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# Maximum likelihood estimation of ordered probabilities II

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## 1. Introduction

The problem treated in this report concerns the maximum likelihood estimation of partially or completely ordered probabilities.

Consider $k$ independent series of independent trials, each trial resulting in a success or a failure. The $i$-th series consists of $n_i$ trials with $\underline{a}_i$[1] successes and $\underline{b}_i = n_i - \underline{a}_i$ failures; $\pi_i$ is the (unknown) probability of a success for each trial of the $i$-th series $(i = 1, 2, \ldots, k)$ and $\pi_1, \pi_2, \ldots, \pi_k$ satisfy the inequalities

$$(1.1) \qquad \alpha_{i,j}(\pi_i - \pi_j) \leqq 0 \qquad (i, j = 1, 2, \ldots, k),$$

where

$$(1.2) \qquad \begin{cases} 1.\ \alpha_{i,j} = -\alpha_{j,i}, \\ 2.\ \alpha_{i,j} = 0 \quad \text{for } m_0 \text{ pairs of values } (i,j) \text{ with } i<j, \\ 3.\ \alpha_{i,j} = 1 \quad \text{for } m_1 \text{ pairs of values } (i,j) \text{ with } i<j, \end{cases}$$

$$(1.3) \qquad \begin{cases} m_0 + m_1 = \binom{k}{2}, \\ m_1 \geqq 1 \end{cases}$$

and, if $i < h < j$ then

$$(1.4) \qquad \alpha_{i,j} = 1 \qquad \text{if} \qquad \alpha_{i,h} = \alpha_{h,j} = 1. \quad \text{(transitivity)}$$

In section 2 and 3 methods will be described by means of which the maximum likelihood estimates of $\pi_1, \pi_2, \ldots, \pi_k$ may be found, i.e. the values of $x_1, x_2, \ldots, x_k$ which maximize

$$(1.5) \qquad L = L(x_1, x_2, \ldots, x_k) \stackrel{\text{def}}{=} \sum_{i=1}^{k} \{ a_i \lg x_i + b_i \lg(1-x_i) \}$$

in the domain

$$(1.6) \qquad D: \quad \begin{cases} \alpha_{i,j}(x_i - x_j) \leqq 0 \\ 0 \leqq x_i \leqq 1 \end{cases} \qquad (i, j = 1, 2, \ldots, k).$$

Unless explicitely stated otherwise $L$ will only be considered in this domain $D$; the maximum likelihood estimates will be denoted by $p_1, p_2, \ldots p_k$,

$$(1.7) \qquad L_i = L_i(x_i) \stackrel{\text{def}}{=} a_i \lg x_i + b_i \lg(1-x_i) \qquad (i = 1, 2, \ldots, k)$$

and

$$(1.8) \qquad f_i \stackrel{\text{def}}{=} \frac{a_i}{n_i} \qquad (i = 1, 2, \ldots, k).$$

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1) Random variables will be distinguished from numbers (e.g. from the value they take in an experiment) by underlining their symbols.

Further the restrictions $\pi_i \leqq \pi_j$ satisfying

(1.9) $\qquad \alpha_{i,h}\,\alpha_{h,j} = 0 \quad$ for each $h$ between $i$ and $j$

will be denoted by $R_1, R_2, \ldots, R_s$. Because of the transitivity relations (1.4) the system $R_1, R_2, \ldots, R_s$ is equivalent to (1.1).

In section 4 some examples will be given.

Remarks:

1. It will be clear that ordered probabilities can always be numbered in such a way that they satisfy (1.1).
2. Every set of restrictions $R_1, R_2, \ldots, R_s$ represents a convex domain.
3. If $m_0 = 0$ then (1.1) is equivalent to

(1.10) $\qquad \pi_1 \leqq \pi_2 \leqq \ldots \leqq \pi_k .$

This case has been solved independently by MIRIAM AYER, H.D. BRUNK, G.M. EWING, W.T. REID, EDWARD SILVERMAN [2] and the present author [1].

