STICHTING
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2e BOERHAAVESTRAAT 49
AMSTERDAM

STATISTISCHE AFDELING

LEIDING: PROF. DR D. VAN DANTZIG
ADVISEUR VOOR STATISTISCHE CONSULTATIE: PROF. DR J. HEMELRIJK

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Some elementary proofs in renewal theory with applications to waiting times

by

H. Kesten and J.Th. Runnenburg

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1) Introduction and summary

Following DOOB [4] [1], we can say that renewal theory in the simplest case deals with the following situation:
A population, consisting of a certain number of individuals is given. As soon as an individual dies, it is replaced by another, whose life starts at the moment of replacement. In this way the total number of living individuals is kept constant. The lifetimes of the individuals are supposed to be independent random variables with the same distribution function. Hence a population of k individuals can be considered as the sum of k independent "one-man" populations each of which consists of the offspring of a single individual. This justifies in many cases the restriction (made in this paper) to populations, consisting of one individual only. For convenience we shall assume that the time interval from the moment 0 to the first occurring death is independent of all lifespans and has the same distribution.
The quantities we are usually concerned with, are:

a) the remaining lifetime of the individual, living at time t ,
b) the age which the individual, living at time t , will reach,
c) the number of deaths in the interval $[0, t]$.

We are interested in renewal theory, because of its applications to waiting time problems.
Let us consider the arrival of busses at a busstop. If we assume that the intervals $\underline{x}_n$ between the arrival of the n^{th} and $(n+1)^{st}$ bus and the interval from zero to the arrival of the first bus, are independently distributed with the same distribution function $F_1(x)$, we have essentially the same situation as described above. If we substitute for an individual in renewal theory a bus and for a lifespan of such an individual the time interval between the arrival of successive busses, results of renewal theory can be interpreted at once. The "translations" of the quantities a), b) and c) are respectively:

a') the time $\underline{w}_t$, a passenger arriving at time t has to wait till the next bus arrives,

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1) Numbers in square brackets refer to the list at the end of this report.

b') the length of that time interval between two consecutive busses, which overlaps the point t,

c') the number of busses $\underline{N}_t$ arriving at the busstop in the time interval $[0, t]$.

In this paper we shall prove some theorems, which can be found in the literature, (3,5) and (5,5) possibly excepted. Here we only consider the case where not all possible values of the arrival interval are integral multiples of a positive constant. The excluded case was investigated by FELLER [7]. In this introduction the random variable $\underline{x}$ has the same distribution as the arrival intervals, i.e. the distribution function $F_1(x)$. Its moments will be denoted by

$$\mu_k \stackrel{\text{def}}{=} E\,\underline{x}^k = \int_{0-}^{\infty} x^k \, dF_1(x) ,$$

while we take $\frac{1}{\mu_1}$ equal to 0, if $\mu_1 = \infty$. The proofs given here depend on the following wellknown theorem which was proved by BLACKWELL [1]:

Theorem 1: Unless all possible values of the arrival interval $\underline{x}$ [2)], with distribution function $F_1(x)$, are integral multiples of some fixed constant, the expected number $U(t)$ of bus arrivals in $[0, t]$ has the property

$$U(t, h) \stackrel{\text{def}}{=} U(t+h) - U(t) \longrightarrow \frac{h}{\mu_1} \quad (t \to \infty)$$

for every $h > 0$.

The analogous theorem, for the case where all possible values of $\underline{x}$ are integral multiples of a fixed constant, was proved by ERDÖS, FELLER and POLLARD [5] (c.f. also [6], p.244,Th.3).

For completeness sake we shall proof th.1 in an appendix. Using this theorem, partial integration, change of order of integration

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2) The random character of a stochastic variable will be denoted by underlining its symbol. The same symbol without underlining will be used for values taken by the stochastic variable.

(Fubini's theorem) and the LEBESGUE-lemma on dominated convergence of integrals [3], we will prove in section 3 the following theorem.

Theorem 2: The waiting time $\underset{\sim}{w}_t$ of a passenger arriving at time t, waiting for the next bus, has the distribution function:

$$H_t(w) = \begin{cases} 0 & \text{for } w < 0 \\ F_1(t+w) + \int_{w+}^{+\infty} \{u(t+w-x) - u(t-)\}\, dF_1(x) & \text{for } w \geq 0 \end{cases}$$

$\lim\limits_{t\to\infty} H_t(w)$ exists and is given by

$$H(w) \overset{\text{def}}{=} \lim_{t\to\infty} H_t(w) = \begin{cases} 0 & \text{for } w < 0 \\ \frac{1}{\mu_1} \int_0^w \{1 - F_1(x)\}\, dx & \text{for } w \geq 0. \end{cases}$$

Evidently $H(w)$ is again a distribution function if $\mu_1 < \infty$. Let $\underset{\sim}{w}$ be a random variable with distribution function $H(w)$. If $\mu_1 < \infty$ the moments of the distribution function $H_t(w)$ tend to the moments of $H(w)$ for $t \to \infty$, for which we find:

$$\lim_{t\to\infty} E\, \underset{\sim}{w}_t^{\,k} = E\, \underset{\sim}{w}^k = \frac{\mu_{k+1}}{(k+1)\mu_1}.$$

In section 4 we give a proof of the asymptotic formula

$$U(t) = \frac{t}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - 1 + o(1) \qquad (t \to \infty)$$

valid when $\mu_2 < \infty$.

