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General theorems on WILCOXON's test for symmetry

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by

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1. Introduction

In this report some properties of F. WILCOXON's test for symmetry will be proved. A description of this test as well as formulas for the expectation and variance of the test statistic under the hypothesis tested and tables of critical values for the case that no ties are present may be found in [15], [16], [17] and [18] [1]. A recursionformula for the distribution of the test statistic under the null hypothesis for the untied case has been derived by J.W. TUKEY [13]. Further the powerfunction of the test has locally been investigated by E.L. LEHMANN [7] and has been compared with the powerfunctions of the sign test and the tests for symmetry of J. HEMELRIJK ([5] and [6]) and N.V. SMIRNOV [11] by E. RUIST [9].

In section 2 a description of the test will be given and in section 3 some properties of the distribution of the test statistic under the null hypothesis will be proved. In section 4 the relation with WILCOXON's two sample test will be given and in section 5 the consistency of the test will be investigated. In section 6 a combination of the sign test and WILCOXON's test for symmetry will be given and section 7 contains a generalization of the test. All theorems in this report hold for the case without ties as well as the case with ties.

2. Description of the test

Consider m independent random variables $\underline{z}_1, \underline{z}_2, \ldots, \underline{z}_m$ [2] and one observation z_i of $\underline{z}_i$ $(i=1,2,\ldots,m)$. By means of WILCOXON's test for symmetry the hypothesis H_0 may then be tested that the probability distributions of $\underline{z}_1, \underline{z}_2, \ldots, \underline{z}_m$ are all symmetrical with respect to zero.

The test statistic $\underline{T}$ is defined as follows. The observations which are equal to zero are omitted. Let the remaining observations consist of $\underline{a}_i$ times the value $\underline{u}_i$ $(i=1,2,\ldots,\underline{k})$, whereas $\underline{u}_1 < \underline{u}_2 < \ldots < \underline{u}_{\underline{k}}$ and $\underline{b}_i$ times the value $-\underline{u}_i$ $(i=1,2,\ldots,\underline{k})$. Let further

$$(2.1)\qquad \begin{cases} \underline{n}_1 \overset{\text{def}}{=} \sum_{i=1}^{\underline{k}} \underline{a}_i \quad , \ \underline{n}_2 \overset{\text{def}}{=} \sum_{i=1}^{\underline{k}} \underline{b}_i \, , \\ \underline{t}_i \overset{\text{def}}{=} \underline{a}_i + \underline{b}_i \ (i=1,2,\ldots,\underline{k}) \quad , \ \underline{n} \overset{\text{def}}{=} \underline{n}_1 + \underline{n}_2 . \end{cases}$$

The absolute values of the observations are replaced by their ranks according to increasing size, i.e. the $\underline{t}_i$ observations which are

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1) The formulas for the variance and the tables of critical values in [15] and [16] contain some mistakes.

2) Random variables will be distinguished from numbers (e.g. from the value they take in an experiment) by underlining their symbols.

equal to $\underline{u}_i$ or $-\underline{u}_i$ are replaced by

(2.2) $$\underline{r}_i \overset{\text{def}}{=} \sum_{j=1}^{i-1} \underline{t}_j + \frac{\underline{t}_i + 1}{2}.$$

The test is executed under the conditions $\underline{k} = k, \underline{t}_1 = t_1, \underline{t}_2 = t_2, \ldots,$ $\underline{t}_k = t_k$ ((k,t) for short) and the test statistic is

(2.3) $$\underline{T} \overset{\text{def}}{=} \sum_{i=1}^{k} (\underline{a}_i - \underline{b}_i)\, r_i.$$

The distribution of $\underline{T}$ under the hypothesis H_0 and under the condition (k,t) is symmetrical with respect to zero and may be calculated by means of a recursion formulae (cf section 3). Let $P\left[\underline{T}=T \mid (k,t);\ H_0\right]$ denote the probability that $\underline{T}$ assumes the value T under the hypothesis H_0 and under the condition (k,t); let further T_α denote the smallest integer satisfying

(2.4) $$P\left[\,\underline{T} \geq T_\alpha \mid (k,t);\ H_0\right] \leq \alpha.$$

Then the following critical regions are used

(2.5) $$\begin{cases} Z_\ell: & T \leq -T_\alpha, \\ Z_r: & T \geq T_\alpha, \\ Z: & |T| \geq T_{\frac{\alpha}{2}}. \end{cases}$$

Table I (p.20) contains the critical values T_α, for the case that $t_i=1$ for each $i=1,2,\ldots,k=n$, for n=3(1)20 and α =0,005; 0,01; 0,025 and 0,05. This table may also be found in [1] (p.28). A table of the distribution of $\underline{T}$ under the hypothesis H_0 and under the condition $\underline{n} = n$ for the case that $t_i=1$ for each $i=1,2,\ldots,n$, may be found in [1] (p.23-27).
For large values of n the distribution of $\underline{T}$ under the hypothesis H_0 and under the condition (k,t) may be approximated by means of a normal distribution (cf section 3).

Remarks:

1. WILCOXON uses as test statistic for his test the sum of the ranks of the positive observations. Denoting this statistic by $\underline{T}_W$ we have

(2.6) $$\underline{T}_W = \sum_{i=1}^{k} \underline{a}_i r_i = \tfrac{1}{2}\underline{T} + \tfrac{1}{4} n(n+1).$$

In order to obtain a test statistic assuming only integers and having expectation 0 under the hypothesis tested, we use $\underline{T}$ instead of $\underline{T}_W$.
Tables of the lefthandsided critical values of $\underline{T}_W$ for the case that $t_i=1$ for each i, may be found in [17] and in [18] .

These critical values are defined as the values of T_W which minimize $|P[\underline{T}_W=T_W|n;H_o]-\alpha|$.

2. If k=1, i.e. if all observations which are ≠0 have the same absolute value, the ranks r_i are equal to $\frac{1}{2}(n+1)$ and

(2.7) $$T = \tfrac{1}{2}(n+1)(n_1-n_2) = (n+1)(n_1-\tfrac{1}{2}n).$$

Thus in this case the test is identical with the sign test.

3. By means of WILCOXON's test for symmetry one may test the hypothesis H_o' that the distribution of $\underline{z}_i$ is symmetrical with respect to a given point a_i (i=1,2,...,m) by applying the test to $z_1-a_1, z_2-a_2, \ldots, z_m-a_m$.

3. Some properties of the distribution of $\underline{T}$ under the hypothesis H_o and under the conditions $\underline{k}=k$, $\underline{t}_1=t_1, \underline{t}_2=t_2, \ldots, \underline{t}_k=t_k$.

Theorem I

(3.1) $$P[\underline{T}=T|k,t_1,t_2,\ldots,t_k;H_o] = 2^{-t_k}\sum_{i=0}^{t_k}\binom{t_k}{i}P[\underline{T}=T-(2i-t_k)r_k|k-1,t_1,t_2,\ldots,t_{k-1};H_o]$$

Proof: Let E_i denote the event that the tie of size t_k consists of i positive and t_k-i negative observations; then

(3.2) $$P[E_i|H_o] = 2^{-t_k}\binom{t_k}{i}.$$

If E_i occurs then the contribution of the observations in the tie of size t_k to the test statistic is

(3.3) $$\{i-(t_k-i)\}r_k = (2i-t_k)r_k.$$

If on the other hand (3.3) is the contribution of the observations in the tie of size t_k to T then this tie must contain exactly i positive and t_k-i negative observations. Thus

(3.4) $$P[\underline{T}=T|k,t_1,t_2,\ldots,t_k;H_o,E_i] = P[\underline{T}=T-(2i-t_k)r_k|k-1,t_1,t_2,\ldots,t_{k-1};H_o].$$

The recursion formula (3.1) then follows from (3.2), (3.4) and

(3.5) $$P[\underline{T}=T|k,t_1,t_2,\ldots,t_k;H_o] = \sum_{i=0}^{t_k}P[E_i|H_o].P[\underline{T}=T|k,t_1,t_2,\ldots,t_k;H_o,E_i].$$

If $t_i=1$ for each i=1,2,...,n then (3.1) reduces to (cf also [~~10~~ 13] p.15)

(3.6) $$2P[\underline{T}=T|n;H_o] = P[\underline{T}=T-n|n-1;H_o]+P[\underline{T}=T+n|n-1;H_o].$$

Remark 4: The recursion formula (3.1) is analogous to the formula derived by L.J. SMID [10] for the distribution of the test statistic of WILCOXON's two sample test.

