Report S 196(VP7)

Maximum likelihood estimation of ordered probabilities

II

by

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4. Introduction

The problem treated in this report concerns the maximum likelihood estimation of partially or completely ordered probabilities.

Consider $k$ independent series of independent trials, each trial resulting in a success or a failure. The $i$-th series consists of $n_i$ trials with $a_i$ successes and $b_i = n_i - a_i$ failures; $\pi_i$ is the (unknown) probability of a success for each trial of the $i$-th series ($i = 1, 2, \ldots, k$) and $\pi_1, \pi_2, \ldots, \pi_k$ satisfy the inequalities

\begin{equation}
\alpha_{i, j} (\pi_i - \pi_j) \leq 0 \quad (i, j = 1, 2, \ldots, k),
\end{equation}

where

\begin{equation}
\begin{cases}
1. \alpha_{i, i} = -\alpha_{i, i}, \\
2. \alpha_{i, j} = 0 \text{ for } m_i \text{ pairs of values } (i, j) \text{ with } i < j, \\
3. \alpha_{i, i} = 1 \text{ for } m_i \text{ pairs of values } (i, i) \text{ with } i < j,
\end{cases}
\end{equation}

\begin{equation}
\begin{cases}
m_i + m_j = \binom{k}{2}, \\
m_i \geq 1
\end{cases}
\end{equation}

and, if $i < h < j$ then

\begin{equation}
\alpha_{i, j} = 1 \quad \text{if} \quad \alpha_{i, h} = \alpha_{h, j} = 1. \quad \text{(transitivity)}
\end{equation}

In section 2 and 3 methods will be described by means of which the maximum likelihood estimates of $\pi_1, \pi_2, \ldots, \pi_k$ may be found, i.e. the values of $x_1, x_2, \ldots, x_k$ which maximize

\begin{equation}
L = L (x_1, x_2, \ldots, x_k) \overset{\text{def}}{=} \sum_{i=1}^{k} \left[ a_i \, l_i(x_i) + b_i \, l_i(1-x_i) \right]
\end{equation}

in the domain

\begin{equation}
D : \left\{ \begin{array}{l}
\alpha_{i, j} (x_i - x_j) \leq 0 \\
o \leq x_i \leq 1
\end{array} \right. \quad (i, j = 1, 2, \ldots, k).
\end{equation}

Unless explicitly stated otherwise, $L$ will only be considered in this domain $D$; the maximum likelihood estimates will be denoted by $\hat{\pi}_1, \hat{\pi}_2, \ldots, \hat{\pi}_k$.

\begin{equation}
L_i = L_i (x_i) \overset{\text{def}}{=} a_i \, l_i(x_i) + b_i \, l_i(1-x_i) \quad (i = 1, 2, \ldots, k)
\end{equation}

and

\begin{equation}
\hat{\pi}_i \overset{\text{def}}{=} \frac{a_i}{n_i} \quad (i = 1, 2, \ldots, k).
\end{equation}

1) Random variables will be distinguished from numbers (e.g. from the value they take in an experiment) by underlining their symbols.
Further the restrictions $\pi_i \leq \pi_j$ satisfying

\[(1.9)\quad \alpha_{i,h} \alpha_{h,j} = 0 \quad \text{for each } h \text{ between } i \text{ and } j\]

will be denoted by $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_s$. Because of the transitivity relations (1.4) the system $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_s$ is equivalent to (1.1).

In section 4 some examples will be given.

Remarks:
1. It will be clear that ordered probabilities can always be numbered in such a way that they satisfy (1.1).
2. Every set of restrictions $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_s$ represents a convex domain.
3. If $m_{ij} = 0$ then (1.1) is equivalent to

\[(1.10)\quad \pi_i \leq \pi_2 \leq \ldots \leq \pi_k.

This case has been solved independently by MIRIAM AYER, H.D. BRUNK, C.M. EWING, W.T. REID, EDWARD SILVERMAN [2] and the present author [1].

2. The maximum likelihood estimates of $\pi_i, \pi_j, \ldots, \pi_k$

In this section the following theorem will be proved.