## 2. The maximum likelihood estimates of $\pi_1, \pi_2, \ldots, \pi_k$

In this section the following theorem will be proved.

Theorem I: $L$ possesses a unique maximum; if $R_\lambda$ implies: $\pi_{i_\lambda} \leqq \pi_{j_\lambda}$ and if $p_1', p_2', \ldots, p_k'$ are the maximum likelihood estimates of $\pi_1, \pi_2, \ldots, \pi_k$ under the restrictions $R_1, \ldots, R_{\lambda-1}, R_{\lambda+1}, \ldots, R_s$ then

$$(2.1) \qquad \begin{cases} 1.\ p_i = p_i' \quad (i = 1, 2, \ldots, k) & \text{if} \quad p_{i_\lambda}' \leqq p_{j_\lambda}' , \\ 2.\ p_{i_\lambda} = p_{j_\lambda} & \text{if} \quad p_{i_\lambda}' > p_{j_\lambda}' . \end{cases}$$

Proof: The uniqueness of the maximum of $L$ will be proved by induction. If $s = 0$ then it is well known that $L$ possesses for any $k$ a unique maximum and

(2.2) $\qquad p_i = f_i \qquad (i = 1, 2, \ldots, k).$

Now suppose that $L$ possesses a unique maximum for the following two cases

$$(2.3) \qquad \begin{cases} 1.\ k \text{ series of trials with } s-1 \text{ restrictions} \\ 2.\ k-1 \text{ series of trials with } s-1 \text{ or less restrictions} \end{cases}$$

where $k \geqq 2$, $s \geqq 1$ and consider a case with $k$ series of trials and $s$ restrictions. Then it follows from (2.3.1) that $L$ possesses a unique maximum under the restrictions $R_1, \ldots, R_{\lambda-1}, R_{\lambda+1}, \ldots, R_s$, i.e. there exists exactly one set $p_1', p_2', \ldots, p_k'$ satisfying these restrictions and maximizing $L$.

Now the following two cases may be distinguished

1. $p'_{i_\lambda} \leqq p'_{j_\lambda}$ ; then $p'_1, p'_2, \ldots, p'_k$ satisfy the restrictions $R_1, R_2, \ldots, R_s$. Therefore in this case $L$ possesses a unique maximum under the restrictions $R_1, R_2, \ldots, R_s$ and

(2.4) $$p_i = p'_i \quad (i = 1, 2, \ldots, k).$$

2. $p'_{i_\lambda} > p'_{j_\lambda}$ ; then (2.1.2) may be proved as follows. If $x_1, x_2, \ldots, x_k$ is any set satisfying the restrictions $R_1, \ldots, R_{\lambda-1}, R_{\lambda+1}, \ldots, R_s$ but different from $p'_1, p'_2, \ldots, p'_k$, then

(2.5) $$L(p'_1, p'_2, \ldots, p'_k) > L(x_1, x_2, \ldots, x_k).$$

Now consider a fixed set $x_1, x_2, \ldots, x_k$ with $x_{i_\lambda} < x_{j_\lambda}$ and satisfying the restrictions $R_1, R_2, \ldots, R_s$. Then if

(2.6) $$\begin{cases} X_i(\beta) \overset{\text{def}}{=} (1-\beta)\, x_i + \beta\, p'_i \\ 0 \leqq X_i(\beta) \leqq 1 \end{cases} \qquad (i = 1, 2, \ldots, k)$$

we have

(2.7) $$X_i(0) = x_i \quad , \quad X_i(1) = p'_i \quad (i = 1, 2, \ldots, k)$$

and for each $\beta$ with $0 \leqq \beta \leqq 1$, $X_1(\beta), X_2(\beta), \ldots, X_k(\beta)$ is a set satisfying the restrictions $R_1, \ldots, R_{\lambda-1}, R_{\lambda+1}, \ldots, R_s$ (cf. remark 2 in the foregoing section). Therefore if

(2.8) $$\beta_0 \overset{\text{def}}{=} \frac{x_{j_\lambda} - x_{i_\lambda}}{x_{j_\lambda} - x_{i_\lambda} + p'_{i_\lambda} - p'_{j_\lambda}}$$

then

(2.9) $$\begin{cases} 1.\ \ 0 < \beta_0 < 1, \\ 2.\ \ X_{i_\lambda}(\beta_0) = X_{j_\lambda}(\beta_0), \end{cases}$$

i.e. $X_1(\beta_0), X_2(\beta_0), \ldots, X_k(\beta_0)$ is a set satisfying the restrictions $R_1, R_2, \ldots, R_s$.