In section 5 we consider the length of the arrival interval $\underset{\sim}{l}_t$ which overlaps the time t. Its distribution function $k_t(l)$ is given by

$$K_t(l) = \begin{cases} 0 & \text{for } l < 0 \\ F_1(t-) - u(t-)\{1 - F_1(l)\} + \int_{0-}^{(t-l)-} U(x)\, dF_1(t-x) & \text{for } l \geq 0 \end{cases}$$

$\lim\limits_{t\to\infty} K_t(l)$ exists and is given by

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3) This lemma can e.g. be formulated in the following way: If $f_n(x)$ is a sequence of μ-measurable functions, such that for all sufficiently large values of n and almost everywhere on E $|f_n(x)| \leq F(x)$ where $F(x)$ is μ-integrable over the μ-measurable set E, and if $\lim\limits_{n\to\infty} f_n(x) = f(x)$ almost everywhere on E, then $\lim\limits_{n\to\infty} \int_E f_n(x)\, d\mu_x = \int_E f(x)\, d\mu_x$.

$$K(l) \overset{\text{def}}{=} \lim_{t \to \infty} K_t(l) = \begin{cases} 0 & \text{for } l < 0 \\ \frac{1}{\mu_1} \int_{0-}^{l+} x \, dF_1(x) & \text{for } l \geq 0 . \end{cases}$$

If $\mu_1 < \infty$, $K(l)$ is again a distribution function. Let $\underline{l}$ be a random variable with distribution function $K(l)$. If $\mu_1 < \infty$ the moments of $K_t(l)$ again tend to the moments of $K(l)$ for $t \to \infty$, for which we find

$$\lim_{t \to \infty} E \, \underline{l}_t^{\,k} = E \, \underline{l}^{\,k} = \frac{\mu_{k+1}}{\mu_1} .$$

Finally a theorem of SMITH [8] is derived:

Theorem 6: If $k(t)$ is any function, which is zero for negative argument, and is bounded, L_1 and non-increasing in $(0, \infty)$, then

$$\lim_{t \to \infty} \int_{-\infty}^{+\infty} k(t-x) \, d\,U(x) = \frac{1}{\mu_1} \int_0^{\infty} k(x) \, dx .$$

2. Simple formulae and lemmas

In what follows we will always assume the intervals between successive busses (including the interval from time 0 to the arrival of the first bus) to be independently distributed with the same distribution function $F_1(x)$. Because the arrival intervals are non-negative we take $F_1(x) = 0$ for $x < 0$ and in addition exclude the case $F_1(0) = 1$ i.e. we take $F_1(0) < 1$. (BLACKWELL [2] gave a generalization of his main theorem in [1] for the case of variables which are not necessarily non-negative). We assume that $F_1(x)$ is not a discrete distribution function with all jumps in integral multiples of a positive constant. This excluded type of distributions was considered by FELLER [7]. The busses can be numbered without ambiguity in accordance with the order in which they arrive, although $F_1(0) > 0$ is allowed. The successive arrivaltimes of the busses will be denoted by [4]

$$\underline{s}_1, \underline{s}_2, \ldots$$

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4) cf. 2) p. 2.

and the arrival intervals by

$$\underline{x}_1 \overset{def}{=} \underline{S}_1$$

and

$$\underline{x}_k \overset{def}{=} \underline{S}_k - \underline{S}_{k-1} \qquad \text{for} \quad k \geqslant 2 .$$

The distribution functions of $\underline{S}_k$ for $k = 2, 3, \dots$ are then given by the following relations [5]

$$(2.1) \qquad F_k(x) \overset{def}{=} P\{\underline{S}_k \leqslant x\} = \begin{cases} 0 & \text{for } x < 0 \\ \int_{0-}^{x+} F_{k-1}(x-y)\, dF_1(y) & \text{for } x \geqslant 0 \end{cases}$$

while furthermore

$$(2.2) \qquad F_{k+l}(x) = \begin{cases} 0 & \text{for } x < 0 \\ \int_{0-}^{x+} F_k(x-y)\, dF_l(y) & \text{for } x \geqslant 0 \end{cases}$$

By means of these functions we can immediately calculate the expected number of arrivals in the interval [6] $[0, t]$. Denoting the number of arrivals in this interval by $\underline{N}_t$, we have

$$(2.3) \qquad E\, \underline{N}_t = \sum_{k=1}^{\infty} k\, P\{\underline{N}_t = k\} = \sum_{k=1}^{\infty} P\{\underline{N}_t \geqslant k\} =$$

$$= \sum_{k=1}^{\infty} P\{\underline{S}_k \leqslant t\} = \sum_{k=1}^{\infty} F_k(t) = U(t).$$

Here we have written for abbreviation

$$(2.4) \qquad U(t) \overset{abb}{=} \sum_{k=1}^{\infty} F_k(t) .$$

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5) We use Lebesgue-Stieltjes integrals with the following notations for intervals of integration:

$$\int_{a+}^{b+} f(x)\, dg(x) \overset{abb}{=} \int_{a<x\leqslant b} f(x)\, dg(x) \; ; \qquad \int_{a-}^{b+} f(x)\, dg(x) \overset{abb}{=} \int_{a\leqslant x\leqslant b} f(x)\, dg(x);$$

$$\int_{a+}^{b-} f(x)\, dg(x) \overset{abb}{=} \int_{a<x<b} f(x)\, dg(x) \; ; \qquad \int_{a-}^{b-} f(x)\, dg(x) \overset{abb}{=} \int_{a\leqslant x<b} f(x)\, dg(x).$$

If the integral happens to be an ordinary Lebesgue-integral these distinctions need not (and will not) be made.

6) [ ] denotes a closed interval, ( ) an open interval.