Now let κ_ν denote the ν^{th} cumulant of the distribution of $\underline{T}$ under the hypothesis H_o and under the condition (k,t) i.e. κ_ν is the coefficient of $\frac{\tau^\nu}{\nu!}$ in the expansion of $\log\mathcal{E}(e^{\tau\underline{T}}|(k,t);H_o)$. Then

Theorem II

(3.7) $$\varkappa_{2\nu+1} = 0 \qquad \nu = 0, 1, \ldots$$

and

(3.8) $$\varkappa_{2\nu} = \frac{2^{2\nu}(2^{2\nu}-1)B_{2\nu}}{2\nu} \sum_{i=1}^{k} t_i \varkappa_i^{2\nu} \qquad \nu = 1, 2, \ldots,$$

where $B_{2\nu}$ are Bernoulli's numbers.

Proof: From (2.3) it follows that

(3.9) $$T = \sum_{i=1}^{k} (2\underline{a}_i - t_i)\varkappa_i$$

and from (3.9) and the fact that $\underline{a}_1\underline{a}_2, \ldots, \underline{a}_k$ are distributed independently follows

(3.10) $$\mathcal{E}(e^{\tau T} \mid (k,t); H_0) = \prod_{i=1}^{k} \mathcal{E}(e^{\tau\varkappa_i(2\underline{a}_i - t_i)} \mid t_i; H_0).$$

Further $\underline{a}_i$ possessing a binomial probability distribution with parameters $(t_i, \frac{1}{2})$, we have

(3.11) $$\mathcal{E}(e^{\tau\varkappa_i(2\underline{a}_i - t_i)} \mid t_i; H_0) = \left\{\frac{e^{\tau\varkappa_i} + e^{-\tau\varkappa_i}}{2}\right\}^{t_i} \qquad (i = 1, 2, \ldots, k).$$

From (3.10) and (3.11) then follows

(3.12) $$\lg \mathcal{E}(e^{\tau T} \mid (k,t); H_0) = \sum_{i=1}^{k} t_i \lg \frac{e^{\tau\varkappa_i} + e^{-\tau\varkappa_i}}{2} = \sum_{i=1}^{k} t_i \lg \cosh \tau\varkappa_i.$$

Further we have

(3.13) $$\lg \cosh x = \int_0^x \operatorname{tgh} u \, du$$

and

(3.14) $$\operatorname{tgh} u = \sum_{\nu=1}^{\infty} \frac{2^{2\nu}(2^{2\nu}-1)B_{2\nu}}{(2\nu)!} u^{2\nu-1},$$

thus

(3.15) $$\lg \cosh x = \sum_{\nu=1}^{\infty} \frac{2^{2\nu}(2^{2\nu}-1)B_{2\nu}}{(2\nu)!} \frac{x^{2\nu}}{2\nu}.$$

From (3.12) and (3.15) then follows

(3.16) $$\lg \mathcal{E}(e^{\tau T} \mid (k,t); H_0) = \sum_{\nu=1}^{\infty} \frac{\tau^{2\nu}}{(2\nu)!} \frac{2^{2\nu}(2^{2\nu}-1)B_{2\nu}}{2\nu} \sum_{i=1}^{k} t_i \varkappa_i^{2\nu}.$$

Thus the coefficient of $\frac{\tau^{2\nu+1}}{(2\nu+1)!}$ is

(3.17) $$\varkappa_{2\nu+1} = 0 \qquad \nu = 0, 1, \ldots$$

and the coefficient of $\frac{\tau^{2\nu}}{(2\nu)!}$ is

(3.18) $$\varkappa_{2\nu} = \frac{2^{2\nu}(2^{2\nu}-1)B_{2\nu}}{2\nu} \sum_{i=1}^{k} t_i \varkappa_i^{2\nu} \qquad \nu = 1, 2, \ldots.$$

From (3.17) it follows that the distribution of $\underline{T}$ under the hypothesis H_o and under the condition (k,t) is symmetrical and

$$(3.19) \qquad \kappa_1 = \mathcal{E}(\underline{T} \mid (k,t); H_o) = 0.$$

From (3.18) it follows that

$$(3.20) \qquad \kappa_2 = \sigma^2(\underline{T} \mid (k,t); H_o) = \frac{4 \cdot 3 \cdot B_2}{2} \sum_{i=1}^{k} t_i \tau_i^2 = \\ = \frac{n^3 - \sum_{i=1}^{k} t_i^3 + 3n(n+1)^2}{12}.$$

If $t_i=1$ for each $i=1,2,\ldots,n$, then (3.20) reduces to

$$(3.21) \qquad \sigma^2(\underline{T} \mid n; H_o) = \tfrac{1}{6} n(n+1)(2n+1) \quad \text{(cf. [17] and [18])}.$$

A table of σ^2 and σ, according to formula (3.21) for n=21(1)100 may be found in [1] (p.30).

Remarks:

5. From (2.6) and (3.19) it follows that

$$(3.22) \qquad \mathcal{E}(\underline{T}_W \mid (k,t); H_o) = \tfrac{1}{4} n(n+1) \quad \text{(cf. [15], [16], [17] and [18])}.$$

6. (3.12) may also be deduced from the recursion formula. From (3.1) it follows that

$$(3.23) \quad \mathcal{E}(e^{\tau \underline{T}} \mid k, t_1, t_2, \ldots, t_k; H_o) = \\ = 2^{-t_k} \sum_{i=0}^{t_k} \binom{t_k}{i} e^{\tau \tau_k (2i - t_k)} \mathcal{E}(e^{\tau \underline{T}} \mid k-1, t_1, t_2, \ldots, t_{k-1}; H_o) = \\ = \left(\frac{e^{\tau\tau_k} + e^{-\tau\tau_k}}{2}\right)^{t_k} \mathcal{E}(e^{\tau \underline{T}} \mid k-1, t_1, t_2, \ldots, t_{k-1}; H_o)$$

and (3.12) follows from (3.23).

Theorem III: The distance d between the successive values which $\underline{T}$ assumes under the conditions $\underline{k} = k, \underline{t}_1 = t_1, \underline{t}_2 = t_2, \ldots, \underline{t}_k = t_k$ is constant and equal to t_1+1 if and only if

$$(3.24) \qquad \begin{cases} 1. \ t_i - (-1)^i \text{ is a multiple of } t_1+1 \quad (i = 2, 3, \ldots, k), \\ 2. \ t_{i+1} \leq \left\{ \sum_{j=1}^{i} t_j \right\}^2 - \sum_{j=2}^{i} t_j \quad (i = 1, 2, \ldots, k-1). \end{cases}$$

Proof: From (2.3) it follows that $\underline{T}$ may be written in the form

$$(3.25) \qquad \underline{T} = 2 \sum_{i=1}^{k} \underline{a}_i \tau_i - \tfrac{1}{2} n(n+1).$$

Now let

$$(3.26) \qquad \underline{T}' \overset{\text{def}}{=} \underline{T} + \tfrac{1}{2} n(n+1) = 2 \sum_{i=1}^{k} \underline{a}_i \tau_i,$$

then it will be proved that the distance d between the successive values $\underline{T}'$ assumes under the condition (k,t) is constant if and only if (3.24) is satisfied.

From (3.26) follows

$$(3.27)\quad \begin{cases} 1.\ \text{if } a_i = 0 \text{ for each } i \text{ then } T' = 0 \\ 2.\ \text{if } a_1 = \sum_{i=1}^{k} a_i = 1 \quad \text{then } T' = 2r_1 = t_1 + 1. \end{cases}$$

Further

$$(3.28)\qquad r_1 < r_2 < \dots < r_k$$

thus $\underline{T}'$ does not assume values between 0 and t_1+1 , i.e. $d = t_1+1$ if d exists.