Further $L\{X_1(\beta), X_2(\beta), \ldots, X_k(\beta)\}$ is for fixed values of $x_1, x_2, \ldots, x_k$ a function of $\beta$, say $g(\beta)$, and

(2.10) $$\frac{d^2 g(\beta)}{d\beta^2} = \sum_{i=1}^{k} (p'_i - x_i)^2 \frac{-n_i X_i(\beta)^2 + 2 a_i X_i(\beta) - a_i}{X_i(\beta)^2 \left\{1 - X_i(\beta)\right\}^2} < 0.$$

From (2.5), (2.7) and (2.10) it follows then that $g(\beta)$ is in the interval $0 \leqq \beta \leqq 1$ an increasing function of $\beta$, i.e.

(2.11) $$L\{X_1(\beta_0), X_2(\beta_0), \ldots, X_k(\beta_0)\} > L(x_1, x_2, \ldots, x_k).$$

Thus for each set $x_1, x_2, \ldots, x_k$ with $x_{i_\lambda} < x_{j_\lambda}$ and satisfying the restrictions $R_1, R_2, \ldots, R_k$ a set $X_1, X_2, \ldots, X_k$ exists satisfying the restrictions $R_1, R_2, \dots, R_s$ and

$$(2.12) \qquad \begin{cases} 1. \ X_{i_\lambda} = X_{j_\lambda}, \\ 2. \ L(X_1, X_2, \dots, X_k) > L(x_1, x_2, \dots, x_k), \end{cases}$$

i.e. $L(x_1, x_2, \dots, x_k)$ attains its maximum under the restrictions $R_1, R_2, \dots, R_s$ for $x_{i_\lambda} = x_{j_\lambda}$. Substituting this in (1.5) the two terms with $i = i_\lambda$ and $i = j_\lambda$ reduce to one term of the form

$(a_{i_\lambda} + a_{j_\lambda}) \lg x_{i_\lambda} + (b_{i_\lambda} + b_{j_\lambda}) \lg (1 - x_{i_\lambda})$.

The uniqueness of the maximum of $L$ under these restrictions then follows from (2.3.2).

By repeatedly applying theorem I and using the well known solution of the problem for the case that $s = 0$ the problem may be solved. This may, however, lead to a rather complicated procedure, which in many cases may be simplified by applying the special theorems mentioned in the following section.

## 3. Some special theorems

Theorem II: If $\alpha_{i,j}(f_i - f_j) \leqq 0$ for each pair of values $(i, j)$ then

$$(3.1) \qquad p_i = f_i \quad (i = 1, 2, \dots, k).$$

Proof: This follows immediately from the fact that in this case the maximum of $L$ in $D$ coincides with the maximum of $L$ in the domain: $0 \leqq x_i \leqq 1 \ (i = 1, 2, \dots, k)$. The theorem also follows from theorem I.

The following theorem will be immediately clear.

Theorem III: If $i_1, i_2, \dots, i_\nu$ is a set of values satisfying

$$(3.2) \qquad \alpha_{i i_1} = \alpha_{i i_2} = \dots = \alpha_{i i_\nu} = 0 \qquad \text{for each } i \neq i_1, i_2, \dots, i_\nu$$

then the maximum likelihood estimates of $\pi_{i_1}, \pi_{i_2}, \dots, \pi_{i_\nu}$ are those values of $x_{i_1}, x_{i_2}, \dots, x_{i_\nu}$ which maximize $L_{i_1} + L_{i_2} + \dots + L_{i_\nu}$ in the domain

$$(3.3) \qquad D': \begin{cases} \alpha_{i_h, i_{h'}} (x_{i_h} - x_{i_{h'}}) \leqq 0, \\ 0 \leqq x_{i_h} \leqq 1 \end{cases} \quad (h, h' = 1, 2, \dots, \nu).$$