$\underline{x}_1, \underline{x}_2, \dots, \underline{x}_{k+l}$ are independent and have the same distribution function. As $\underline{x}_1 + \dots + \underline{x}_{k+l} \leq t$ implies $\underline{x}_1 + \dots + \underline{x}_k \leq t$ and $\underline{x}_{k+1} + \dots + \underline{x}_{k+l} \leq t$ we have

$$(2.5) \quad F_{k+l}(t) \leq F_k(t)\; P\{\underline{x}_{k+1} + \dots + \underline{x}_{k+l} \leq t\} = F_k(t) \cdot F_l(t).$$

It follows from the assumption $F_1(0) < 1$ that there exists a $\delta > 0$ with $F_1(\delta) < 1$. Now, $\underline{S}_n \leq n\delta$ implies $\min\limits_{1 \leq i \leq n} \underline{x}_i \leq \delta$, therefore

$$(2.6) \qquad F_n(n\delta) \leq 1 - \{1 - F_1(\delta)\}^n < 1.$$

Thus, for fixed $t > 0$, $F_n(t) < 1$ for $n > \frac{t}{\delta}$ and from (2.4)

$$(2.7) \quad U(t) = \sum_{k=1}^{\infty} F_k(t) = \sum_{l=0}^{\infty} \sum_{k=1}^{n} F_{ln+k}(t) \leq \sum_{k=1}^{n} F_k(t) \sum_{l=0}^{\infty} F^l(t) < \infty,$$

proving that $U(t)$ is finite for every $t \geq 0$.
STEIN [9] proved inequalities analogous to (2.6) and (2.7) in a more general case.
$U(t)$ is a monotonic non-decreasing function of t, as follows immediately from the definition (2.4), or the interpretation of $U(t)$ as $E\,\underline{N}_t$.

Lemma 1. For any positive integer k, $U(t)$ is a solution of the integral equation

$$(2.7) \quad U(t) = \sum_{r=1}^{k} F_r(t) + \int_{0-}^{t+} U(t-x)\, dF_k(x).$$

Proof: From (2.2) we have

$$\int_{0-}^{t+} U(t-x)\, dF_k(x) = \int_{0-}^{t+} \sum_{r=1}^{\infty} F_r(t-x)\, dF_k(x) =$$

$$= \sum_{r=1}^{\infty} \int_{0-}^{t+} F_r(t-x)\, dF_k(x) = \sum_{r=k+1}^{\infty} F_r(t) = U(t) - \sum_{r=1}^{k} F_r(t).$$

We define

$$(2.8) \qquad U(t,h) \overset{def}{=} U(t+h) - U(t);$$

BLACKWELL [1], lemma 3, proved

(2.9) $\qquad \mathcal{U}(t,h) \leq \mathcal{U}(h) + 1 .$

This follows from the interpretation of $\mathcal{U}(t)$ as $E\,\underline{N}_t$. $\mathcal{U}(t,h)$ is the expectation of the number of busses arriving in the interval $(t, t+h]$. The time available after the first arrival in $(t, t+h]$ for the other busses which arrive in that interval, is smaller than h , so (2.9) holds.

Equation (2.9) can be used to find an upper bound for $\mathcal{U}(t)$. In fact for every fixed h

$$(2.10) \qquad \mathcal{U}(t) = \sum_{k=1}^{[\frac{t}{h}]} \left\{ \mathcal{U}(t-(k-1)h) - \mathcal{U}(t-kh) \right\} + \mathcal{U}(t - [\tfrac{t}{h}]h) \leq$$
$$\leq \sum_{k=1}^{[\frac{t}{h}]} \mathcal{U}(t-kh, h) + \mathcal{U}(h) \leq$$
$$\leq \frac{t}{h} \left\{ \mathcal{U}(h) + 1 \right\} + \mathcal{U}(h)$$

which leads to

$$(2.11) \qquad \mathcal{U}(t) = O(t) \qquad (t \to \infty) .$$

Using the fact that $\mathcal{U}(t)$ is a monotonic non-decreasing function of t , we can derive from (2.7) the inequality

$$(2.12) \qquad \mathcal{U}(t,h) \leq \frac{F_1(t+h)}{1 - F_1(h)} \leq \frac{1}{1 - F_1(h)} .$$

Formula (2.11) could have been derived from (2.12) instead of (2.9). The result of the next lemma will be a useful formula and is derived by means of partial integration only.

<u>Lemma 2.</u>

$$(2.13) \qquad \int_0^t \mathcal{U}(x) \left\{ 1 - F_1(t-x) \right\} dx = \left(\int_0^t \mathcal{U}(t-x) \left\{ 1 - F_1(x) \right\} dx = \right)$$
$$= \int_{0-}^{t+} (t-x)\, d F_1(x) .$$

<u>Proof</u>: By integrating (2.7) for $k = 1$ we find:

$$\int_0^t \mathcal{U}(x)\, dx = \int_0^t F_1(x)\, dx + \int_0^t dx \int_{0-}^{x+} \mathcal{U}(x-y)\, dF_1(y) =$$
$$= \left[- F_1(x)(t-x) \right]_{0-}^{t+} + \int_{0-}^{t+} (t-x)\, d F_1(x) + \int_{0-}^{t+} dF_1(y) \int_y^t \mathcal{U}(x-y)\, dx =$$

(introduce $z = t - x + y$ in the third term)

$$= \int_{0-}^{t+} (t-x)\, dF_1(x) + \int_{0-}^{t+} dF_1(y) \int_y^t U(t-z)\, dz =$$

$$= \int_{0-}^{t+} (t-x)\, dF_1(x) + \int_0^t U(t-z)\, dz \int_{0-}^{z+} dF_1(y) =$$

$$= \int_{0-}^{t+} (t-x)\, dF_1(x) + \int_0^t U(x)\, F_1(t-x)\, dx ,$$

from which (2.13) follows.