Further if T'_0 is a value $\underline{T}'$ assumes, then $\underline{T}'$ also assumes the values T'_0+2r_i and (or) T'_0-2r_i. Thus a necessary condition for the existence of d is that $2r_i$ is a multiple of t_1+1 (i=2,3,...,k) and it may easily be verified that this is equivalent with (3.24.1).

Further if (3.24.1) is satisfied then all values $\underline{T}'$ assumes are multiples of t_1+1 .

Now suppose that (3.24.1) is satisfied then it will be proved that (3.24.2) is a necessary and sufficient condition for the occurrence of <u>all</u> multiples of t_1+1 between $T'_{min}=0$ and $T'_{max}=n(n+1)$.

We first prove that (3.24.2) is a necessary condition.

Consider for any fixed value of i the following two cases

1. $a_j = 0$ for each $j > i$,
2. $a_j \geq 1$ for at least one value of $j > i$.

These two cases are mutually exclusive and one of the two must occur. The greatest value $\underline{T}'$ assumes in case 1 is $2\sum_{j=1}^{i} t_j r_j$ and the smallest value in case 2 is $2r_{i+1}$. Thus if $2r_{i+1} > 2\sum_{j=1}^{i} t_j r_j$ then no values between these two can be assumed by $\underline{T}'$. This means that the difference $2r_{i+1} - 2\sum_{j=1}^{i} t_j r_j$ should not be larger than $d = t_1+1$, or

$$(3.29)\qquad 2r_{i+1} \leqq 2\sum_{j=1}^{i} t_j r_j + t_1 + 1 \qquad (i = 1,2,\dots,k-1),$$

which is equivalent with (3.24.2).

The sufficiency of condition (3.24.2) will be proved by induction. Suppose that it has been proved, for a certain value of i, that in case 1 the distance between the successive values of $\underline{T}'$ are constant and equal to t_1+1 . Then $\underline{T}'$ assumes in this case the values

$$(3.30)\qquad \ell(t_1+1) \qquad\qquad \ell = 0,1,\dots,\frac{2}{t_1+1}\sum_{j=1}^{i} t_j r_j.$$

For i=1 this is true, because $r_1 = \frac{t_1+1}{2}$. Further the contribution of the tie of size t_{i+1} to $\underline{T}'$ equals

$$(3.31)\qquad 2h\,r_{i+1} \qquad\qquad h = 0,1,\dots,t_{i+1}.$$

Thus if $a_j = 0$ for each $j > i+1$ then T' assumes the values

$$(3.32)\qquad 2h\,r_{i+1} + \ell(t_1+1) \qquad \begin{cases} \ell = 0,1,\dots,\frac{2}{t_1+1}\sum_{j=1}^{i} t_j r_j, \\ h = 0,1,\dots,t_{i+1}. \end{cases}$$

For each possible value of h and l these values are multiples of t_1+1 and for any fixed value of h, say h_0, the distance between these values for $l=0,1,\ldots,\frac{2}{t_1+1}\sum_{j=1}^{i} t_j r_j$ is constant and equal to t_1+1. Thus it remains to prove that no gap can arise by raising h from, say h_0 to h_0+1. The smallest value $\underline{T}'$ assumes if $h=h_0+1$ is $2(h_0+1)r_{i+1}$ and the greatest value of $\underline{T}'$ if $h=h_0$ is $2h_0 r_{i+1}+2\sum_{j=1}^{i} t_j r_j$. Thus if

$$(3.33) \qquad 2r_{i+1} \leq 2\sum_{j=1}^{i} t_j r_j + t_1 + 1$$

then no gap arises if h is raised from h_0 to h_0+1, i.e. (3.24.2) is a sufficient condition for the occurrence of all multiples of t_1+1 between 0 and $n(n+1)$.

The condition (3.24) is e.g. satisfied if

$$(3.34) \qquad \begin{cases} t_{2i+1} = t_1 & \text{for} \quad i = 1,2,\ldots,[\frac{k-1}{2}], \\ t_{2i} = 1 & \text{for} \quad i = 1,2,\ldots,[\frac{k}{2}] \end{cases}$$

and a special case of (3.34) is the case that $t_i = 1$ for each $i=1,2,\ldots,k$.

In order to prove the conditional asymptotic normality of $\underline{T}$ under the hypothesis H_0 we consider a sequence $\{\underline{z}_\lambda\}$ $(\lambda=1,2,\ldots)$ of independent random variables.
Let

$$(3.35) \qquad \pi_\lambda \overset{\text{def}}{=} P[\underline{z}_\lambda \neq 0],$$

then if

$$(3.36) \qquad \sum_{\lambda=1}^{\infty} \pi_\lambda = \infty$$

the sequence $\{\underline{z}_\lambda\}$ has, according to the BOREL-CANTELLI lemma (cf e.g. W. FELLER [3] , p.155), probability one of containing infinitely many elements $\neq 0$. Thus, omitting the elements which on observation assume the value 0, an infinite sequence remains.
We further denote by $\underline{k}_\lambda$ the number of different non-zero values assumed by $|\underline{z}_1|,|\underline{z}_2|,\ldots,|\underline{z}_\lambda|$ and by $\underline{t}_{i,\lambda}$ the number of times the i^{th} value of these $\underline{k}_\lambda$ values is assumed. Let further $\underline{T}_\lambda$ be the test statistic for $\underline{z}_1,\underline{z}_2,\ldots,\underline{z}_\lambda$.

$$(3.37) \qquad \underline{n}_\lambda \overset{\text{def}}{=} \sum_{i=1}^{k_\lambda} \underline{t}_{i,\lambda}$$

and

$$(3.38) \qquad \sigma_{0,\lambda}^2 \overset{\text{def}}{=} \sigma^2\{\underline{T}_\lambda \mid k_\lambda, t_{1,\lambda}, t_{2,\lambda},\ldots,t_{k_\lambda,\lambda}; H_0\}.$$

Then $\underline{n}_\lambda \to \infty$ with λ except for a probability 0.

Theorem IV: Let $\{k_\lambda\}, \{n_\lambda\}$ and $\{t_{1,\lambda}\}, \{t_{2,\lambda}\}, \ldots, \{t_{k_\lambda,\lambda}\}$ be arbitrary sequences of non negative integers with $n_\lambda = \sum_{i=1}^{k_\lambda} t_{i,\lambda}$ and let (3.36) be satisfied, then the random variable $\frac{T_\lambda}{\sigma_{0,\lambda}}$ is, under the hypothesis H_0 and under the conditions $\underline{k}_\lambda = k_\lambda$, $\underline{t}_{1,\lambda} = t_{1,\lambda}$, $\underline{t}_{2,\lambda} = t_{2,\lambda}, \ldots$, $\underline{t}_{k_\lambda,\lambda} = t_{k_\lambda,\lambda}$, for $\lambda \to \infty$ asymptotically normally distributed with zero mean and variance 1.

Proof: The notation will be simplified by omitting the index λ. It will be proved that

(3.39) $$\lim_{\lambda\to\infty} \frac{\kappa_{2\nu}}{\sigma_0^{2\nu}} = 0 \qquad \text{for } \nu = 2, 3, \ldots .$$

From (2.2) it follows that

(3.40) $$\varkappa_i < 2n \qquad (i = 1, 2, \ldots, k),$$

thus (cf.(3.8))

(3.41) $$\kappa_{2\nu} < \frac{2^{4\nu}(2^{2\nu}-1)B_{2\nu}}{2\nu} n^{2\nu+1} \sum_{i=1}^{k} \frac{t_i}{n} = \frac{2^{4\nu}(2^{2\nu}-1)B_{2\nu}}{2\nu} n^{2\nu+1}.$$

Further (cf.(3.20))

(3.42) $$\sigma_0^2 \geq \frac{1}{4} n(n+1)^2,$$

thus

(3.43) $$\lim_{n\to\infty} \frac{\kappa_{2\nu}}{\sigma_0^{2\nu}} = 0 \qquad \nu = 2, 3, \ldots .$$

and (3.39) then follows from the fact that n tends to infinity with λ.