For the proof of the theorems IV and V we need the following lemma

Lemma I: If $x_1, x_2, \dots, x_k$ is any set in $D$ with

$$(3.4) \qquad \begin{cases} 1. \quad x_i < x_j, \\ 2. \quad f_i \geqq f_j, \\ 3. \quad \alpha_{h,i} \geqq \alpha_{h,j} & \text{for each } h < j, \\ 4. \quad \alpha_{i,h} \leqq \alpha_{j,h} & \text{for each } h > i. \end{cases}$$

for any given pair of values $(i,j)$ then a number $x$ exists satisfying

$$(3.5)\quad \begin{cases} 1.\ x_1, x_2, \dots, x_k \text{ is also a set in } D \text{ if } x \text{ is substituted for } x_i \text{ and } x_j\ , \\ 2.\ L_i(x) + L_j(x) > L_i(x_i) + L_j(x_j). \end{cases}$$

Proof:

The following cases may be distinguished

$$(3.6)\qquad x_i < x_j \leqq \xi_i \quad ;\ \text{then take } x = x_j.$$

$$(3.7)\qquad \xi_i \leqq x_i < x_j \quad ;\ \text{then take } x = x_i.$$

$$(3.8)\qquad x_i < \xi_i < x_j \quad ;\ \text{then take } x = \xi_i.$$

It may easily be proved that this number $x$ satisfies (3.5.2). For (3.6) e.g. we have

$$(3.9)\qquad L_j(x) = L_j(x_j)$$

and

$$(3.10)\qquad x_i < x \leqq \xi_i .$$

From (3.10) follows

$$(3.11)\qquad L_i(x) > L_i(x_i)$$

and (3.5.2) follows from (3.9) and (3.11).

For the cases (3.7) and (3.8) it may be proved in an analogous way by means of (3.4.2) that $x$ satisfies (3.5.2).

In order to prove that this number $x$ satisfies (3.5.1) it is sufficient to prove that

$$(3.12)\quad \begin{cases} 1.\ \alpha_{h,i}(x_h - x) \leqq 0 \text{ for each } h < i, \\ 2.\ \alpha_{i,h}(x - x_h) \leqq 0 \text{ for each } h > i, \\ 3.\ \alpha_{h,j}(x_h - x) \leqq 0 \text{ for each } h < j, \\ 4.\ \alpha_{j,h}(x - x_h) \leqq 0 \text{ for each } h > j. \end{cases}$$

From the fact that $x$ satisfies

$$(3.13)\qquad x_i \leqq x \leqq x_j$$

and the fact that $x_1, x_2, \dots, x_k$ is a set in $D$ it follows that

$$(3.14)\quad \begin{cases} 1.\ \alpha_{h,i}(x_h - x) \leqq \alpha_{h,i}(x_h - x_i) \leqq 0 \text{ for each } h < i, \\ 2.\ \alpha_{j,h}(x - x_h) \leqq \alpha_{j,h}(x_j - x_h) \leqq 0 \text{ for each } h > j. \end{cases}$$

Further it follows from (3.4.3) and (3.4.4) that

$$(3.15)\quad \begin{cases} 1.\ \alpha_{i,h}(x-x_h)=\alpha_{j,h}(x-x_h) & \text{for each } h>i \text{ with } \alpha_{i,h}=1, \\ 2.\ \alpha_{h,j}(x_h-x)=\alpha_{h,i}(x_h-x) & \text{for each } h<j \text{ with } \alpha_{h,j}=1 \end{cases}$$

and (3.12) follows from (3.14) and (3.15).