In the next section we need the theorem of Blackwell mentioned above. Introducing the moments of $F_1(x)$, i.e.

$$(2.14) \qquad \mu_k \overset{\text{def}}{=} E\, \underline{x}^k = \int_{0-}^{\infty} x^k\, dF_1(x) ,$$

the theorem can be given the following form:

<u>Theorem 1.</u> If $F_1(x)$ is not a discrete distribution function, with all jumps in points which are integral multiples of a constant, if $F_1(x) = 0$ for $x < 0$ and $F_1(0) < 1$, then

$$\lim_{t \to \infty} U(t, h) = \frac{h}{\mu_1}$$

for every $h > 0$. (If $\mu_1 = \infty$, then $\lim_{t \to \infty} U(t, h) = 0$ for every $h > 0$).

A proof of this theorem is given in the appendix.

3. <u>The limiting distribution and moments of the waiting time</u>.

A passenger arriving at time t at the busstop will take the first bus arriving at or after t. If we denote the waiting time of a passenger arriving at t by $\underline{w}_t$ and the distribution function of this waiting time by $H_t(w)$, we have for $w \geq 0$ and $t \geq 0$:

$$1 - H_t(w) = P\{\underline{w}_t > w\} = P\{\underline{s}_1 > t+w\} + \sum_{k=1}^{\infty} P\{\underline{s}_k < t \,\&\, \underline{s}_{k+1} > t+w\} =$$

$$= 1 - F_1(t+w) + \sum_{k=1}^{\infty} \int_{x<t} \int_{y > t+w-x} dF_1(y)\, dF_k(x) ,$$

therefore, using [7] (2.4)

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[7] By $U(t-)$ we denote $\lim_{\varepsilon \downarrow 0} U(t-\varepsilon)$ and in $\iint dx\, dy$ we first integrate with respect to x, then with respect to y.

$$(3.1)\quad \begin{aligned} H_t(w) &= F_1(t+w) - \sum_{k=1}^{\infty} \int_{x<t} \int_{y>t+w-x} dF_1(y)\, dF_k(x) = \\ &= F_1(t+w) - \sum_{k=1}^{\infty} \int_{w+}^{\infty} \int_{0-}^{t-} dF_k(x)\, dF_1(y) + \sum_{k=1}^{\infty} \int_{w+}^{(t+w)+} \int_{0-}^{(t+w-y)+} dF_k(x)\, dF_1(y) = \\ &= F_1(t+w) - \sum_{k=1}^{\infty} F_k(t-) \int_{w+}^{\infty} dF_1(y) + \sum_{k=1}^{\infty} \int_{w+}^{\infty} F_k(t+w-y)\, dF_1(y) = \\ &= F_1(t+w) + \int_{w+}^{\infty} \{U(t+w-y) - U(t-)\}\, dF_1(y). \end{aligned}$$

Applying the Lebesgue lemma (cf. footnote 3)), which is allowed because of (2.9) and (2.11), we have, for $\mu_1 < \infty$, according to Th.1

$$(3.2)\quad \begin{aligned} H(w) &\overset{\text{def}}{=} \lim_{t\to\infty} H_t(w) = 1 + \lim_{t\to\infty} \int_{w+}^{\infty} \{U(t+w-y) - U(t-)\}\, dF_1(y) = \\ &= 1 - \int_{w+}^{\infty} \frac{y-w}{\mu_1}\, dF_1(y) = \frac{1}{\mu_1} \int_{0-}^{\infty} y\, dF_1(y) - \frac{1}{\mu_1} \int_{w+}^{\infty} y\, dF_1(y) + \\ &+ \frac{w}{\mu_1} \{1 - F_1(w)\} = \frac{1}{\mu_1} \int_{0-}^{w+} y\, dF_1(y) + \frac{w}{\mu_1} \{1 - F_1(w)\} = \\ &= \frac{1}{\mu_1} \int_0^{w} \{1 - F_1(y)\}\, dy. \end{aligned}$$

It is easily seen, that $H(w)$ is again a distribution function. If $\mu_1 = \infty$ we cannot use the Lebesgue lemma in this derivation, but then

$$\lim_{t\to\infty} H_t(w) = 0$$

follows immediately for every finite w, for in this case:

$$\begin{aligned} H_t(w) &= P\,\{\text{at least one bus arrives in the interval } [t, t+w]\} \leq \\ &\leq \text{expectation of the number of arrivals in } [t, t+w] \leq \\ &\leq U(t+w) - U(t-1) = U(t-1, w+1), \end{aligned}$$

and therefore, from theorem 1

$$(3.3)\qquad 0 \leq \lim_{t\to\infty} H_t(w) \leq \lim_{t\to\infty} U(t-1, w+1) = 0.$$

We have thus proved (cf. (3.1), (3.2) and (3.3)):

Theorem 2. The waiting time $\underline{w}_t$ of a passenger arriving at time t, waiting for the next bus, has the distribution function

$$H_t(w) = \begin{cases} 0 & \text{for } w < 0 \\ F_1(t+w) + \int_{w+}^{\infty} \{U(t+w-x) - U(t-)\}\, dF_1(x) & \text{for } w \geq 0 \end{cases}$$

$\lim_{t\to\infty} H_t(w)$ exists and is given by

$$H(w) = \begin{cases} 0 & \text{for } w < 0 \\ \frac{1}{\mu_1} \int_0^w \{1 - F_1(x)\}\, dx & \text{for } w \geq 0 \end{cases}$$

(If $\mu_1 = \infty$, $H(w) = 0$ for every w).