4. The relation with WILCOXON's two sample test

From (2.3) it follows that $\underline{T}$ may be written in the form

(4.1) $$\begin{aligned} \underline{T} &= 2\sum_{i=1}^{k} \underline{a}_i\Big(\varkappa_i - \frac{1}{n}\sum_{j=1}^{k} t_j \varkappa_j\Big) + (\underline{n}_1 - \tfrac{1}{2}n)(n+1) = \\ &= \sum_{i=1}^{k} \underline{a}_i\Big(\sum_{j<i} t_j - \sum_{j>i} t_j\Big) + (\underline{n}_1 - \tfrac{1}{2}n)(n+1) = \\ &= \underline{\tilde{W}} + (\underline{n}_1 - \tfrac{1}{2}n)(n+1), \end{aligned}$$

with

(4.2) $$\underline{\tilde{W}} \overset{\text{def}}{=} \underline{W} - \underline{n}_1 \underline{n}_2,$$

where $\underline{W}$ is the test statistic of WILCOXON's two sample test applied to the positive observations as the first and the absolute values of the negative observations as the second sample [3].

Further the hypothesis H_0 implies, under the conditions (k,t) and $\underline{n}_1 = n_1$, the hypothesis H_0'' that the positive observations are a random sample without replacement from the absolute values of all

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3) The test statistic of WILCOXON's two sample test for the samples $x_1, x_2, \ldots, x_{n_1}$ and $y_1, y_2, \ldots, y_{n_2}$ is defined here as two times the number of pairs (x_i, y_j) with $x_i > y_j$ increased by the number of pairs (x_i, y_j) with $x_i = y_j$ $(i = 1, 2, \ldots, n_1, j = 1, 2, \ldots, n_2)$ (cf [14]).

observations (cf [6] p.71 and [2] p.307). The mean and variance of $\underline{T}$ under the hypothesis H_o and under the condition (k,t) thus also follow from the well known formulae for the mean and variance of $\underline{W}$ under the hypothesis H_o''. We have

(4.3) $$\mathcal{E}(\underline{T} \mid (k,t), n_1 ; H_o) = (n+1)(n_1 - \tfrac{1}{2} n)$$

and

(4.4) $$\sigma^2(\underline{T} \mid (k,t), n_1 ; H_o) = \frac{n_1 n_2 (n^3 - \sum_{i=1}^{k} t_i^3)}{3n(n-1)} .$$

From (4.3) and (4.4) then follows

(4.5) $$\mathcal{E}(\underline{T} \mid (k,t); H_o) = \mathcal{E}\{ \mathcal{E}(\underline{T} \mid (k,t), \underline{n}_1 ; H_o) \mid (k,t); H_o\} = (n+1)\, \mathcal{E}(\underline{n}_1 - \tfrac{1}{2} n \mid (k,t); H_o) = 0 \qquad \text{(cf (3.19))}$$

and

(4.6) $$\sigma^2(\underline{T} \mid (k,t); H_o) =$$
$$= \sigma^2\{ \mathcal{E}(\underline{T} \mid (k,t), \underline{n}_1 ; H_o) \mid (k,t); H_o\} + \mathcal{E}\{\sigma^2(\underline{T} \mid (k,t), \underline{n}_1 ; H_o) \mid (k,t); H_o\} =$$
$$= (n+1)^2 \sigma^2(\underline{n}_1 - \tfrac{1}{2} n \mid (k,t); H_o) + \frac{n^3 - \sum_{i=1}^{k} t_i^3}{3n(n-1)} \mathcal{E}(\underline{n}_1 \underline{n}_2 \mid (k,t); H_o) =$$
$$= \frac{3n(n+1)^2 + n^3 - \sum_{i=1}^{k} t_i^3}{12} \qquad \text{(cf. (3.20))}.$$

5. The consistency of the test

We again consider the sequence $\{\underline{z}_\lambda\}$ and an alternative hypothesis stating that the distributions of $\underline{z}_\lambda$ under the condition $\underline{z}_\lambda \neq 0$ are, for $\lambda = 1, 2, \ldots$, identical. Let $x_1, x_2, \ldots, x_{n_{1,\lambda}}$ denote the positive observations and $y_1, y_2, \ldots, y_{n_{2,\lambda}}$ the negative observations, with $n_{1,\lambda} + n_{2,\lambda} = n_\lambda$; let further

(5.1) $$\begin{cases} p \overset{\text{def}}{=} P[\underline{z}_\lambda > 0 \mid \underline{z}_\lambda \neq 0] \\ q \overset{\text{def}}{=} 1 - p \end{cases} \qquad \lambda = 1, 2, \ldots$$

and

(5.2) $$\theta \overset{\text{def}}{=} P[\underline{x}_\lambda > \underline{y}_\mu] - P[\underline{x}_\lambda < \underline{y}_\mu] \qquad \lambda, \mu = 1, 2, \ldots .$$

<u>Theorem V: If (3.36) is satisfied then the test based on the critical region Z (cf.(2.5)) is, for $\lambda \to \infty$, consistent for the class of alternative hypotheses</u>

(5.3) $$| p - \tfrac{1}{2} + pq\theta | > 0.$$

<u>The tests based on the critical regions Z_ℓ and Z_r respectively are consistent for the classes of alternative hypotheses</u>

(5.4) $$p - \tfrac{1}{2} + pq\theta < 0$$

and

(5.5) $$p - \tfrac{1}{2} + pq\theta > 0$$

respectively and not consistent for the classes of alternative hypotheses

(5.6) $$p - \tfrac{1}{2} + pq\theta > 0$$

and

(5.7) $$p - \tfrac{1}{2} + pq\theta < 0$$

respectively.

All tests mentioned are, for sufficiently small α, not consistent for the class of alternative hypotheses

(5.8) $$p - \tfrac{1}{2} + pq\theta = 0.$$

Proof: The notation will be simplified by omitting the index λ.

From (4.1) it follows that

(5.9) $$\underline{T} = \sum_{i=1}^{\underline{n}_1}\sum_{j=1}^{\underline{n}_2} \operatorname{sgn}(\underline{x}_i - \underline{y}_j) + (n+1)(\underline{n}_1 - \tfrac{1}{2}n).$$

Let H be an alternative hypothesis stating that the distributions of $\underline{z}_\lambda$ under the condition $\underline{z}_\lambda \neq 0$ are, for $\lambda = 1, 2, \ldots$, identical. Then it follows from (5.1), (5.2) and (5.9) that

(5.10) $$\mathcal{E}(\underline{T} \mid n, n_1; H) = n_1 n_2 \theta + (n+1)(n_1 - \tfrac{1}{2}n),$$

thus

(5.11) $$\mu \overset{\text{def}}{=} \mathcal{E}(\underline{T} \mid n; H) = \mathcal{E}\{\mathcal{E}(\underline{T} \mid n, \underline{n}_1; H) \mid n; H\} =$$
$$= n(n-1)pq\theta + n(n+1)(p - \tfrac{1}{2}).$$

Let further

(5.12) $$\begin{cases} 1. & \underline{s}^2 \overset{\text{def}}{=} \sigma^2\{\underline{T} \mid n, t_1, t_2, \ldots, t_k; H_0\}, \\ 2. & c_1^2 \overset{\text{def}}{=} \tfrac{1}{4}n(n+1)^2, \\ 3. & c_2^2 \overset{\text{def}}{=} \tfrac{1}{6}n(n+1)(2n+1), \end{cases}$$

then

(5.13) $$c_1^2 \leqq \underline{s}^2 \leqq c_2^2.$$

Further

(5.14) $$\sigma^2 \overset{\text{def}}{=} \sigma^2\{\underline{T} \mid n; H\} =$$
$$= \mathcal{E}\{\sigma^2(\underline{T} \mid n, \underline{n}_1; H) \mid n; H\} + \sigma^2\{\mathcal{E}(\underline{T} \mid n, \underline{n}_1; H) \mid n; H\}.$$