Theorem IV: If for any pair of values $(i,j)$ with $i<j$

$$(3.16)\qquad \alpha_{i,j}(f_i-f_j)>0$$

and

$$(3.17)\quad \begin{cases} 1.\ \alpha_{i,h}=\alpha_{h,j}=0 & \text{for each } h \text{ between } i \text{ and } j, \\ 2.\ \alpha_{h,i}=\alpha_{h,j} & \text{for each } h<i, \\ 3.\ \alpha_{i,h}=\alpha_{j,h} & \text{for each } h>j, \end{cases}$$

then

$$(3.18)\qquad p_i=p_j.$$

Proof: Suppose $x_1,x_2,\ldots,x_k$ is a set in $\mathfrak{D}$ with

$$(3.19)\qquad x_i<x_j.$$

Further we have (cf. (3.16) and (3.17))

$$(3.20)\quad \begin{cases} 1.\ f_i>f_j, & \\ 2.\ \alpha_{h,i}=\alpha_{h,j} & \text{for each } h<j, \\ 3.\ \alpha_{i,h}=\alpha_{j,h} & \text{for each } h>i. \end{cases}$$

From lemma I and (3.20) it follows then that a number $x$ exists such that $x_1,\ldots,x_{i-1},x,x_{i+1},\ldots,x_{j-1},x,x_{j+1},\ldots,x_k$ is a set in $\mathfrak{D}$ and

$$(3.21)\qquad L_i(x)+L_j(x)>L_i(x_i)+L_j(x_j).$$

Thus for each set $x_1,x_2,\ldots,x_k$ in $\mathfrak{D}$ with $x_i<x_j$ a set $x_1',x_2',\ldots,x_k'$ in $\mathfrak{D}$ exists with

$$(3.22)\quad \begin{cases} 1.\ x_i'=x_j', \\ 2.\ L(x_1',x_2',\ldots,x_k')>L(x_1,x_2,\ldots,x_k), \end{cases}$$

i.e. $L$ attains its maximum for $x_i=x_j$ and (3.18) then follows from the uniqueness of this maximum.

Remarks:

4. This theorem also follows from theorem I. If $R_\lambda$ represents the restriction $\pi_i\leqq\pi_j$ it follows from (3.22) that $L$ attains its maximum under the restrictions $R_1,\ldots,R_{\lambda-1},R_{\lambda+1},\ldots,R_k$ for $x_i\geqq x_j$, giving $p_i'\geqq p_j'$; from (2.1) then follows: $p_i=p_j$.

5. If $m_0=0$, i.e. if the probabilities $\pi_1,\pi_2,\ldots,\pi_k$ satisfy the inequalities

$$(3.23)\qquad \pi_1\leqq\pi_2\leqq\ldots\leqq\pi_k,$$

then each pair of values $(i,j)$ with $j=i+1$ satisfies (3.17). Therefore in this case we have

(3.24) $$p_i = p_{i+1} \qquad \text{for each } i \text{ with } f_i > f_{i+1}.$$

From theorem IV it follows that if there is a pair of values $(i,j)$ satisfying (3.16) and (3.17) then the problem may be reduced to the case of $k-1$ series of trials with $s-1$ (or less) restrictions by substituting $x_i = x_j$ in $L(x_1, x_2, \ldots, x_k)$ i.e. by pooling the $i$-th and $j$-th series of trials.

Theorem V: If $(i,j)$ is a pair of values satisfying

(3.25) $$f_i \leqq f_j$$

and

(3.26) $$\begin{cases} 1.\ \alpha_{i,j} = 0, & \\ 2.\ \alpha_{h,i} \leqq \alpha_{h,j} & \text{for each } h < i, \\ 3.\ \alpha_{i,h} \geqq \alpha_{j,h} & \text{for each } h > j, \end{cases}$$

then

(3.27) $$p_i \leqq p_j.$$

Proof: Suppose $x_1, x_2, \ldots, x_k$ is a set in $D$ with

(3.28) $$x_i > x_j.$$

Further we have (cf. (3.25) and (3.26))

(3.29) $$\begin{cases} 1.\ f_i \leqq f_j, & \\ 2.\ \alpha_{h,i} \leqq \alpha_{h,j} & \text{for each } h < i, \\ 3.\ \alpha_{i,h} \geqq \alpha_{j,h} & \text{for each } h > j. \end{cases}$$