Let us now define for all $k \geq 0$:

$$\nu_k(t) \overset{\text{def}}{=} E\,\underline{w}_t^{\,k} = \int_{0-}^{\infty} w^k\, dH_t(w),$$

$$\nu_k \overset{\text{def}}{=} \int_{0-}^{\infty} w^k\, dH(w).$$

An easy calculation shows

$$\nu_k = \frac{\mu_{k+1}}{(k+1)\mu_1}.$$

We now prove

Theorem 3. If $\mu_1 < \infty$ the moments of the distribution function $H_t(w)$ tend to the moments of $H(w)$ for $t \to \infty$, i.e.

$$\lim_{t\to\infty} E\,\underline{w}_t^{\,k} = \lim_{t\to\infty} \nu_k(t) = \nu_k = \frac{\mu_{k+1}}{(k+1)\mu_1}.$$

If $\mu_1 = \infty$,

$$\lim_{t\to\infty} E\,\underline{w}_t^{\,k} = \lim_{t\to\infty} \nu_k(t) = \infty \quad \text{for every} \quad k > 0.$$

Proof: From (3.1) we find (if $\mu_k < \infty$):

$$(3.4) \qquad \nu_k(t) = \int_{0+}^{\infty} w^k\, dF_1(t+w) + \int_{0+}^{\infty} w^k\, d_w \left\{ \int_{w+}^{\infty} (U(t+w-x) - U(t-))\, dF_1(x) \right\} =$$

$$= \int_{t+}^{\infty} (x-t)^k\, dF_1(x) + \left[w^k \int_{w+}^{\infty} (U(t+w-x) - U(t-))\, dF_1(x) \right]_{0+}^{\infty} +$$

$$- k \int_0^{\infty} w^{k-1} \left\{ \int_{w+}^{\infty} (U(t+w-x) - U(t-))\, dF_1(x) \right\} dw =$$

$$= \int_{t+}^{\infty} (x-t)^k \, dF_1(x) - k \int_0^{\infty} w^{k-1} \left\{ \int_{w+}^{\infty} \left(U(t+w-x) - U(t-)\right) dF_1(x) \right\} dw .$$

If $\mu_{k+1} < \infty$, we find by means of (2.9), (2.11) and the Lebesgue lemma:

$$\lim_{t \to \infty} \nu_k(t) = -k \int_0^{\infty} w^{k-1} \left\{ \int_{w+}^{\infty} \frac{w-x}{\mu_1} \, dF_1(x) \right\} dw =$$

$$= \frac{k}{\mu_1} \int_{0+}^{\infty} \left\{ \int_0^{x} w^{k-1} (x-w) \, dw \right\} dF_1(x) =$$

$$= \frac{k}{\mu_1} \int_{0+}^{\infty} \left\{ \left[x \frac{w^k}{k} - \frac{w^{k+1}}{k+1} \right]_0^x \right\} dF_1(x) =$$

$$= \frac{k}{\mu_1} \int_{0+}^{\infty} \frac{x^{k+1}}{k(k+1)} \, dF_1(x) = \frac{\mu_{k+1}}{(k+1)\mu_1} .$$

Therefore, we have for $k \geq 0$ and $\mu_{k+1} < \infty$:

$$(3.5) \qquad \lim_{t \to \infty} \nu_k(t) = \lim_{t \to \infty} E \, \underline{w}_t^k = \nu_k = \frac{\mu_{k+1}}{(k+1)\mu_1} .$$

If $\mu_{k+1} = \infty$, then

$$(3.6) \qquad \lim_{t \to \infty} \nu_k(t) = \infty .$$

We distinguish between the two cases $\mu_1 = \infty$ and $\mu_1 < \infty$:
If $\mu_1 = \infty$, then (3.6) follows immediately from (3.3), as

$$\nu_k(t) = \int_{0-}^{\infty} w^k \, dH_t(w) \geq \int_A^{\infty} w^k \, dH_t(w) \geq A^k \left\{ 1 - H_t(A) \right\}$$

for every non-negative A.
If $\mu_1 < \infty$, we get for every positive A, using (2.9), (2.11) and the Lebesgue lemma:

$$\lim_{t\to\infty} \nu_k(t) \geqslant \lim_{t\to\infty} \int_{0+}^{A+} w^k dw \left\{ \int_{w+}^{\infty} (U(t+w-x) - U(t-)) dF_1(x) \right\} =$$

$$= \lim_{t\to\infty} \left[w^k \int_{w+}^{\infty} (U(t+w-x) - U(t-) dF_1(x) \right]_{0+}^{A+} +$$

$$- \lim_{t\to\infty} k \int_0^A w^{k-1} \left\{ \int_{w+}^{\infty} (U(t+w-x) - U(t-)) dF_1(x) \right\} dw =$$

$$= \frac{A^{k+1}}{(k+1)\mu} \int_{A+}^{\infty} dF_1(x) + \frac{1}{(k+1)\mu_1} \int_{0-}^{A+} x^{k+1} dF_1(x),$$

and this becomes arbitrarily large if $A \to \infty$.
This completes the proof of Theorem 3.