Theorem I: $L$ possesses a unique maximum; if $\mathcal{R}_s$ implies $\pi_i, \pi_j$, and if $p'_1, p'_2, \ldots, p'_k$ are the maximum likelihood estimates of $\pi_1, \pi_2, \ldots, \pi_k$ under the restrictions $\mathcal{R}_1, \ldots, \mathcal{R}_s, \mathcal{R}_{s+1}, \ldots, \mathcal{R}_k$, then

\[(2.1)\quad \begin{cases} 1. & p'_i = p'_j \quad (i = 1, 2, \ldots, k) \quad \text{if} \quad p'_i \geq p'_j, \\ 2. & p'_i = p'_j \quad \text{if} \quad p'_i > p'_j. \end{cases}

Proof: The uniqueness of the maximum of $L$ will be proved by induction. If $s = 0$ then it is well known that $L$ possesses for any $k$ a unique maximum and

\[(2.2)\quad p_i = \frac{f_i}{\sum f_i} \quad (i = 1, 2, \ldots, k).

Now suppose that $L$ possesses a unique maximum for the following two cases

\[(2.3)\quad \begin{cases} 1. & k \text{ series of trials with } s-1 \text{ restrictions} \\ 2. & k-1 \text{ series of trials with } s-1 \text{ or less restrictions} \end{cases}

where $k \geq 2, s \geq 1$ and consider a case with $k$ series of trials and $s$ restrictions. Then it follows from (2.3.1) that $L$ possesses a unique maximum under the restrictions $\mathcal{R}_1, \ldots, \mathcal{R}_{k-1}, \mathcal{R}_{k+1}, \ldots, \mathcal{R}_s$, i.e. there exists exactly one set $p'_1, p'_2, \ldots, p'_k$ satisfying these restrictions and maximizing $L$. 
Now the following two cases may be distinguished

1. \( F_i \equiv F_{i\alpha} \); then \( p_i, p_i^\prime, \ldots, p_k \) satisfy the restrictions \( R_1, R_2, \ldots, R_6 \). Therefore in this case \( L \) possesses a unique maximum under the restrictions \( R_1, R_2, \ldots, R_6 \) and

\[
(2.4) \quad p_i = p_i^\prime \quad (i = 1, 2, \ldots, k).
\]

2. \( F_i > F_{i\alpha} \); then (2.1.2) may be proved as follows. If \( x_1, x_2, \ldots, x_k \) is any set satisfying the restrictions \( R_1, R_2, R_{1\alpha}, R_{2\alpha}, \ldots, R_6 \), but different from \( p_i, p_i^\prime, \ldots, p_k^\prime \), then

\[
(2.5) \quad L(\ p_i, p_i^\prime, \ldots, p_k^\prime\ ) > L(\ x_1, x_2, \ldots, x_k\ ).
\]

Now consider a fixed set \( x_1, x_2, \ldots, x_k \) with \( x_i < x_{i\alpha} \) and satisfying the restrictions \( R_1, R_2, \ldots, R_6 \). Then if

\[
(2.6) \quad \begin{cases}
X_i(\beta) \overset{\text{def}}{=} (1 - \beta) x_i + \beta p_i^\prime \\
0 \equiv X_i(\beta) \leq 1
\end{cases} \quad (i = 1, 2, \ldots, k)
\]

we have

\[
(2.7) \quad X_i(0) = x_i \quad , \quad X_i(1) = p_i^\prime \quad (i = 1, 2, \ldots, k)
\]

and for each \( \beta \) with \( 0 \leq \beta \leq 1 \), \( X_i(\beta), X_2(\beta), \ldots, X_k(\beta) \) is a set satisfying the restrictions \( R_1, \ldots, R_{1\alpha}, R_{2\alpha}, \ldots, R_6 \) (cf. remark 2 in the foregoing section). Therefore if

\[
(2.8) \quad \beta_0 \overset{\text{def}}{=} \frac{x_i - x_{i\alpha}}{x_i - x_{i\alpha} + p_i - p_{i\alpha}}
\]

then

\[
(2.9) \quad \begin{cases}
1. \quad 0 < \beta_0 < 1, \\
2. \quad X_i(\beta_0) = X_i(\beta),
\end{cases}
\]

i.e. \( X_i(\beta_0), X_2(\beta_0), \ldots, X_k(\beta_0) \) is a set satisfying the restrictions \( R_1, R_2, \ldots, R_6 \).