From lemma I it follows then in the same way as in theorem IV that for each set $x_1, x_2, \ldots, x_k$ in $D$ with $x_i > x_j$ a set $x'_1, x'_2, \ldots, x'_k$ in $D$ exists with

(3.30) $$\begin{cases} 1.\ x'_i = x'_j, \\ 2.\ L(x'_1, x'_2, \ldots, x'_k) > L(x_1, x_2, \ldots, x_k), \end{cases}$$

i.e. $L$ attains its maximum for $x_i \leqq x_j$; (3.27) then follows from the uniqueness of the maximum.

By means of theorem V a new restriction may be introduced. This is sometimes useful as may be seen from example 2 of section 4. In the following section some examples will be given. In these examples the theorem II-V will be applied if possible and theorem I will only be used where the theorems II-V cannot be applied.

## 4. Examples

**Example 1:**

Suppose $k = 4$, $m_0 = 0$ $(\pi_1 \leqq \pi_2 \leqq \pi_3 \leqq \pi_4)$ and

$$(4.1)\qquad \begin{cases} i & 1 & 2 & 3 & 4 \\ a_i & 4 & 3 & 10 & 8 \\ n_i & 10 & 5 & 30 & 15 \\ f_i & 0{,}4 & 0{,}6 & 0{,}33 & 0{,}53\ . \end{cases}$$

From (4.1) and (3.24) it follows that

$$(4.2)\qquad p_2 = p_3$$

and the problem is reduced to the case of $k-1=3$ series of trials by pooling the second and third series of trials:

$$(4.3)\qquad \begin{cases} i & 1 & 2(+3) & 4 \\ a_i' & 4 & 13 & 8 \\ n_i' & 10 & 35 & 15 \\ f_i' & 0{,}4 & 0{,}37 & 0{,}53\ . \end{cases}$$

From (4.3) and (3.24) it then follows that

$$(4.4)\qquad p_1 = p_2$$

and the problem is reduced to the case of $k-2=2$ series of trials with

$$(4.5)\qquad \begin{cases} i & 1(+2+3) & 4 \\ a_i'' & 17 & 8 \\ n_i'' & 45 & 15 \\ f_i'' & 0{,}38 & 0{,}53. \end{cases}$$

From (4.5), (4.2), (4.4) and theorem I then follows

$$(4.6)\qquad p_1 = p_2 = p_3 = 0{,}38\ ,\ p_4 = 0{,}53.$$

**Example 2:**

Suppose $k = 5$, $m_1 = 6$, $m_0 = 4$

$$(4.7)\qquad \begin{cases} i & 1 & 2 & 3 & 4 & 5 \\ a_i & 7 & 13 & 15 & 2 & 12 \\ n_i & 10 & 20 & 30 & 5 & 15 \\ f_i & 0{,}7 & 0{,}65 & 0{,}5 & 0{,}4 & 0{,}8 \end{cases}$$

and

$$(4.8)\qquad \alpha_{1,2} = \alpha_{1,2} = \alpha_{2,4} = \alpha_{3,5} = 1.$$

Then the pair of values $i = 2$, $j = 4$ satisfies (3.16) and (3.17). Therefore we have

$$(4.9)\qquad p_2 = p_4$$

and the problem is reduced to the case of $k-1=4$ series of trials with $m_1' = 4$, $m_0' = 2$,

(4.10)
$$\begin{cases} i & 1 & 2(+4) & 3 & 5 \\ a_i' & 7 & 15 & 15 & 12 \\ n_i' & 10 & 25 & 30 & 15 \\ f_i' & 0,7 & 0,6 & 0,5 & 0,8 \end{cases}$$

and

(4.11) $$\alpha_{1,2}' = \alpha_{1,3}' = \alpha_{3,5}' = 1.$$

For these 4 series of trials the pair $i=3, j=2$ and the pair $i=2, j=5$ satisfy (3.25) and (3.26). From theorem V then follows that $L$ attains its maximum for