If the last bus arriving before t came at time $t - \underline{\vartheta}_t$ to the busstop, we may ask for

$$P\left\{\underline{\vartheta}_t > \vartheta \text{ and } \underline{W}_t > W\right\}.$$

Analogous to (3.2) we find

$$(3.7) \quad \lim_{t\to\infty} P\left\{\underline{\vartheta}_t > \vartheta \text{ and } \underline{W}_t > W\right\} = 1 - \frac{1}{\mu_1} \int_0^{\vartheta + W} \left\{1 - F_1(y)\right\} dy.$$

4. Asymptotic formula for $U(t) = E\underline{N}_t$

Theorem 4: If $\mu_2 < \infty$,

$$U(t) = E\underline{N}_t = \frac{t}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - 1 + o(1) \quad (t \to \infty).$$

Proof: For $\mu_1 < \infty$ we obtain from lemma 1 the following relation:

$$\mu_1\left\{1 + U(t-)\right\} - t = \mu_1 U(t-) - \int_0^t U(x)\left\{1 - F_1(t-x)\right\} dx +$$

$$+ \int_{t+}^{\infty} (x-t)\, dF_1(x).$$

But $\mu_1\left\{1 + U(t-)\right\} - t = E\underline{W}_t = \nu_1(t)$, because we have from (3.4):

$$\nu_1(t) = \int_{t+}^{\infty} (x-t)\, dF_1(x) - \int_0^{\infty}\int_{w+}^{\infty} \left\{U(t+w-x) - U(t-)\right\} dF_1(x)\, dw =$$

$$= \int_{t+}^{\infty} (x-t)\, dF_1(x) - \int_0^{\infty}\int_{w+}^{\infty} U(t+w-x)\, dF_1(x)\, dw +$$

$$+ U(t-) \int_0^{\infty} \left\{1 - F_1(w)\right\} dw = \int_{t+}^{\infty} (x-t)\, dF_1(x) +$$

$$- \int_0^{t} U(x) \left\{1 - F_1(t-x)\right\} dx + U(t-) \int_0^{\infty} \left\{1 - F_1(w)\right\} dw =$$

$$= \mu_1 \left\{1 + U(t-)\right\} - t ,$$

for

$$\int_0^{t} U(x) \left\{1 - F_1(t-x)\right\} dx = \int_0^{\infty} U(t-x) \left\{ \int_{x+}^{\infty} dF_1(y) \right\} dx =$$

$$= \int_{0+}^{\infty} \int_0^{y} U(t-x)\, dx\, dF_1(y) = \int_{0+}^{\infty} \int_0^{y} U(t+w-y)\, dw\, dF_1(y) =$$

$$= \int_0^{\infty} \int_{w+}^{\infty} U(t+w-y)\, dF_1(y)\, dw .$$

For $\mu_1 < \infty$ we have thus from (3.5):

$$\lim_{t\to\infty} \left\{\mu_1 (1 + U(t-) - t\right\} = \lim_{t\to\infty} \nu_1(t) = \frac{\mu_2}{2\mu_1}$$

or

$$(4.1) \qquad U(t) = E\,\underline{N}_t = \frac{t}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - 1 + o(1) \qquad (t \to \infty).$$

TÄCKLIND [10] and [11] proved, that

$$U(t) = \frac{t}{\mu_1} + o(t^{2-\delta}) \text{ if } \mu_\delta < \infty \text{ and } 1 \leq \delta < 2,$$

while SMITH [8] gave the following formula for $\mathrm{var}\left\{\underline{N}_t\right\}$:

$$\mathrm{var}\left\{\underline{N}_t\right\} = \frac{\mu_2 - \mu_1^2}{\mu_1^3}\, t + o(t) \quad \text{if} \quad \mu_2 < \infty .$$

5. Some other results

Analogous to the derivation of the distribution of $\underline{w}_t$, we can find the distribution of the arrival interval containing the point t and the limit of this distribution for $t \to \infty$. We denote by $\underline{l}_t$ the length of the time interval between the last bus arriving before t and the first bus arriving at or after t and define

$$K_t(l) \overset{def}{=} P\{\underline{l}_t \leq l\}.$$

Then we have for $t > l \geq 0$:

$$(5.1) \quad 1 - K_t(l) = 1 - F(t-) + \sum_{k=1}^{\infty} \int_{0-}^{(t-l)-} \{1 - F_1(t-x)\} dF_k(x) +$$

$$+ \{1 - F_1(l)\} \sum_{k=1}^{\infty} \int_{(t-l)-}^{t-} dF_k(x) =$$

$$= 1 - F(t-) + U(t-)\{1 - F_1(l)\} + \int_{0-}^{(t-l)-} U(x)\, dF_1(t-x).$$

Using (5.1) one can prove the analogues of Theorems 2 and 3.

Theorem 5: The length $\underline{l}_t$ of the arrival interval overlapping t has the distribution function

$$(5.2) \quad K_t(l) = \begin{cases} 0 & \text{for } l < 0 \\ F_1(t-) - U(t-)\{1 - F_1(l)\} + \int_{0-}^{(t-l)-} U(x)\, dF_1(t-x) & \text{for } l \geq 0. \end{cases}$$

$\lim_{t\to\infty} K_t(l)$ exists and is given by [8])

$$(5.3) \quad K(l) \overset{def}{=} \lim_{t\to\infty} K_t(l) = \begin{cases} 0 & \text{for } l < 0 \\ \frac{1}{\mu_1} \int_{0-}^{l+} x\, dF_1(x) & \text{for } l \geq 0. \end{cases}$$

(If $\mu_1 < \infty$, $\lim_{t\to\infty} K_t(l) = 0$ for every l)

$$(5.4) \quad E\,\underline{l}_t^{\,k} = \int_{0-}^{\infty} l^k \{U(t-) - U((t-l)-)\}\, dF_1(l).$$

If $\mu_1 < \infty$, then $K(l)$ is again a distribution function and

$$\lim_{t\to\infty} E\,\underline{l}_t^{\,k} = \int_{0-}^{\infty} l^k\, dK(l) = \frac{\mu_{k+1}}{\mu_1}.$$

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8) As $\underline{l}_t = \underline{v}_t + \underline{w}_t$, (5.3) is an easy consequence of (3.7).