Further \( L\{X_i(\beta), X_2(\beta), \ldots, X_k(\beta)\} \) is for fixed values of \( x_1, x_2, \ldots, x_k \) a function of \( \beta \), say \( q(\beta) \), and

\[
(2.10) \quad \frac{d^2 q(\beta)}{d\beta^2} = \sum_{i=1}^{k} \left( p_i - x_i \right)^2 \frac{-X_i(\beta)}{X_i(\beta)^2} \cdot \frac{a_i X_i(\beta) - a_i}{X_i(\beta)^2} \leq 0.
\]

From (2.5), (2.7) and (2.10) it follows then that \( q(\beta) \) is in the interval \( 0 \leq \beta \leq 1 \) an increasing function of \( \beta \), i.e.

\[
(2.11) \quad L\{X_i(\beta_0), X_2(\beta_0), \ldots, X_k(\beta_0)\} > L(\ x_1, x_2, \ldots, x_k\ ).
\]

Thus for each set \( x_1, x_2, \ldots, x_k \) with \( x_i < x_{i\alpha} \) and satisfying the restrictions \( R_1, R_2, \ldots, R_k \) a set \( X_i, X_2, \ldots, X_k \) exists satisfying
the restrictions $R_1, R_2, \ldots, R_b$ and

\[
\begin{align*}
1. \quad X_i &= X_{i}^*, \\
2. \quad L(X_1, X_2, \ldots, X_b) &> L(x_1, x_2, \ldots, x_b),
\end{align*}
\]

(2.12)

1.e. $L(x_1, x_2, \ldots, x_b)$ attains its maximum under the restrictions $R_1, R_2, \ldots, R_b$ for $x_i^* = x_{i}^*$. Substituting this in (1.5) the two terms with $i = i$ and $i = ji$ reduce to one term of the form

\[
(\alpha_{i} + \alpha_{ji}) q_{i} x_{i} + (b_{i} + b_{ji}) q_{i} (1 - X_{i}).
\]

The uniqueness of the maximum of $L$ under these restrictions then follows from (2.32).

By repeatedly applying theorem I and using the well known solution of the problem for the case that $S = 0$ the problem may be solved. This may, however, lead to a rather complicated procedure, which in many cases may be simplified by applying the special theorems mentioned in the following section.

3. Some special theorems

**Theorem II:** If $\alpha_{i,j}(f_i - f_j) \equiv 0$ for each pair of values $(i, j)$ then

\[
p_i = f_i \quad (i = 1, 2, \ldots, k).
\]

(3.1)

**Proof:** This follows immediately from the fact that in this case the maximum of $L$ in $D$ coincides with the maximum of $L$ in the domain: $0 \equiv x_i \equiv 1 (i = 1, 2, \ldots, k)$. The theorem also follows from theorem I.

The following theorem will be immediately clear.

**Theorem III:** If $\tau_1, \tau_2, \ldots, \tau_b$ is a set of values satisfying

\[
\alpha_{i,i} = \alpha_{i,ij} = \ldots = \alpha_{i,ii} = 0
\]

for each $i \neq i, i_1, \ldots, i_b$

then the maximum likelihood estimates of $\tau_1, \tau_2, \ldots, \tau_b$ are those values of $x_{i_1}, x_{i_2}, \ldots, x_{i_b}$ which maximize $L_{i_1} + L_{i_2} + \ldots + L_{i_b}$ in the domain

\[
D': \quad \left\{ \begin{array}{l}
\alpha_{h',h} (X_{h'} - X_{h}) \equiv 0, \quad (h, h' = 1, 2, \ldots, v)\\
0 \leq x_{ih} \leq 1
\end{array} \right.
\]