(4.12) $$x_1 \leqq x_3 \leqq x_2 \leqq x_5$$

and from (4.9), (4.10) and (4.12) follows

(4.13) $$p_1 = p_3 = 0,55 \;,\; p_2 = p_4 = 0,6 \;,\; p_5 = 0,8.$$

Example 3:

Suppose $k=4$, $m_0 = m_1 = 3$,

(4.14)
$$\begin{cases} i & 1 & 2 & 3 & 4 \\ a_i & 7 & 18 & 13 & 10 \\ n_i & 10 & 30 & 20 & 25 \\ f_i & 0,7 & 0,6 & 0,65 & 0,4 \end{cases}$$

and

(4.15) $$\alpha_{1,2} = \alpha_{1,4} = \alpha_{3,4} = 1.$$

For this case the theorems II-V cannot be applied and therefore we use theorem I.

Take $i_\lambda = 1$ and $j_\lambda = 4$ (i.e. omit the restriction $\pi_1 \leqq \pi_4$), then $p_1', p_2', p_3', p_4'$ are those values of $x_1, x_2, x_3, x_4$ which maximize $L$ in the domain

(4.16)
$$\begin{cases} x_1 \leqq x_2 \;,\; x_3 \leqq x_4 \\ 0 \leqq x_i \leqq 1 \qquad (i=1,2,3,4). \end{cases}$$

From theorem III and IV then follows

(4.17) $$p_1' = p_2' = 0,63 \;,\; p_3' = p_4' = 0,51$$

and from theorem I and (4.17) (cf. (2.1.2))

(4.18) $$p_1 = p_4.$$

In this way the problem is reduced to the case of $k-1=3$ series of trials with

$$(4.19)\quad \begin{cases} i & 3 & 1(+4) & 2 \\ a'_i & 13 & 17 & 18 \\ n'_i & 20 & 35 & 30 \\ f'_i & 0,65 & 0,49 & 0,6 \end{cases}$$

and

$$(4.20)\qquad \alpha'_{3,1} = \alpha'_{1,2} = 1.$$

From (4.18), (4.19) and (4.20) follows

$$(4.21)\qquad p_1 = p_3 = p_4 = 0,55 \quad , \; p_2 = 0,6.$$

<u>Example 4</u>:

Suppose $k = 8$, $m_0 = 13$, $m_1 = 15$

$$(4.22)\quad \begin{cases} i & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ a_i & 8 & 22 & 13 & 25 & 20 & 21 & 32 & 2 \\ n_i & 10 & 40 & 20 & 50 & 30 & 50 & 50 & 5 \\ f_i & 0,8 & 0,55 & 0,65 & 0,5 & 0,67 & 0,42 & 0,64 & 0,4 \end{cases}$$

and

$$(4.23)\qquad \alpha_{1,2} = \alpha_{1,4} = \alpha_{3,4} = \alpha_{4,5} = \alpha_{5,6} = \alpha_{5,8} = \alpha_{7,8} = 1.$$

For this case the theorems II-V cannot be applied and therefore we use theorem I.

If we take $i_\lambda = 4$ and $j_\lambda = 5$ then it follows from theorem III that $p'_1, p'_2, \ldots, p'_8$ are found by maximizing $\sum_{i=1}^{4} L_i$ under the restrictions: $x_1 \leqq x_2$, $x_1 \leqq x_4$, $x_3 \leqq x_4$ and $\sum_{i=5}^{8} L_i$ under the restrictions: $x_5 \leqq x_6$, $x_5 \leqq x_8$, $x_7 \leqq x_8$.