If $\mu_1 = \infty$

$$\lim_{t \to \infty} E\, L_t^{\kappa} = \infty .$$

The following theorem, here given only for the special case of stochastic variables, whose distribution function $F_1(x)$ satisfies the restrictions made in section 2, is due to SMITH (see [8] , page 16, Theorem 1):

Theorem 6: If $k(t)$ is any function which is zero for negative argument and is bounded, L_1 , and non-increasing in $(0, \infty)$, then

$$(5.5) \qquad \lim_{t \to \infty} \int_{-\infty}^{+\infty} k(t-x)\, dU(x) = \frac{1}{\mu_1} \int_0^{\infty} k(x)\, dx .$$

If $\mu_1 = \infty$, the right hand side of this equation is to be taken as zero.

Proof: This theorem can also be proved by means of partial integration[9] using theorem 1:

If $\int_0^{\infty} k(x)\, dx$ exists, then $k(+\infty) = 0$ and so $\int_{-\infty}^{+\infty} dK(x) = 0.$

Therefore,

$$\int_{-\infty}^{+\infty} k(t-x)\, dU(x) = \int_{-\infty}^{+\infty} U(t-x)\, dK(x) = -U(t)\int_{-\infty}^{+\infty} dK(x) + \int_{-\infty}^{+\infty} U(t-x)\, dK(x) =$$

$$= -\int_{-\infty}^{+\infty} \{U(t) - U(t-x)\}\, dK(x).$$

In addition we have:

$$\int_0^{\infty} k(x)\, dx = \Big[x\, k(x)\Big]_0^{\infty} - \int_{0-}^{\infty} x\, dk(x) = -\int_{0-}^{\infty} x\, dk(x),$$

because $k(x) = o(x^{-1})$ for $x \to \infty$ (which follows from $k(x) \downarrow 0$ and $k(x) \in L_1$).

(5.5) now follows from Theorem 1 (2.9), (2.11) and the Lebesgue lemma. Smith used the last theorem (which he proved, starting

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9) Partial integration is easily justified if $k(x)$ is continuous from the right. If $k(x)$ is not continuous from the right, then we change $k(x)$ into $k^*(x) = \lim_{\epsilon \downarrow 0} k(x+\epsilon)$ in our proof, for $\lim_{t\to\infty} \int_{-\infty}^{+\infty} k(t-x)\, dU(x) = \lim_{t\to\infty} \int_{-\infty}^{+\infty} k^*(t-x)\, dU(x)$ and $\int_0^{\infty} k(x)\, dx = \int_0^{\infty} k^*(x)\, dx$.

from a special Tauberian theorem) to prove a.o. (3.2), (4.1) and Theorem 1.

6. Acknowledgements

A.R. VAN DER BURG [3] recently gave without proof the value $\frac{\mu_2}{2\mu_1}$ for the mean waiting time of a passenger. This report arose out of the authors' attempts to supply the proof of that formula.

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Appendix

Proof of Theorem 1

Lemma 3. If $F_1(x)$ is not a discrete distribution function with all jumps in points, which are integral multiples of a constant, there exists for every $\varepsilon > 0$ and every positive integer l a real number a and a positive integer k such that

$$\frac{1}{k}\sum_{n=1}^{k} \int_{a+j\varepsilon < x \leq a+(j+1)\varepsilon} dF_n(x) > 0 \quad \text{for } j = 0, 1, \ldots, l-1.$$

Proof: (cf. the proof of lemma 4 of BLACKWELL [1]).
Denote by V the set of all points v such that for every open interval I containing v $P\{\underline{x} \in I\} > 0$ 10).
Because of our assumptions on $F_1(x)$ there exist points $v_1, v_2 \in V$ and non-negative integers k_1 and k_2 such that $0 < |k_1 v_1 - k_2 v_2| < \varepsilon$.
The lemma is proved if we can find a non-negative integer m and a real number a, such that

$$a < m \cdot \min(k_1 v_1, k_2 v_2) < a + \varepsilon$$

$$a + (l-1)\varepsilon < m \cdot \max(k_1 v_1, k_2 v_2),$$

for then we can take $k = m \cdot \max(k_1, k_2)$. Therefore the numbers

$$k \overset{\text{def}}{=} \left\{\left[\frac{(l-1)\varepsilon}{|k_1 v_1 - k_2 v_2|}\right] + 1\right\} \max(k_1, k_2),$$

$$a \overset{\text{def}}{=} \left\{\left[\frac{(l-1)\varepsilon}{|k_1 v_1 - k_2 v_2|}\right] + 1\right\} \min(k_1 v_1, k_2 v_2) - \frac{\varepsilon}{2},$$

satisfy the requirements of the lemma.
We will now give an elementary proof of the following theorem, which is a translation of the proof, which Erdös, Feller and Pollard [5] gave for the case which we excluded. BLACKWELL gave in [1] an elementary proof of this theorem:

Theorem 1. If $F_1(x)$ is not a discrete distribution function, with all jumps in points, which are integral multiples of a constant, if $F_1(x) = 0$ for $t < 0$ and $F_1(0) < 1$, then

$$\lim_{t\to\infty} U(t, h) = \frac{h}{\mu_1},$$

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10) Then $P\{\underline{S}_N \in I\} > 0$ as well, for every open interval I which contains $n_1 v_1 + \ldots + n_k v_k$, where $v_1, \ldots, v_k \in V$ and $\sum_{j=1}^{k} n_j = N$.

for every $h > 0$. (If $\mu_1 = \infty$, then $\lim_{t\to\infty} U(t,h) = 0$ for every $h > 0$).