From (5.10) it follows that

(5.15) $$\sigma^2\{\mathcal{E}(\underline{T} \mid n, \underline{n}_1; H) \mid n; H\} = \sigma^2\{\theta \underline{n}_1 \underline{n}_2 + (n+1)(\underline{n}_1 - \tfrac{1}{2}n) \mid n; H\} = O(n^3)$$

and the coefficient of n^3 in (5.15) is

(5.16) $$pq(\theta + 1 - 2pq\theta)^2.$$

Further (cf (5.9))

(5.17) $$\sigma^2(\underline{T} \mid n, n_1; H) = \sigma^2\Big\{\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \operatorname{sgn}(\underline{x}_i - \underline{y}_j) \mid n, n_1; H\Big\}$$

and from D.J. STOKER ([12] p.67-68) it follows that

(5.18) $$\sigma^2\left\{\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \operatorname{sgn}(\underline{x}_i-\underline{y}_j)\,|\,n,n_1;H\right\} \leqq n_1 n_2 (n+1),$$

thus

(5.19) $$\mathcal{E}\left\{\sigma^2(\underline{T}\,|\,n,\underline{n}_1;H)\,|\,n;H\right\} \leqq n(n^2-1)pq.$$

Thus (cf (5.14))

(5.20) $$\sigma^2(\underline{T}\,|\,n;H) = \mathcal{O}(n^3)$$

and the coefficient of n^3 in (5.20) is

(5.21) $$\leqq pq(\theta+1-2pq\theta)^2 + pq \leqq 5/4.$$

Now first consider the case that

(5.22) $$p-\tfrac{1}{2}+pq\theta<0.$$

We have

(5.23) $$\lim_{\lambda\to\infty} P[\underline{T}\notin Z_\ell\,|\,n;H] = \lim_{\lambda\to\infty} P[\underline{T} > -\xi_\alpha \underline{s}\,|\,n;H] \leqq$$
$$\leqq \lim_{\lambda\to\infty} P\left[\frac{\underline{T}-\mu}{\sigma} > -\frac{\xi_\alpha c_2+\mu}{\sigma}\,\Big|\,n;H\right],$$

where ξ_α is defined by

(5.24) $$\frac{1}{\sqrt{2\pi}}\int_{\xi_\alpha}^{\infty} e^{-\frac{1}{2}x^2}\,dx = \alpha.$$

From (5.12), (5.20), (5.22) and the fact that n tends to infinity with λ it follows that $-\frac{\xi_\alpha c_2+\mu}{\sigma}$ is positive for sufficiently large λ; thus according to the inequality of Bienaymé-Tschebycheff

(5.25) $$\lim_{\lambda\to\infty} P[\underline{T}\notin Z_\ell\,|\,n;H] \leqq \lim_{\lambda\to\infty}\frac{\sigma^2}{(\xi_\alpha c_2+\mu)^2} = 0.$$

Thus the test based on the critical region Z_ℓ is, for $\lambda\to\infty$, consistent for the class of alternative hypotheses (5.22).

If

(5.26) $$p-\tfrac{1}{2}+pq\theta>0$$

then

(5.27) $$\lim_{\lambda\to\infty} P[\underline{T}\in Z_\ell\,|\,n;H] = \lim_{\lambda\to\infty} P[\underline{T}\leqq -\xi_\alpha \underline{s}\,|\,n;H] \leqq$$
$$\leqq \lim_{\lambda\to\infty} P\left[\frac{\underline{T}-\mu}{\sigma} \leqq -\frac{\xi_\alpha c_1+\mu}{\sigma}\,\Big|\,n;H\right] \leqq \lim_{\lambda\to\infty}\frac{\sigma^2}{(\xi_\alpha c_1+\mu)^2} = 0,$$

$-\frac{\xi_\alpha c_1+\mu}{\sigma}$ being negative for sufficiently large λ. Thus the test based on Z_ℓ is, for $\lambda\to\infty$, not consistent for the class of alternatives (5.26).

Finally if

(5.28) $$p-\tfrac{1}{2}+pq\theta=0$$

then

$$(5.29)\quad \lim_{\lambda\to\infty} P[\,\underline{T}\in Z_\ell \mid n;H] = \lim_{\lambda\to\infty} P[\,\underline{T} \leqq -\xi_\alpha \underline{s} \mid n;H] \leqq$$
$$\leqq \lim_{\lambda\to\infty} P\Big[\frac{\underline{T}-\mu}{\sigma} \leqq -\frac{\xi_\alpha c_1+\mu}{\sigma} \mid n;H\Big] \leqq \lim_{\lambda\to\infty}\frac{\sigma^2}{(\xi_\alpha c_1+\mu)^2} = \lim_{\lambda\to\infty}\frac{\sigma^2}{(\xi_\alpha c_1)^2}.$$

Thus if

$$(5.30)\qquad \xi_\alpha > \lim_{\lambda\to\infty}\frac{\sigma}{c_1}$$

then the test based on Z_ℓ is for $\lambda\to\infty$ not consistent for the class of alternative hypotheses (5.28) and from (5.12) and (5.21) follows

$$(5.31)\qquad \lim_{\lambda\to\infty}\frac{\sigma}{c_1} \leqq \sqrt{5}.$$

The proofs for the tests based on Z_κ and Z are analogous.

<u>Theorem VI: If the distributions of $\underline{z}_1, \underline{z}_2, \ldots, \underline{z}_m$ are identical and symmetrical with respect to a then</u>

$$(5.32)\quad \begin{cases} 1.\ \ p-\tfrac{1}{2}+pq\theta = 0 \ \text{ if } \ a=0, \\ 2.\ \ a(p-\tfrac{1}{2}+pq\theta) > 0 \ \text{ if } \ a\neq 0. \end{cases}$$

<u>Proof</u>: Let

$$(5.33)\qquad H(z) \overset{\text{def}}{=} P[\underline{z}_i \leqq z]$$

and (cf.(3.35))

$$(5.34)\qquad \pi \overset{\text{def}}{=} P[\underline{z}_i \neq 0].$$

Then (cf (5.1))

$$(5.35)\qquad p = \frac{\int_{+0}^{\infty} dH(z)}{\pi}, \qquad q = \frac{\int_{-\infty}^{-0} dH(z)}{\pi}.$$

We first consider the case that $a>0$; then $p\geqq\frac{1}{2}$. From the fact that the distribution of $\underline{z}_i$ is symmetrical with respect to a it follows that

$$(5.36)\qquad q = \frac{\int_{2a+0}^{\infty} dH(z)}{\pi}.$$

If further

$$(5.37)\qquad F(x) \overset{\text{def}}{=} P[\underline{x}_i \leqq x], \qquad G(y) \overset{\text{def}}{=} P[\underline{y}_i \leqq y]$$

then

$$(5.38)\qquad dF(x) = \frac{dH(x)}{p}, \qquad F(x) = \frac{\int_{+0}^{x} dH(u)}{p}$$

and from the symmetry of the distribution of $\underline{z}_i$ it follows that

$$(5.39)\qquad dG(y) = \frac{dH(y+2a)}{q}, \qquad G(y) = \frac{\int_{2a+0}^{2a+y} dH(u)}{q}.$$