(3.3)

For the proof of the theorems IV and V we need the following lemma

**Lemma I:** If $x_1, x_2, \ldots, x_k$ is any set in $D$ with

\[
\begin{align*}
1. \quad x_i &< x_j, \\
2. \quad f_i &\geq f_j, \\
3. \quad \alpha_{h,i} &\equiv \alpha_{h,j}, \\
4. \quad \alpha_{h,i} &\leq \alpha_{h,j}
\end{align*}
\]

(3.4)

for each $h < i$,

for each $h > i$. 

for each $i$. 


for any given pair of values \((i,j)\), then a number \(x\) exists satisfying

\[
\begin{align*}
1. & \quad x_1, x_2, \ldots, x_k \text{ is also a set in } D \text{ if } x \text{ is substituted} \\
& \quad \text{for } x_i \text{ and } x_j, \\
2. & \quad L_i(x) + L_j(x) > L_i(x_i) + L_j(x_j).
\end{align*}
\]

**Proof:**

The following cases may be distinguished

\[
\begin{align*}
(3.6) & \quad x_i < x_j \leq f_i \quad ; \text{ then take } x = x_j, \\
(3.7) & \quad f_i \leq x_i < x_j \quad ; \text{ then take } x = x_i, \\
(3.8) & \quad x_i < f_j < x_j \quad ; \text{ then take } x = f_j.
\end{align*}
\]

It may easily be proved that this number \(x\) satisfies \((3.5.2)\).

For \((3.6)\) e.g. we have

\[
(3.9) \quad L_i(x) = L_i(x_j)
\]

and

\[
(3.10) \quad x_i < x \leq f_i.
\]

From \((3.10)\) follows

\[
(3.11) \quad L_i(x) > L_i(x_i)
\]

and \((3.5.2)\) follows from \((3.9)\) and \((3.11)\).

For the cases \((3.7)\) and \((3.8)\) it may be proved in an analogous way by means of \((3.4.2)\) that \(x\) satisfies \((3.5.2)\).

In order to prove that this number \(x\) satisfies \((3.5.1)\) it is sufficient to prove that

\[
\begin{align*}
1. & \quad \alpha_{h,i}(x_h - x) \leq 0 \text{ for each } h < i, \\
2. & \quad \alpha_{i,h}(x - x_h) \leq 0 \text{ for each } h > i, \\
3. & \quad \alpha_{h,i}(x_h - x) \leq 0 \text{ for each } h < i, \\
4. & \quad \alpha_{i,h}(x - x_h) \leq 0 \text{ for each } h > i.
\end{align*}
\]

From the fact that \(x\) satisfies

\[
(3.12) \quad x_i \leq x \leq x_j
\]

and the fact that \(x_1, x_2, \ldots, x_k\) is a set in \(D\) it follows that

\[
(3.13) \quad x_i \leq x \leq x_j
\]

Further it follows from \((3.4.3)\) and \((3.4.4)\) that
(3.15) \[
\begin{aligned}
1. \quad \alpha_{i,h}(x-x_h) &= \alpha_{i,h}(x-x_h) \quad \text{for each } h \geq i \text{ with } \alpha_{i,h} = 1, \\
2. \quad \alpha_{h,i}(x_h-x) &= \alpha_{h,i}(x_h-x) \quad \text{for each } h < i \text{ with } \alpha_{h,i} = 1
\end{aligned}
\]
and (3.12) follows from (3.14) and (3.15).