Proof: From (2.7) and (2.8) follows

$$(A.1) \quad U(t,h) = \int_{0-}^{t+} \{U(t+h-x) - U(t-x)\}\, dF_k(x) +$$

$$+ \int_{t+}^{(t+h)+} U(t+h-x)\, dF_k(x) + \sum_{r=1}^{k} \{F_r(t+h) - F_r(t)\}.$$

Therefore, with $G_k(x) \overset{def}{=} \sum_{l=1}^{k} \frac{F_l(x)}{k}$, we have:

$$(A.2) \quad U(t,h) = \int_{0-}^{t+} U(t-x)\, dG_k(x) + \varepsilon_k(t,h),$$

where $\lim_{t\to\infty} \varepsilon_k(t,h) = 0$ for every positive integer k .
From (2.9) and (2.13) follows

$$(A.3) \quad \lim_{t\to\infty} \int_0^t U(t-x,h)\{1-F_1(x)\}\, dx = h,$$

$$(A.4) \quad \lim_{t\to\infty} \int_0^t U^*(t-x,h)\{1-F_1(x)\}\, dx = h,$$

if we define

$$U^*(t,h) \overset{def}{=} U(t) - U(t-h).$$

As $U(t,h)$ is bounded according to (2.9),

$$(A.5) \quad \limsup_{t\to\infty} U(t,h) = U < \infty .$$

Let t_n be a sequence of real numbers, for which

$$(A.6) \quad \lim_{n\to\infty} U(t_n,h) = U .$$

If $\int_{a+}^{b+} dG_k(x) > 0$, then we can prove:

$$(A.7) \quad \lim_{n\to\infty} \int_{a+}^{b+} U(t_n-x,h)\, dG_k(x) = U \int_{a+}^{b+} dG_k(x) .$$

Suppose that for an $\eta > 0$

$$\limsup_{n\to\infty} \int_{a+}^{b+} U(t_n - x, h)\, dG_k(x) \le (U-\eta)\int_{a+}^{b+} dG_k(x),$$

then from (2.9) and (A.2) we have for every $\varepsilon > 0$ and sufficiently large n with appropriate choice of c:

$$U(t_n, h) \le \int_{0-}^{a+} U(t_n - x, h)\, dG_k(x) + (U-\eta)\int_{a+}^{b+} dG_k(x) +$$

$$+ \int_{b+}^{c+} U(t_n - x, h)\, dG_k(x) + \{1 + U(h)\}\int_{c+}^{t_n+} dG_k(x) + \varepsilon_k(t_n, h) \le$$

$$\le (U+\varepsilon)\int_{0-}^{c+} dG_k(x) - \eta\int_{a+}^{b+} dG_k(x) + \{1+U(h)\}\int_{c+}^{t_n+} dG_k(x) + \varepsilon_k(t_n,h) \le$$

$$\le U + 2\varepsilon - \eta\int_{a+}^{b+} dG_k(x),$$

which contradicts (A.5), for sufficiently small ε. Therefore (A.7) is true.

For an arbitrary $\varepsilon > 0$ we can choose A such that 11)

(A.8) $$\int_0^A \{1 - F_1(x)\}\, dx \ge \mu_1 - \varepsilon$$

and according to lemma 3 we can choose a real number a and a positive integer k, such that

$$\frac{1}{k}\int_{a+j\varepsilon < x < a+(j+1)\varepsilon} dG_k(x) > 0$$

for $j = 0, 1, \ldots, l = \left[\frac{A}{\varepsilon}\right]$,

so that from (A.7) for sufficiently large n every interval $a + j\varepsilon < x \le a + (j+1)\varepsilon$ $(j = 0, 1, \ldots, l)$ contains an x_j such that

(A.9) $$U(t_n - x_j, h) \ge U - \varepsilon .$$

If $a + j\varepsilon < x \le a + (j+1)\varepsilon$, then $|x - x_j| < \varepsilon$ and

(A.10) $$|U(t_n - x, h) - U(t_n - x_j, h)| \le |U(t_n - x) - U(t_n - x_j)| +$$

$$+ |U(t_n + h - x) - U(t_n + h - x_j)| \le U(t_n - x, \varepsilon) + U^*(t_n - x, \varepsilon) +$$

$$+ U(t_n + h - x, \varepsilon) + U^*(t_n + h - x, \varepsilon).$$

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11) If $\mu_1 = \infty$ we choose an arbitrary large M and A such that $\int_0^A \{1 - F_1(x)\}\, dx \ge M$. The other necessary changes in the proof if $\mu_1 = \infty$ are clear.

From (A.3) we find

$$h \geqslant \lim_{n\to\infty} \int_0^{(l+1)\varepsilon} U(t_n + a - x, h)\{1 + F_1(x)\}\,dx =$$

$$= \lim_{n\to\infty} \sum_{j=0}^{l} \int_{j\varepsilon}^{(j+1)\varepsilon} U(t_n + a - x, h)\{1 - F_1(x)\}\,dx.$$

Thus, with (A.3), (A.4), (A.8), (A.9) and (A.10)

$$h \geqslant (\mu_1 - \varepsilon)(U - \varepsilon) - \lim_{n\to\infty} \int_0^{(l+1)\varepsilon} \{U(t_n - x, \varepsilon) + U^*(t_n - x, \varepsilon) + U(t_n + h - x, \varepsilon) + U^*(t_n + h - x, \varepsilon)\}.$$

$$\cdot \{1 - F_1(x)\}\,dx \geqslant (\mu_1 - \varepsilon)(U - \varepsilon) - 4\varepsilon.$$

As this is valid for arbitrary $\varepsilon > 0$

$$U \leqslant \frac{h}{\mu_1}.$$

In the same way we can show

$$\liminf_{t\to\infty} U(t, h) \geqslant \frac{h}{\mu_1},$$ so that

$$\lim_{t\to\infty} U(t, h) = \frac{h}{\mu_1} \qquad \text{for every } h > 0,$$

as was to be proved.

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