If $q>0$ then

$$(5.40)\qquad \theta = P[\underline{x}_i > \underline{y}_j] - P[\underline{x}_i < \underline{y}_j] >$$
$$> P[\underline{x}_i > \underline{y}_j + 2a] - P[\underline{x}_i < \underline{y}_j + 2a] =$$

$$= \int_{2a+0}^{\infty} \frac{dH(x)}{p} \int_{2a+0}^{x} \frac{dH(u)}{q} - \int_{+0}^{\infty} \frac{dH(x+2a)}{q} \int_{+0}^{x+2a} \frac{dH(u)}{p},$$

thus if $q>0$ then

$$(5.41) \quad pq\theta > \int_{2a+0}^{\infty} dH(x) \int_{2a+0}^{x} dH(u) - \int_{+0}^{\infty} dH(x+2a) \int_{+0}^{x+2a} dH(u) =$$

$$= \int_{2a+0}^{\infty} dH(x) \int_{2a+0}^{x} dH(u) - \int_{2a+0}^{\infty} dH(x) \int_{+0}^{x} dH(u) =$$

$$= \int_{2a+0}^{\infty} dH(x) \int_{2a+0}^{\infty} dH(u) - \int_{2a+0}^{\infty} dH(x) \int_{+0}^{\infty} dH(u) =$$

$$= \pi^2 q^2 - \pi^2 pq = \pi^2 q (q-p).$$

Thus if $q>0$ then

$$(5.42) \quad p - \tfrac{1}{2} + pq\theta > p - \tfrac{1}{2} + \pi^2 q(q-p) = (p-q)(\tfrac{1}{2} - \pi^2 q) >$$

$$> (p-q)(\tfrac{1}{2} - q) = \tfrac{1}{2}(p-q)^2 \geqq 0.$$

Further if $q=0$ then $p=1$ and then

$$(5.43) \quad p - \tfrac{1}{2} + pq\,\theta = p - \tfrac{1}{2} > 0.$$

Thus $p-\frac{1}{2}+pq\theta$ is positive if a is positive.

The proof for $a<0$ is analogous.

From the theorems V and VI it follows that if the distributions of z_λ are, for $\lambda = 1, 2, \ldots$, identical and symmetrical with respect to a then the test based on Z is, for $\lambda \to \infty$, consistent for the class of alternative hypotheses with

$$(5.44) \quad a \neq 0.$$

The tests based on Z_ℓ and Z_r respectively are consistent for the classes of alternative hypotheses with

$$(5.45) \quad a < 0$$

and

$$(5.46) \quad a > 0$$

respectively and not consistent for the classes of alternative hypotheses with

$$(5.47) \quad a > 0$$

and

$$(5.48) \quad a < 0$$

respectively.

6. A combination of the sign test and WILCOXON's test for symmetry.

In this section a test for the hypothesis H_0 will be described which is a combination of the sign test and WILCOXON's test for symmetry.

Let $n_{1,\alpha}$ denote the smallest integer satisfying

(6.1) $$P[\underline{n}_1 \geq n_{1,\alpha} \mid n; H_0] \leq \alpha,$$

then the following critical regions are used (cf.(2.4))

(6.2) $$\begin{cases} Z'_\ell & : \; n_1 \leq n - n_{1,\alpha_1} \text{ and(or) } T \leq -T_{\alpha_2}, \\ Z'_r & : \; n_1 \geq n_{1,\alpha_1} \text{ and(or) } T \geq T_{\alpha_2}, \\ Z' & : \; |n_1 - \frac{1}{2}n| \geq n_{1,\frac{\alpha_1}{2}} - \frac{1}{2}n \text{ and(or)} |T| \geq T_{\frac{\alpha_2}{2}}. \end{cases}$$

Now let

(6.3) $$\begin{cases} \varepsilon_1 \stackrel{\text{def}}{=} P[\underline{n}_1 \geq n_{1,\alpha_1} \mid n; H_0], \\ \varepsilon_2 \stackrel{\text{def}}{=} P[\underline{T} \geq T_{\alpha_2} \mid (k,t); H_0] \end{cases}$$

and let ε denote the size of the critical region Z'_r, then

(6.4) $$\varepsilon = \varepsilon_1 + (1-\varepsilon_1) P[\underline{T} \geq T_{\alpha_2} \mid \underline{n}_1 < n_{1,\alpha_1}, (k,t); H_0] =$$
$$= \varepsilon_1 + (1-\varepsilon_1) \sum_{i=0}^{n_{1,\alpha_1}-1} \frac{2^{-n}\binom{n}{i}}{1-\varepsilon_1} P[\underline{T} \geq T_{\alpha_2} \mid \underline{n}_1 = i, (k,t); H_0].$$

Analogous formulae hold for the other onesided and for the twosided test.

Thus, $\underline{T} - (n+1)(n_1 - \frac{1}{2}n)$ possessing under the hypothesis H_0 and under the conditions (k,t) and $\underline{n}_1 = n_1$ the same probability distribution as the test statistic $\underline{\tilde{W}}$ of WILCOXON's twosample test under the hypothesis H_0'' (cf. section 4), ε may be calculated from (6.4) for each α_1, α_2 and n .

On the other hand the critical regions Z'_ℓ, Z'_r and Z' are not uniquely determined by ε and n . One may now proceed e.g. in one of the following two ways.

I Say one wants to test the hypothesis H_0 by means of the combination of the sign test and WILCOXON's test for symmetry with level of significance α . Then for each $\varepsilon_1 < \alpha$ let $\varepsilon_{2,max}$ denote the largest value of ε_2 satisfying $\varepsilon \leq \alpha$. This value $\varepsilon_{2,max}$ may be found from (6.4).

Further, for this value $\varepsilon_{2,max}$ of ε_2, let $\varepsilon_{1,max}$ denote the largest value of ε_1 satisfying $\varepsilon \leq \alpha$; of these pairs ($\varepsilon_{1,max}$, $\varepsilon_{2,max}$) choose the one with the smallest difference $|\varepsilon_{1,max} - \varepsilon_{2,max}|$.

If two pairs of values have the same value of $|\varepsilon_{1,max} - \varepsilon_{2,max}|$ then choose the pair with the largest value of ε .

II Take $\alpha_1 = \alpha_2$ and choose the largest value of $\alpha_1 = \alpha_2 \leq \alpha$ satisfying $\varepsilon \leq \alpha$.

These two procedures do not always give the same critical values, but if they give different results then in general the first procedure gives a larger value of ε. Further it will be clear that the two procedures are asymptotically, for $n \to \infty$, identical.

Table II (p.21) contains the critical values of $Z'_\varkappa$, for the case that $t_i = 1$ for each $i = 1, 2, \ldots, k$, for n=5(1)20 and α =0,005; 0,01; 0,025 and 0,05. In this table we used $\underline{v} \overset{\text{def}}{=} \underline{n}_1 - \underline{n}_2$ as test statistic for the sign test instead of $\underline{n}_1$; then $\underline{v}$ is, under the hypothesis H_0, distributed symmetrically with respect to zero. This table may also be found in [1] (p.31).

In the following an approximation for α will be given for large values of n. First we prove the following theorems.

<u>Theorem VII</u>: <u>If</u> $\varkappa_{s,r}$ ($s = 0, 1, \ldots; r = 0, 1, \ldots; s + r > 0$) <u>are the cumulants of the simultaneous probability distribution of $\underline{T}$ and $\underline{n}_1 - \frac{1}{2}n$ under the hypothesis H_0 and under the condition (k,t) then</u>

(6.5) $$\varkappa_{s,2\nu+1-s} = 0 \qquad \nu \geqq 0, s \geqq 0$$

and

(6.6) $$\varkappa_{s,2\nu-s} = \frac{2^s(2^{2\nu}-1)B_{2\nu}}{2\nu} \sum_{i=1}^{k} t_i \varkappa_i^s \qquad \nu > 0, s \geqq 0.$$

<u>Proof</u>: In the same way as in section 3 we find

(6.7) $$\lg \mathcal{E}(e^{\tau_1 \underline{T} + \tau_2(\underline{n}_1 - \frac{1}{2}n)} \mid (k,t); H_0) =$$
$$= \sum_{i=1}^{k} t_i \lg \cosh(\tau_1 \varkappa_i + \tfrac{1}{2}\tau_2) =$$
$$= \sum_{\nu=1}^{\infty} \frac{2^{2\nu}(2^{2\nu}-1)B_{2\nu}}{2\nu} \sum_{i=1}^{k} t_i (\tau_1 \varkappa_i + \tfrac{1}{2}\tau_2)^{2\nu} =$$
$$= \sum_{\nu=1}^{\infty} \frac{(2^{2\nu}-1)B_{2\nu}}{2\nu} \sum_{j=0}^{2\nu} \frac{\tau_1^j \tau_2^{2\nu-j}}{j!(2\nu-j)!} 2^j \sum_{i=1}^{k} t_i \varkappa_i^j.$$