In order to find $p'_1, p'_2, p'_3, p'_4$ we apply theorem I; this results in (cf. example 3)

$$(4.24)\qquad p'_1 = p'_2 = p'_3 = p'_4 = 0,57.$$

In an analogous way we find by means of theorem I

$$(4.25)\qquad p'_5 = p'_6 = 0,51 \;,\; p'_7 = p'_8 = 0,62.$$

From (4.24), (4.25) and theorem I then follows (cf. (2.1.2))

$$(4.26)\qquad p_4 = p_5$$

and the problem is reduced to the case of $k-1 = 7$ series of trials with $m'_0 = 10$, $m'_1 = 11$

$$(4.27)\quad \begin{cases} i & 1 & 2 & 3 & 4(+5) & 6 & 7 & 8 \\ a'_i & 8 & 22 & 13 & 45 & 21 & 32 & 2 \\ n'_i & 10 & 40 & 20 & 80 & 50 & 50 & 5 \\ f'_i & 0,8 & 0,55 & 0,65 & 0,56 & 0,42 & 0,64 & 0,4 \end{cases}$$

and

(4.28) $\qquad \alpha'_{1,2} = \alpha'_{1,4} = \alpha'_{3,4} = \alpha'_{4,6} = \alpha'_{4,8} = \alpha'_{7,8} = 1.$

For these 7 series of trials we again apply theorem I; taking $i_\lambda = 1$ and $j_\lambda = 4$ one finds by means of the theorems II-V

(4.29) $\qquad p'_1 = p'_2 = 0,6 , \qquad p'_3 = p'_4 = p'_6 = 0,53, \quad p'_7 = p'_8 = 0,62.$

From (4.29) and theorem I then follows

(4.30) $$p_1 = p_4$$

and the problem is reduced to the case of $k-2=6$ series of trials with $m''_0 = 7$, $m''_1 = 8$

(4.31)

| | | | | | | |
|---|---|---|---|---|---|---|
| $i$ | 3 | 1(+4+5) | 2 | 6 | 7 | 8 |
| $a''_i$ | 13 | 53 | 22 | 21 | 32 | 2 |
| $n''_i$ | 20 | 90 | 40 | 50 | 50 | 5 |
| $f''_i$ | 0,65 | 0,59 | 0,55 | 0,42 | 0,64 | 0,4 |

and

(4.32) $\qquad \alpha''_{3,1} = \alpha''_{1,2} = \alpha''_{1,6} = \alpha''_{1,8} = \alpha''_{7,8} = 1.$

From (4.31), (4.32) and theorem IV then follows

(4.33) $$p_3 = p_1$$

and the problem is reduced to the case of $k-3=5$ series of trials with $m'''_0 = 6$, $m'''_1 = 4$

(4.34)

| | | | | | |
|---|---|---|---|---|---|
| $i$ | 1(+3+4+5) | 2 | 6 | 7 | 8 |
| $a'''_i$ | 66 | 22 | 21 | 32 | 2 |
| $n'''_i$ | 110 | 40 | 50 | 50 | 5 |
| $f'''_i$ | 0,6 | 0,55 | 0,42 | 0,64 | 0,4 |

and

(4.35) $\qquad \alpha'''_{1,2} = \alpha'''_{1,6} = \alpha'''_{1,8} = \alpha'''_{7,8} = 1.$

From theorem V it follows then that $L$ attains its maximum for $x_1 \leq x_7$ . Introducing this new restriction it follows from theorem IV that

(4.36) $$p_7 = p_8$$

and the problem is reduced to the case of $k-4=4$ series of trials with $m''''_0 = m''''_1 = 3$

(4.37)

| | | | | |
|---|---|---|---|---|
| $i$ | 1(+3+4+5) | 2 | 6 | 7(+8) |
| $a''''_i$ | 66 | 22 | 21 | 34 |
| $n''''_i$ | 110 | 50 | 40 | 55 |
| $f''''_i$ | 0,6 | 0,55 | 0,42 | 0,62 |

and

(4.38) $\alpha''''_{1,2} = \alpha''''_{1,6} = \alpha''''_{1,7} = 1.$

From theorem V then follows that $L$ attains its maximum for $x_1 \leqq x_6 \leqq x_2 \leqq x_7$; thus we have

(4.39) $p_1 = p_3 = p_4 = p_5 = p_6 = 0,54$; $p_2 = 0,56$; $p_7 = p_8 = 0,62$.

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