Theorem IV: If for any pair of values \((i,j)\) with \(i < j\)

\[
\alpha_{i,j}(\frac{p_i}{\beta} - \frac{p_j}{\beta}) > 0
\]

and

\[
\begin{aligned}
1. \quad \alpha_{i,h} = \alpha_{h,i} = 0 \quad &\text{for each } h \text{ between } i \text{ and } j, \\
2. \quad \alpha_{h,i} = \alpha_{h,j} \quad &\text{for each } h < i, \\
3. \quad \alpha_{i,h} = \alpha_{i,j} \quad &\text{for each } h > j,
\end{aligned}
\]

then

\[
p_i = p_j.
\]

Proof: Suppose \(x_1, x_2, \ldots, x_k\) is a set in \(\mathbb{D}\) with

\[
x_i < x_j.
\]

Further we have (cf. (3.16) and (3.17))

\[
\begin{aligned}
1. \quad \frac{p_i}{\beta} > \frac{p_j}{\beta}, \\
2. \quad \alpha_{h,i} = \alpha_{h,j} \quad &\text{for each } h < j, \\
3. \quad \alpha_{i,h} = \alpha_{i,j} \quad &\text{for each } h > i.
\end{aligned}
\]

From lemma I and (3.20) it follows then that a number \(x\) exists such that \(x, x_1, x, x_2, \ldots, x_3, x, x_4, \ldots, x_k\) is a set in \(\mathbb{D}\) and

\[
L(x) + L(x_j) > L(x_i) + L(x_j).
\]

Thus for each set \(x_1, x_2, \ldots, x_k\) in \(\mathbb{D}\) with \(x_i < x_j\) a set \(x_1', x_2', \ldots, x_k'\) in \(\mathbb{D}\) exists with

\[
\begin{aligned}
1. \quad x_i' = x_i, \\
2. \quad L(x_1', x_2', \ldots, x_k') > L(x_1, x_2, \ldots, x_k),
\end{aligned}
\]

i.e. \(L\) attains its maximum for \(x_i = x_j\) and (3.18) then follows from the uniqueness of this maximum.

Remarks:

4. This theorem also follows from theorem I. If \(R_\lambda\) represents the restriction \(\pi_i \triangleq \pi_j\) it follows from (3.22) that \(L\) attains its maximum under the restrictions \(R_i, R_{i+1}, R_{i+2}, \ldots, R_k\) for \(x_i \geq x_j\), giving \(p_i \geq p_j\); from (2.1) then follows: \(p_i = p_j\).

5. If \(m_n = 0\), i.e. if the probabilities \(\pi_i, \pi_2, \ldots, \pi_k\) satisfy the inequalities

\[
\pi_i \leq \pi_2 \leq \ldots \leq \pi_k
\]
then each pair of values \((i,j)\) with \(i = j + 1\) satisfies (3.17). Therefore in this case we have

\[(3.24) \quad p_i = p_{i+1}, \quad \text{for each } i \text{ with } f_i > f_{i+1}.\]

From theorem IV it follows that if there is a pair of values \((i,j)\) satisfying (3.16) and (3.17) then the problem may be reduced to the case of \(k-1\) series of trials with \(s-1\) (or less) restrictions by substituting \(x_i = x_j\) in \(L(x_i, x_2, \ldots, x_k)\) i.e. by pooling the \(i\)-th and \(j\)-th series of trials.

**Theorem V:** If \((i,j)\) is a pair of values satisfying

\[(3.25) \quad f_i \leq f_j,\]

and

\[(3.26) \begin{align*}
1. \quad \alpha_{i,j} &= 0, \\
2. \quad \alpha_{k,i} &\leq \alpha_{k,j} \quad \text{for each } h < i, \\
3. \quad \alpha_{i,h} &\leq \alpha_{j,h} \quad \text{for each } h > j,
\end{align*}\]

then

\[(3.27) \quad p_i \leq p_j.\]

**Proof:** Suppose \(x_1, x_2, \ldots, x_k\) is a set in \(D\) with

\[(3.28) \quad x_i > x_j.\]

Further we have (cf. (3.25) and (3.26),)

\[(3.29) \begin{align*}
1. \quad f_i &\leq f_j, \\
2. \quad \alpha_{k,i} &\leq \alpha_{k,j} \quad \text{for each } h < i, \\
3. \quad \alpha_{i,h} &\leq \alpha_{j,h} \quad \text{for each } h > j.
\end{align*}\]

From lemma I it follows then in the same way as in theorem IV that for each set \(x_1, x_2, \ldots, x_k\) in \(D\) with \(x_i > x_j\) a set \(x_1, x_2, \ldots, x_k\) in \(D\) exists with

\[(3.30) \begin{align*}
1. \quad x_i &\leq x_j, \\
2. \quad L(x_i, x_2, \ldots, x_k) > L(x_1, x_2, \ldots, x_k),
\end{align*}\]

i.e. \(L\) attains its maximum for \(x_i = x_j\); (3.27) then follows from the uniqueness of the maximum.