Thus the coefficient of $\dfrac{\tau_1^s \tau_2^{2\nu+1-s}}{s!(2\nu+1-s)!}$ is

(6.8) $$\varkappa_{s,2\nu+1-s} = 0 \qquad \nu \geqq 0, s \geqq 0$$

and the coefficient of $\dfrac{\tau_1^s \tau_2^{2\nu-s}}{s!(2\nu-s)!}$ is

(6.9) $$\varkappa_{s,2\nu-s} = \frac{2^s(2^{2\nu}-1)B_{2\nu}}{2\nu} \sum_{i=1}^{k} t_i \varkappa_i^s.$$

From (6.9) it follows that

(6.10) $$\varkappa_{2,0} = \sigma^2(\underline{T} \mid (k,t); H_0) = \frac{n^3 - \sum_{i=1}^{k} t_i^3 + 3n(n+1)^2}{12} \text{ (cf. 3.20)),}$$

(6.11) $$\varkappa_{0,2} = \sigma^2(\underline{n}_1 \mid n; H_0) = \tfrac{1}{4} n$$

and

(6.12) $$\varkappa_{1,1} = \operatorname{cov}(\underline{T}, \underline{n}_1 \mid (k,t); H_0) = \tfrac{1}{4} n(n+1).$$

In order to prove the conditional asymptotic normality of the simultaneous distribution of $\underline{T}$ and $\underline{n}_1$ under the hypothesis H_0 we again consider the sequence $\{z_\lambda\}$ $(\lambda = 1, 2, \ldots)$.

Theorem VIII: If $\{k_\lambda\}$ and $\{t_{1,\lambda}\},\{t_{2,\lambda}\},\dots,\{t_{k_\lambda,\lambda}\}_{k_\lambda}$ are arbitrary sequences of non negative integers with $n_\lambda = \sum_{i=1} t_{i,\lambda}$, if (3.36) is satisfied and if moreover

$$\lim_{\lambda\to\infty} \sum_{i=1}^{k_\lambda} \frac{t_{i,\lambda}^3}{n_\lambda^3} \tag{6.13}$$

exists and is <1, then the random variables

$$\frac{T_\lambda}{\sigma_{0,\lambda}} \quad \text{and} \quad \frac{\mathfrak{B}_{1,\lambda} - \frac{1}{2}n_\lambda}{\frac{1}{2}\sqrt{n_\lambda}} \tag{6.14}$$

possess, under the hypothesis H_0 and under the conditions $\underline{k}_\lambda = k_\lambda$, $\underline{t}_{1,\lambda} = t_{1,\lambda}, \underline{t}_{2,\lambda} = t_{2,\lambda}, \dots, \underline{t}_{k_\lambda,\lambda} = t_{k_\lambda,\lambda}$, asymptotically for $\lambda\to\infty$ a two dimensional normal probability distribution with zero means, variances 1 and correlation coefficient

$$\rho \overset{\text{def}}{=} \lim_{\lambda\to\infty} \frac{1}{\sqrt{1 + \dfrac{n_\lambda^3 - \sum_{i=1}^{k_\lambda} t_{i,\lambda}^3}{3n_\lambda(n_\lambda+1)^2}}}\,. \tag{6.15}$$

Proof: The index λ is omitted.
It will be proved that

$$\lim_{\lambda\to\infty} \frac{\kappa_{s,2\nu-s}}{(\kappa_{2,0})^{s/2}(\kappa_{2,0})^{\nu-s/2}} = 0 \quad \text{for } \nu>1 \text{ and } s\geq 0 \tag{6.16}$$

and that

$$\rho = \lim_{\lambda\to\infty} \frac{\kappa_{1,1}}{(\kappa_{2,0})^{1/2}(\kappa_{0,2})^{1/2}} \tag{6.17}$$

exists and is <1.
From (6.6) it follows (cf. also (3.41))

$$\kappa_{s,2\nu-s} \leq \frac{2^{2s}(2^{2\nu}-1)B_{2\nu}}{2\nu}\, n^{s+1}; \tag{6.18}$$

further

$$\kappa_{2,0} \geq \tfrac{1}{4}n(n+1)^2\,, \quad \kappa_{0,2} = \tfrac{1}{4}n. \tag{6.19}$$

Thus

$$\lim_{n\to\infty} \frac{\kappa_{s,2\nu-s}}{(\kappa_{2,0})^{s/2}(\kappa_{0,2})^{\nu-s/2}} = 0 \quad \text{for } \nu>1 \text{ and } s\geq 0\,. \tag{6.20}$$

Then (6.16) follows from the fact that n tends to infinity with λ.
Further

$$\frac{\kappa_{1,1}}{(\kappa_{2,0})^{1/2}(\kappa_{0,2})^{1/2}} = \frac{1}{\sqrt{1 + \dfrac{n^3 - \sum_{i=1}^{k} t_i^3}{3n(n+1)^2}}}\,, \tag{6.21}$$

thus (6.17) follows from the fact that n tends to infinity with λ and that $\lim_{\lambda\to\infty} \sum_{i=1}^{k} \frac{t_i^3}{n^3}$ exists and is < 1.

From theorem VIII it follows that, for Z'_n and $\alpha_1=\alpha_2=\alpha'$, α may be approximated by

$$(6.22) \qquad \alpha \approx 2\alpha' - \frac{1}{2\pi\sqrt{1-r^2}} \int_{\xi_{\alpha'}}^{\infty} \int_{\xi_{\alpha'}}^{\infty} e^{-\frac{1}{2}\frac{x^2+y^2-2rxy}{1-r^2}} \, dx\, dy,$$

where

$$(6.23) \qquad r \overset{\text{def}}{=} \frac{1}{\sqrt{1+\frac{n^3-\sum_{i=1}^{k} t_i^3}{3n(n+1)^2}}} \geqq \frac{1}{2}\sqrt{3} = 0{,}866.$$

Analogous formulas hold for the other onesided and for the two-sided test.

Thus an approximation for α may be found by means of a table of the two dimensional normal distribution with correlation coefficient r (cf e.g. [8] ,p.52-57). Table 1 contains this approximation for the level of significance of Z'_r (and thus of Z'_ℓ) for some values of α' and r. For the smallest value of r we take 0.85; the table not containing the value $r = \frac{1}{2}\sqrt{3}$.

Table 1

Approximation for α for some values of α' and r

r \ α'	0,005	0,01	0,025	0,05
0,85	0,008	0,015	0,037	0,072
0,90	0,007	0,015	0,035	0,068
0,95	0,007	0,013	0,032	0,063

Further an approximation for α' may be found from (6.23) for given value of α ; table 2 contains this approximation for the onesided test for some values of α and r.

Table 2

Approximation for α' for some values of α and r

r \ α	0,01	0,025	0,05
0,85	0,0064	0,0165	0,034
0,90	0,0068	0,0175	0,036
0,95	0,0075	0,0193	0,040

In [1] (p.32-33) a table is given for the approximate critical values of Z'_r for n =21(1)100, α =0,01; 0,025; 0,05 and r =0,85 (i.e. for α' =0,0064; 0,0165; 0,034).

We now consider the sequence $\{z_\lambda\}$ and an alternative hypothesis stating that the distributions of z_λ under the condition $z_\lambda \neq 0$ are identical. Then if p,q and θ are defined by (5.1) and (5.2) it follows from theorem V and the properties of the sign test that the following theorem holds.

