} \stackrel{\text{def}}{=} \mathrm{var}\,(\boldsymbol{U} \mid F, G) = \sigma_1^2 \qquad \text{(cf. D. van Dantzig)}$$

and

$$\Delta(m, n; F, G) = \frac{1.88 \max\,(n\sqrt{\Gamma_4/\Gamma_2};\; m\sqrt{\Lambda_4/\Lambda_2})}{\sqrt{mn(m+n)} \cdot \sigma_2} + \min_{\delta}\left[\frac{\sigma_1^2 - \sigma_2^2}{\delta^2} + \frac{1}{(2\pi)^{\frac{1}{2}}\sigma_2}\{\delta + \max\,(\delta, \sigma_1 - \sigma_2)\}\right].$$

Thus $\Delta(m, n; F, G)$ tends to 0 when m and n both tend to infinity, no matter how, unless σ_2 is equal to 0 or tends to 0.

Although better results have been obtained, we only state here the above upper bound because of its simplicity. Also further improvements of the upper bound have been obtained for values of ξ far from 0.

The method used to obtain an upper bound for the deviation from normality of Wilcoxon's statistic, can also be applied to other statistics of the class of statistics introduced by Hoeffding (1948) and Lehmann (1951).

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