By means of theorem V a new restriction may be introduced. This is sometimes useful as may be seen from example 2 of section 4. In the following section some examples will be given. In these examples the theorem II-V will be applied if possible and theorem I will only be used where the theorems II-V cannot be applied.
4. Examples

Example 1:

Suppose \( k = 4, m_o = 0 \) (\( m_1 \leq m_2 \leq m_3 \leq m_4 \)) and

\[
\begin{pmatrix}
  i & 1 & 2 & 3 & 4 \\
  a_i & 4 & 3 & 10 & 8 \\
  n_i & 10 & 5 & 30 & 15 \\
  f_i & 0.4 & 0.6 & 0.33 & 0.53 \\
\end{pmatrix}
\]

(4.1)

From (4.1) and (3.24) it follows that

(4.2) \[ p_2 = p_3 \]

and the problem is reduced to the case of \( k - 1 = 3 \) series of trials by pooling the second and third series of trials:

\[
\begin{pmatrix}
  i & 1 & 2(+3) & 4 \\
  a_i & 4 & 13 & 8 \\
  n_i & 10 & 35 & 15 \\
  f_i & 0.4 & 0.37 & 0.53 \\
\end{pmatrix}
\]

(4.3)

From (4.3) and (3.24) it then follows that

(4.4) \[ p_1 = p_2 \]

and the problem is reduced to the case of \( k - 2 = 2 \) series of trials with

\[
\begin{pmatrix}
  i & 1(+2+3) & 4 \\
  a_i & 17 & 8 \\
  n_i & 45 & 15 \\
  f_i & 0.38 & 0.53 \\
\end{pmatrix}
\]

(4.5)

From (4.5), (4.2), (4.4) and theorem I then follows

(4.6) \[ p_1 = p_2 = p_3 = 0.38, \ p_4 = 0.53. \]

Example 2:

Suppose \( k = 5, m_1 = 6, m_o = 4 \)

\[
\begin{pmatrix}
  i & 1 & 2 & 3 & 4 & 5 \\
  a_i & 7 & 13 & 15 & 2 & 12 \\
  n_i & 10 & 20 & 30 & 5 & 15 \\
  f_i & 0.7 & 0.65 & 0.5 & 0.4 & 0.8 \\
\end{pmatrix}
\]

(4.7)

and

(4.8) \[ \alpha_{12} = \alpha_{13} = \alpha_{24} = \alpha_{35} = 1. \]

Then the pair of values \( i = 2, \ i = 4 \) satisfies (3.16) and (3.17). Therefore we have

(4.9) \[ p_2 = p_4 \]
and the problem is reduced to the case of \( k-1=4 \) series of trials with \( m'_1 = 4, m'_a = 2 \).

\[
\begin{align*}
\begin{cases}
  \alpha'_1 & \quad 7 & \quad 15 & \quad 12 \\
  m'_1 & \quad 10 & \quad 25 & \quad 30 & \quad 15 \\
  f'_1 & \quad 0.7 & \quad 0.6 & \quad 0.5 & \quad 0.8
\end{cases}
\end{align*}
\]

and

\[
(4.11) \quad \alpha'_{1,3} = \alpha'_{1,5} = \alpha'_{3,5} = 1.
\]

For these 4 series of trials the pair \( i = 3, j = 2 \) and the pair \( i = 2, j = 5 \) satisfy (3.25) and (3.26). From theorem V then follows that \( L \) attains its maximum for

\[
(4.12) \quad x_1 \leq x_3 \leq x_5 \leq x_8
\]

and from (4.9), (4.10) and (4.12) follows

\[
(4.13) \quad p_1 = p_3 = 0.65, \quad p_2 = p_4 = 0.6, \quad p_5 = 0.8.
\]