Theorem IX: If (3.36) is satisfied then the test based on the critical region Z' is, for $\lambda \to \infty$, consistent for the class of alternative hypotheses

(6.24) $$p \neq \tfrac{1}{2} \text{ and(or) } \theta \neq 0$$

and, for sufficiently small α, not consistent for the class of alternative hypotheses

(6.25) $$p = \tfrac{1}{2}, \theta = 0.$$

The test based on Z'_{ℓ} is, for $\lambda \to \infty$, consistent for the classes of alternatives

(6.26) $$\begin{cases} 1. & p < \tfrac{1}{2}, \\ 2. & p \geqq \tfrac{1}{2}, \ p - \tfrac{1}{2} + pq\theta < 0, \end{cases}$$

not consistent for the class of alternatives

(6.27) $$p \geqq \tfrac{1}{2}, \quad p - \tfrac{1}{2} + pq\theta > 0$$

and, for sufficiently small α, not consistent for the class of alternatives

(6.28) $$p \geqq \tfrac{1}{2}, \quad p - \tfrac{1}{2} + pq\theta = 0.$$

The test based on Z'_{r} is, for $\lambda \to \infty$, consistent for the classes of alternatives

(6.27) $$\begin{cases} 1. & p > \tfrac{1}{2}, \\ 2. & p \leqq \tfrac{1}{2}, \ p - \tfrac{1}{2} + pq\theta > 0, \end{cases}$$

not consistent for the class of alternatives

(6.28) $$p \leqq \tfrac{1}{2}, \quad p - \tfrac{1}{2} + pq\theta < 0$$

and, for sufficiently small α, not consistent for the class of alternatives

(6.29) $$p \leqq \tfrac{1}{2}, \quad p - \tfrac{1}{2} + pq\theta = 0.$$

This test has two advantages

1. If n_1 falls in the critical region then the statistic T need not be computed,

2. The test is consistent for a larger class of alternatives then WILCOXON's test for symmetry.

Further the test is analogous to the test for symmetry of HEMELRIJK (cf. [6], p.69-81), which is based on n_1 and the test statistic W of WILCOXON's two sample test applied to the positive observations as the first and the absolute values of the negative observations as the second sample (cf section 4). The critical regions differ only slightly from the ones indicated here, but the computations are more complicated. The two-sided test of HEMELRIJK is consistent for the same class of alternatives as the two-sided test described in this section, but other critical regions are also indicated, which are consistent for other alternatives, e.g. for $p<\frac{1}{2}$, for $\theta<0$, etc.

7. A generalization of the test for symmetry

The test statistic of WILCOXON's test for symmetry is a special case of the test statistic

(7.1) $$T = \sum_{i=1}^{k} q_i (a_i - b_i)$$

where $q_1, q_2, \ldots, q_k$ are given numbers.

For the distribution of this statistic under the hypothesis and under the condition (k,t) the following recursion formula may be obtained (cf. theorem I).

(7.2) $$P[T=T|k,t_1,t_2,\ldots,t_k;H_0] = 2^{-t_k}\sum_{i=0}^{t_k}\binom{t_k}{i}P[T=T-(2i-t_k)q_k|k-1,t_1,t_2,\ldots,t_{k-1};H_0].$$

Further if $\varkappa_\nu$ $(\nu=1,2,\ldots)$ are the cumulants of the distribution of T under the hypothesis H_0 and under the condition (k,t) then (cf. theorem II)

(7.3) $$\varkappa_{2\nu+1} = 0 \qquad \nu=0,1,\ldots$$

and

(7.4) $$\varkappa_{2\nu} = \frac{2^{2\nu}(2^{2\nu}-1)B_{2\nu}}{2\nu}\sum_{i=1}^{k} t_i q_i^{2\nu} \qquad \nu=1,2,\ldots.$$

Further (cf. theorem IV) if (3.36) is satisfied and if moreover

(7.5) $$\lim_{n\to\infty}\frac{\sum_{i=1}^{k} t_i q_i^{2\nu}}{\left\{\sum_{i=1}^{k} t_i q_i^{2}\right\}^{\nu}} = 0 \quad \text{for} \quad \nu>1$$

then the random variable $\dfrac{T_\lambda}{\sigma(T_\lambda|(k,t);H_0)}$ is, under the hypothesis H_0 and under the conditions $k_\lambda = k_\lambda$, $t_{1,\lambda}=t_{1,\lambda}$, $t_{2,\lambda}=t_{2,\lambda}$, $\ldots$, $t_{k_\lambda,\lambda}=t_{k_\lambda,\lambda}$ for $\lambda\to\infty$, asymptotically normally distributed with zero mean and variance 1.

A special case of the test statistic (7.1) is the statistic of R.A. FISHER's randomization test for symmetry (cf. [4], p. 43-47) with

(7.6) $$q_i = u_i \qquad (i=1,2,\ldots,k),$$

where $u_1, u_2, \ldots, u_k$ are the non-zero values assumed by $|z_1|, |z_2|, \ldots, |z_m|$ (cf. section 2).

Two other test for symmetry may be obtained, based on the two sample tests of M.E. TERRY [19] and B.L. VAN DER WAERDEN [20], i.e. by taking

$$(7.7) \quad q_i = \frac{1}{t_i} \sum_{h=h_1}^{h_2} \mathcal{E}\, Z_{n,h} \; (i = 1, 2, \ldots, k),$$

or

$$(7.8) \quad q_i = \frac{1}{t_i} \sum_{h=h_1}^{h_2} \psi\left(\frac{h}{n+1}\right) \; (i = 1, 2, \ldots, k),$$

$$h_1 \overset{\text{def}}{=} \sum_{j=1}^{i-1} t_j + 1, \qquad h_2 \overset{\text{def}}{=} \sum_{j=1}^{i} t_j,$$

where $\mathcal{E}\, Z_{n,h}$ is the expectation of the h^{th} order statistic of a random sample of size n from a standard normal distribution and where $\psi\left(\frac{h}{n+1}\right)$ is defined by

$$(7.9) \qquad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\psi\left(\frac{h}{n+1}\right)} e^{-\frac{1}{2}x^2}\, dx = \frac{h}{n+1}.$$

If we take $q_i = 1$ for each i then the test based on the statistic (7.1) is identical with the sign test.

Table I

Critical values T_α of Z_α for the case that $t_i = 1$ for each $i = 1, 2, \ldots, k$. [4]

(cf. section 2)

n \ α	0,005	0,01	0,025	0,05
3	-	-	-	-
4	-	-	-	-
5	-	-	-	15
6	-	-	21	17
7	-	28	24	22
8	36	34	30	26
9	43	39	35	29
10	49	45	39	35
11	56	52	46	40
12	64	60	52	44
13	73	67	57	49
14	81	75	63	55
15	90	82	70	60
16	98	90	78	66
17	107	99	85	71
18	117	107	91	77
19	126	116	98	84
20	136	124	106	90

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4) "-" means that $P[T \geq T \mid n; H_0] > \alpha$ for each value of T with $-\frac{1}{2}n(n+1) \leq T \leq \frac{1}{2}n(n+1)$.

Table II

Critical values v_{α_1} and T_{α_2} of Z'_r for the case that $t_i = 1$ for each $i = 1, 2, \ldots, k$.[5)] (cf section 6)

α	0,005		0,01		0,025		0,05	
n	v_{α_1}	T_{α_2}	v_{α_1}	T_{α_2}	v_{α_1}	T_{α_2}	v_{α_1}	T_{α_2}
5	-	-	-	-	-	-	5	15
6	-	-	-	-	6	21	6	17
7	-	-	7	28	7	24	7	22
8	8	36	8	34	8	30	6	28
9	9	43	9	39	7	37	7	31
10	10	49	10	45	8	41	8	35
11	11	56	9	54	9	46	7	42
12	10	66	10	60	8	56	8	44
13	11	73	11	67	9	59	7	59
14	12	81	10	77	10	65	8	57
15	11	94	11	82	9	76	9	60
16	12	100	12	90	10	80	8	74
17	13	107	11	103	11	85	9	75
18	12	125	12	109	10	97	10	79
19	13	128	13	116	11	106	9	90
20	14	138	12	130	10	120	10	94

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5) "-" means that $P[\underline{v} \geqq v \text{ or } \underline{T} \geqq T \mid n; H_0] > \alpha$ for each pair of values (v, T) with $-n \leqq v \leqq n$, $-\frac{1}{2}n(n+1) \leqq T \leqq \frac{1}{2}n(n+1)$; $(\underline{v} = \underline{n}_1 - \underline{n}_2)$.

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