Example 3:
Suppose \( k = 4, m_a = m_1 = 3 \),

\[
\begin{align*}
\begin{cases}
  \alpha & \quad 7 & \quad 18 & \quad 13 & \quad 10 \\
  m & \quad 10 & \quad 30 & \quad 20 & \quad 25 \\
  f & \quad 0.7 & \quad 0.6 & \quad 0.65 & \quad 0.4
\end{cases}
\end{align*}
\]

and

\[
(4.15) \quad \alpha_{1,3} = \alpha_{1,4} = \alpha_{3,4} = 1.
\]

For this case the theorems II-V cannot be applied and therefore we use theorem I.

Take \( \lambda = 1 \) and \( \lambda = 4 \) (i.e. omit the restriction \( \pi_1 \neq \pi_4 \)), then \( p'_1, p'_2, p'_3, p'_4 \) are those values of \( x_1, x_2, x_3, x_4 \) which maximize \( L \) in the domain

\[
(4.16) \quad \begin{cases}
  x_1 \leq x_2 \leq x_3 \leq x_4 \\
  0 \leq x_i \leq 1 \quad (i=1,2,3,4).
\end{cases}
\]

From theorem III and IV then follows

\[
(4.17) \quad p'_1 = p'_2 = 0.63, \quad p'_3 = p'_4 = 0.51
\]

and from theorem I and (4.17) (cf. (2.1.2))

\[
(4.18) \quad p_i = p_i.
\]

In this way the problem is reduced to the case of \( k-1=3 \) series of trials with
\[
\begin{align*}
\left\{ \begin{array}{c}
i \quad 3 \quad 1(+4) \quad 2 \\
\alpha_i \quad 13 \quad 17 \quad 18 \\
\eta_i \quad 20 \quad 35 \quad 30 \\
f_i \quad 0,65 \quad 0,49 \quad 0,6 
\end{array} \right.
\end{align*}
\]

and

\[
\alpha_{i,1} = \alpha_{i,2} = 1.
\]

From (4.18), (4.19) and (4.20) follows

\[
\psi_1 = \psi_2 = \psi_3 = 0,55 \quad \psi_4 = 0,6.
\]

**Example 4:**

Suppose \(k = 8\), \(n_0 = 13\), \(n_1 = 15\)

\[
\begin{align*}
\left\{ \begin{array}{c}
i \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \\
\alpha_i \quad 8 \quad 22 \quad 13 \quad 25 \quad 20 \quad 21 \quad 32 \quad 2 \\
\eta_i \quad 10 \quad 40 \quad 20 \quad 50 \quad 30 \quad 50 \quad 50 \quad 5 \\
f_i \quad 0,8 \quad 0,55 \quad 0,65 \quad 0,5 \quad 0,67 \quad 0,42 \quad 0,64 \quad 0,4
\end{array} \right.
\end{align*}
\]

and

\[
\alpha_{1,2} = \alpha_{2,4} = \alpha_{3,6} = \alpha_{4,8} = \alpha_{5,8} = \alpha_{7,8} = 1.
\]

For this case the theorems II-V cannot be applied and therefore we use theorem I.

If we take \(i_{2,4} = 4\) and \(j_{1,5} = 5\) then it follows from theorem III that \(p'_1, p'_2, \ldots, p'_8\) are found by maximizing \(\sum_{i} L_i\) under the restrictions: \(x_1 \leq x_2, x_2 \leq x_4, x_3 \leq x_4\) and \(\sum_{i} L_i\) under the restrictions: \(x_5 \leq x_6, x_6 \leq x_8, x_7 \leq x_8\).

In order to find \(p'_1, p'_2, p'_3, p'_4\) we apply theorem I; this results in (cf. example 3)

\[
\psi_1 = \psi_2 = \psi_3 = \psi_4 = 0,57.
\]

In an analogous way we find by means of theorem I

\[
\psi_1 = \psi_2 = 0,51, \quad \psi_1 = \psi_2 = 0,62.
\]

From (4.24), (4.25) and theorem I then follows (cf. (2.1.2))

\[
\psi_1 = \psi_2
\]

and the problem is reduced to the case of \(k-1 = 7\) series of trials with

\[
\begin{align*}
\left\{ \begin{array}{c}
i \quad 1 \quad 2 \quad 3 \quad 4(+5) \quad 6 \quad 7 \quad 8 \\
\alpha_i \quad 8 \quad 22 \quad 13 \quad 45 \quad 21 \quad 32 \quad 2 \\
\eta_i \quad 10 \quad 40 \quad 20 \quad 80 \quad 50 \quad 50 \quad 5 \\
f_i \quad 0,8 \quad 0,55 \quad 0,65 \quad 0,56 \quad 0,42 \quad 0,64 \quad 0,4
\end{array} \right.
\end{align*}
\]
and

\[ (4.28) \quad \alpha_i = \alpha_{i,4} = \alpha_{i,2} = \alpha_{i,4} = \alpha_{i,6} = \alpha_{i,8} = 1. \]

For these 7 series of trials we again apply theorem I; taking \( \lambda = 1 \) and \( \lambda = 4 \) one finds by means of the theorems II-V

\[ (4.29) \quad p_1 = p_2 = 0.6, \quad p_3 = p_4 = p_5 = 0.5, \quad p_7 = p_8 = 0.5. \]

From (4.29) and theorem I then follows

\[ (4.30) \quad p_1 = p_4 \]

and the problem is reduced to the case of \( k = 6 \) series of trials with \( m_0 = 7, m_i = 8 \)

\[
\begin{array}{ccccccc}
\lambda & 2 & 6 & 7 & 8 \\
\alpha_i & 13 & 53 & 22 & 21 & 32 & 2 \\
m_i & 20 & 90 & 40 & 50 & 50 & 5 \\
d_i & 0.65 & 0.59 & 0.55 & 0.42 & 0.64 & 0.4 \\
\end{array}
\]

and

\[ (4.32) \quad \alpha_{i,1} = \alpha_{i,2} = \alpha_{i,6} = \alpha_{i,8} = \alpha_{i,7} = 1. \]

From (4.31), (4.32) and theorem IV then follows

\[ (4.33) \quad p_1 = p_4 \]

and the problem is reduced to the case of \( k = 8 \) series of trials with \( m_0 = 6, m_i = 4 \)

\[
\begin{array}{ccccccc}
\lambda & 1(3+4+5) & 2 & 6 & 7 & 8 \\
\alpha_i & 66 & 22 & 21 & 32 & 2 \\
n_i & 110 & 40 & 50 & 50 & 5 \\
di & 0.6 & 0.55 & 0.42 & 0.64 & 0.4 \\
\end{array}
\]

and

\[ (4.35) \quad \alpha_{i,1} = \alpha_{i,2} = \alpha_{i,9} = \alpha_{i,8} = 1. \]

From theorem V it follows then that \( L \) attains its maximum for \( x_1 = x_2 \). Introducing this new restriction it follows from theorem IV that

\[ (4.36) \quad p_7 = p_5 \]

and the problem is reduced to the case of \( k = 4 \) series of trials with \( m_0 = m_i = 3 \)

\[
\begin{array}{ccccccc}
\lambda & 1(3+4+5) & 2 & 6 & 7(8) \\
\alpha_i & 66 & 22 & 21 & 34 \\
m_i & 110 & 50 & 40 & 55 \\
di & 0.6 & 0.55 & 0.42 & 0.62 \\
\end{array}
\]
and

\[ (4.38) \quad \alpha_{i,2} = \alpha_{i,6} = \alpha_{i,7} = 1. \]

From theorem V then follows that \( L \) attains its maximum for \( x, x_6, x_2 \leq x_7 \); thus we have

\[ (4.39) \quad p_1 = p_3 = p_4 = p_5 = p_6 = 0.54; \quad p_2 = 0.56; \quad p_7 = p_8 = 0.62. \]

References:
