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Priority in waiting line problems

by

H. Kesten and J.Th. Runnenburg

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1. Introduction[1)]

The object of this paper is to give a more detailed account of the situation, discussed in the first part of Cobham's article [2] . We shall consider here the situation where customers of different priorities arrive at one counter to be served.

2. Description of the system

We distinguish r priorities by the priority numbers $1,2,\dots,r$, where 1 stands for the highest and r for the lowest priority. Customers of priority number k will be called k-customers in the sequel. At time zero the counter is opened for servicing. At that moment, with probability $p_0(a_1,\dots,a_r)$ a queue consisting of a_1 1-customers,..., a_r r-customers is present (with $a_1 \geqq 0,\dots,a_r \geqq 0$, $p_0(a_1,\dots,a_r) \geqq 0$, $\sum[a_1 \geqq 0,\dots,a_r \geqq 0]\, p_0(a_1,\dots,a_r) = 1$). 2)
New k-customers arrive $(k \in \{1,\dots,r\})$ according to the following law: the interval from time zero to the first arrival of a k-customer, and the intervals between arrivals of successive k-customers are mutually independent random variables with distributionfunction

$$(2.1) \qquad G_k(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 1-e^{-\lambda_k x} & \text{for} \quad x \geqq 0, \end{cases}$$

where we assume $\lambda_k > 0$ for $k \in \{1,\dots,r\}$. The servicetime is also stochastic and has the same distributionfunction $F_k(t)$ (continuous from the right) for all k-customers. All arrival intervals (including the intervals from time zero to the arrival of the first k-customer) and all servicetimes are mutually independent.

Servicing takes place for each priority in the order of arrival. If customers of different priorities are present when the counter becomes free to serve a new customer, that one with highest priority which came first to the counter, is the next to be served. If the counter becomes empty the next customer to be served is the first newly arriving customer. Servicing of a customer is never interrupted to make way for another customer.

Following D.G. Kendall [10] we consider the moments at which customers leave the counter at the end of their servicetime. The customers are numbered (1,2,...) in the order in which they leave the counter, and

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1) Questions, put to us by the N.V. Philips' Gloeilampenfabrieken, Eindhoven, Holland, gave rise to the present investigation.

2) If a summation is extended over a rather involved set of indices, this set is given in [ ] brackets, directly after the Σ sign. Summation is always over non-negative integers.

(2.2) $$p_{k,n}(a_1, \dots, a_r)$$

is defined as the probability that the n^{th} departing customer is a k-customer and leaves a queue consisting of a_1 1-customers, ..., a_r r-customers at the counter ($k \in \{1, \dots, r\}$, $n \in \{1, 2, \dots\}$ and $a_j \in \{0, 1, \dots\}$ for $j \in \{1, \dots, r\}$).
We introduce the generating functions

(2.3) $$f_{k,n}(X_1, \dots, X_r) \stackrel{\text{def}}{=} \sum \lfloor a_1 \geq 0, \dots, a_r \geq 0 \rfloor p_{k,n}(a_1, \dots, a_r) X_1^{a_1} \dots X_r^{a_r}$$

for $|X_1| \leq 1, \dots, |X_r| \leq 1$, the functions $\varphi_k(\alpha)$ and the moments of $F_k(t)$, defined by [1)]

(2.4) $$\varphi_k(\alpha) \stackrel{\text{def}}{=} \int_{0-}^{\infty} e^{-\alpha t} \, d F_k(t)$$

for $Re\, \alpha \geq 0$ and

(2.5) $$\mu_k^{(\ell)} \stackrel{\text{def}}{=} \int_{0-}^{\infty} t^{\ell} \, d F_k(t).$$

We exclude the case where $F_k(0) = 1$ for some k , i.e. we have $\mu_k^{(\ell)} > 0$ for all k and all ℓ and $\varphi_k(\alpha) < 1$ for all k and all $\alpha > 0$.

Finally let

(2.6) $$H_{k,n}(t)$$

be the conditional distributionfunction of the waiting time of the n^{th} departing customer, given that the n^{th} departing customer is a k-customer, and

(2.7) $$\psi_{k,n}(\alpha) \stackrel{\text{def}}{=} \int_{0-}^{\infty} e^{-\alpha t} \, d H_{k,n}(t)$$

for $k \in \{1, \dots, r\}$ and $n \in \{1, 2, \dots\}$.

We distinguish two cases:

the case of nonsaturation, defined by $\sum_1^r \lambda_i \mu_i^{(1)} < 1$

and

the case of saturation, defined by $\sum_1^r \lambda_i \mu_i^{(1)} \geq 1$.

For the case of nonsaturation we prove that the limits of $p_{k,n}(a_1, \dots, a_r)$ and $f_{k,n}(X_1, \dots, X_r)$ for $n \to \infty$ exist and that $H_{k,n}(t)$ tends to a distributionfunction $H_k(t)$ for $n \to \infty$. All these limits are independent of the initial situation, i.e. the probabilitydistribution $\{p_0(a_1, \dots, a_r)\}$. $H_k(t)$ is the distributionfunction of the waiting time of an arbitrary k-customer in the stationary situation.

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1) The integrals are Lebesque-Stieltjesintegrals over the interval $0 \leq t < \infty$.

Using D. van Dantzig's "method of collective marks" ([5] , [6] and [7]), we derive recurrence relations (3.12) between the generating functions $f_{k,n}(X_1,\dots,X_r)$ together with relations (3.16) connecting the $f_{k,n}(X_1,\dots,X_r)$, $\psi_{k,n}(\alpha)$ and $\varphi_k(\alpha)$. From these relations we derive the relations (5.2) for the

$$f_k(X_1,\dots,X_r) \overset{\text{def}}{=} \lim_{n\to\infty} f_{k,n}(X_1,\dots,X_r),$$

which are then solved. From the relation (3.16) we derive (5.3), connecting $f_k(X_1,\dots,X_r)$ and

$$\psi_k(\alpha) \overset{\text{def}}{=} \lim_{n\to\infty} \psi_{k,n}(\alpha).$$

Once the $f_k(X_1,\dots,X_r)$ are solved, they are used, together with the last relation, to compute the first two moments of $H_k(t)$ and to derive an expression for $\psi_k(\alpha)$, for $k\in\{1,\dots,r\}$. The first moment of $H_k(t)$ was given by Cobham [2], but we did not understand his proof.

For the case of saturation we only state some results without proof.

We shall use some abbreviations to keep the formulae from becoming awkwardly long. With the understanding that on both sides of the equalitysign in (2.8) up to and including (2.14) indices may be added to the function symbols, we write [1)]

(2.8) $$f(X) \overset{\text{abb}}{=} f(X_1,\dots,X_r),$$

(2.9) $$g(X) \overset{\text{abb}}{=} \sum_1^r f_i(X),$$

(2.10) $$f(u^k X v^\ell) \overset{\text{abb}}{=} f(u,\dots,u,X_{k+1},\dots,X_{r-\ell},v,\dots,v),$$

i.e. the first k variables in (2.10) are equal to u, the last ℓ variables are equal to v and the remaining variables (if any) are equal to the corresponding variables of $f(X)$ (we shall always have $k+\ell\leqq r$). In the same way

(2.11) $$f(\mathcal{U}^{(k)}X) \overset{\text{abb}}{=} f(\mathcal{U}_1,\dots,\mathcal{U}_{k-1},X_k,\dots,X_r),$$

(2.12) $$f(\mathcal{U}^{(k)}Xv^\ell) \overset{\text{abb}}{=} f(\mathcal{U}_1,\dots,\mathcal{U}_{k-1},X_k,\dots,X_{r-\ell},v,\dots,v),$$

(2.13) $$f(y_{(k)}X) \overset{\text{abb}}{=} f(y_{k,1},\dots,y_{k,k-1},X_k,\dots,X_r),$$

(2.14) $$f(y_{(k)}Xv^\ell) \overset{\text{abb}}{=} f(y_{k,1},\dots,y_{k,k-1},X_k,\dots,X_{r-\ell},v,\dots,v).$$

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1) $\overset{\text{abb}}{=}$ is used, when on the left hand side of an equalitysign an abbreviation is introduced for an expression on the right hand side.

We use

$$\lim_{X \to 1} f(X) \qquad (|X|<1)$$

if we want to take

$$\lim_{X_1 \to 1} \lim_{X_2 \to 1} \dots \lim_{X_r \to 1} f(X),$$

where $X_1, \dots, X_r$ must remain inside the unit circle. The order in which the latter limits are taken is irrelevant unless otherwise stated.

Finally [1)]

(2.15) $$pX \stackrel{\text{abb}}{=} \sum_1^r p_i X_i ,$$

and for all $k \in \{1, \dots, r\}$

(2.16) $$p(u^k, X) \stackrel{\text{abb}}{=} \sum_1^k p_i u + \sum_{k+1}^r p_i X_i ,$$

(2.17) $$p(U^{(k)}, X) \stackrel{\text{abb}}{=} \sum_1^{k-1} p_i U_i + \sum_k^r p_i X_i ,$$

(2.18) $$p(y_{(k)}, X) \stackrel{\text{abb}}{=} \sum_1^{k-1} p_i y_{k,i} + \sum_k^r p_i X_i .$$

3. Recurrence relations for the system

In order to apply the method of collective marks of D. van Dantzig [5] and [6], we introduce an event E, which happens with probability $1-X_k$ whenever a k-customer arrives, thus

(3.1) $$0 \leqq X_k \leqq 1 \qquad \text{for each} \qquad k \in \{1, \dots, r\}.$$

The events E are independent for all customers. Any event E is called a "catastrophe" in D. van Dantzig's papers, but its nature is irrelevant. As only probabilities of other events, together with non-occurence of any "catastrophe" are considered, it is irrelevant whether under occurence of an event E the process continues or not.

We can now interprete $f_{k,n}(X)$ as a probability for

(3.2) $$p_{k,n}(a_1, \dots, a_r)\, X_1^{a_1} \dots X_r^{a_r}$$

is the probability, that at the n^{th} departure, $n \in \{1, 2, \dots\}$, one k-customer leaves the counter, a_1 1-customers, ..., a_r r-customers remain at the counter and with respect to none of the remaining customers the event E happened. Therefore

(3.3) $$f_{k,n}(X) = \sum [a_1 \geqq 0, \dots, a_r \geqq 0]\, p_{k,n}(a_1, \dots, a_r)\, X_1^{a_1} \dots X_r^{a_r}$$

is the probability that at the n^{th} departure $n = \{1, 2, \dots\}$ a k-customer leaves the counter and with respect to none of those remaining at the counter the event E happened. Further

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1) If $k=1$ the first sum on the right hand side of (2.17) and (2.18) equals zero, if $k=r$ the last sum of (2.16).

(3.4) $$p_{i,n}(0,\dots,0,a_k,\dots,a_r)X_k^{a_k}\dots X_r^{a_r}$$

is the probability, that at the n^{th} departure, $n \in \{1,2,\dots\}$, an i-customer leaves the counter, a_k k-customers,..., a_r r-customers remain at the counter and with respect to none of the customers remaining at the counter the event E happened. If $a_k > 0$ the next customer to be served is a k-customer, therefore for $k \in \{1,\dots,r\}$ 1) (using (2.10))

(3.5) $$f_{i,n}(0^{k-1}X) - f_{i,n}(0^k X) = \sum[a_k \geq 1, a_{k+1} \geq 0, \dots, a_r \geq 0]\, p_{i,n}(0,\dots,0,a_k,\dots,a_r)X_1^{a_1}\dots$$

is the probability, that at the n^{th} departure an i-customer leaves the counter, service on a k-customer starts and with respect to none of the customers left by the departing i-customer the event E happened.

Put

(3.6) $$\lambda \overset{def}{=} \lambda_1 + \dots + \lambda_r .$$

Now

$$f_{i,n}(0^r) = p_{i,n}(0,\dots,0)$$

is the probability, that at the n^{th} departure an i-customer leaves and the counter becomes empty, while

(3.7) $$p_k \overset{def}{=} \frac{\lambda_k}{\lambda}$$

is the probability, that the first customer arriving after a given moment is a k-customer, therefore (using (2.9) and (2.10))

(3.8) $$p_k X_k\, g_n(0^r)$$

is the probability, that at the n^{th} departure, $n \in \{1,2,\dots\}$, the counter becomes empty and the next arriving customer is a k-customer, with respect to which the event E does not happen.

(3.9) $$\int_0^\infty e^{-\lambda_1 t}\frac{(\lambda_1 t)^{a_1}}{a_1!}\dots e^{-\lambda_r t}\frac{(\lambda_r t)^{a_r}}{a_r!}\,dF_k(t)$$

is the probability, that during the servicetime of a k-customer exactly a_1 1-customers,..., a_r r-customers arrive, so (using (2.15))

(3.10) $$\varphi_k(\lambda(1-pX)) = \sum[a_1 \geq 0,\dots,a_r \geq 0]\, X_1^{a_1}\dots X_r^{a_r}\int_0^\infty e^{-\lambda_1 t}\frac{(\lambda_1 t)^{a_1}}{a_1!}\dots e^{-\lambda_r t}\frac{(\lambda_r t)^{a_r}}{a_r!}\,dF_k(t)$$

is the probability, that with respect to none of the customers, arriving during the servicetime of a k-customer, the event E happened.

Analogously

(3.11) $$\varphi_k(\lambda(1-p(1^{k-1},X)))$$

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1) If $k = r$ then $f_{i,n}(0^k X)$ stands for $f_{i,n}(0^r)$.

is the probability, that with respect to none of the customers with priority number $\geq \kappa$, arriving during the servicetime of a κ-customer, the event E happened.

Now the probability that at the $(n+1)^{st}$ departure a κ-customer leaves and that neither to him nor to those remaining at the counter the event E happened is equal to the probability that at the n^{th} departure either an i-customer leaves the counter (for i equal to 1,2,... or r), service on a κ-customer starts and to those remaining at the counter (the κ-customer under service included) the event E did not happen or the counter becomes empty and the first customer arriving is a κ-customer, with respect to whom the event E did not happen and (in any case) during the servicetime of that κ-customer no customers, with respect to whom the event E happened, arrive. This equality can be written in the following way, using (3.3), (3.5), (3.8) and (3.10) with their interpretations

$$(3.12) \qquad X_\kappa f_{\kappa,n+1}(X) = \{ g_n(o^{\kappa-1}X) - g_n(o^\kappa X) + p_\kappa X_\kappa g_n(o^r) \} \varphi_\kappa(\lambda(1-pX)).$$

This relation is valid for $\kappa \in \{1,\dots,r\}$, $n \in \{1,2,\dots\}$ and all real X_κ satisfying $0 \leq X_\kappa \leq 1$, because of the arbitrariness of the event E. If at the moment the counter is opened for service, with probability $p_0(a_1,\dots,a_r)$ a queue consisting of a_1 1-customers,..., a_r r-customers is present and

$$(3.13) \qquad g_0(X) \stackrel{def}{=} \sum \lfloor a_1 \geq 0,\dots,a_r \geq 0 \rfloor \, p_0(a_1,\dots,a_r) X_1^{a_1} \cdots X_r^{a_r},$$

then (3.12) is true for $n=0$ as well.

For $0 \leq X_i \leq 1$, $i \neq \kappa$ and $0 < X_\kappa \leq 1$ we can solve (3.12) for $f_{\kappa,n+1}(X)$ once $g_n(X)$ is known for those values of X. But then we can find $f_{\kappa,n+1}(X)$ (and $g_n(X)$)) for all X satisfying $|X_1| \leq 1,\dots,|X_r| \leq 1$ by analytic continuation for each $\kappa \in \{1,\dots,r\}$. Therefore (3.12) holds generally for each $\kappa \in \{1,\dots,r\}$, $n \in \{0,1,2,\dots\}$ and $|X_1| \leq 1,\dots,|X_r| \leq 1$.

We might try to express $f_{\kappa,n+1}(X)$ as a function of $g_0(X)$ only, by repeated application of (3.12) and so eliminating $g_\ell(X)$ with $\ell \geq 1$. This is however not practicable, the more so as $f_{\kappa,n+1}(X)$ for $X_\kappa = 0$ can be found from (3.12) only by dividing both sides by X_κ for $X_\kappa \neq 0$ and taking the limits for $X_\kappa \to 0$, which leads to partial differential quotients in the expression for $f_\kappa(X)$ for $X_\kappa = 0$.

Analogous to (3.11) and its interpretation we have

(3.14) $$\psi_{k,n}(\lambda_k(1-X_k))$$

is the probability, that if at the n^{th} departure a k-customer leaves the counter, with respect to none of the customers with priority number k arriving during his waiting time, the event E happened. Finally

(3.15) $$f_{k,n}(1^{k-1}\, X\, 1^{r-k})$$

is the probability, that at the n^{th} departure a k-customer leaves the counter and with respect to none of the customers with priority number k which remain at the counter the event E happened. Now this is equal to the probability that at the n^{th} departure a k-customer leaves and that with respect to none of the customers with priority number k arriving either during his waiting time or during his service time the event E happened.

Therefore we have

(3.16) $$f_{k,n}(1^{k-1} X\, 1^{r-k}) = f_{k,n}(1^r)\, \psi_{k,n}(\lambda_k(1-X_k))\, \varphi_k(\lambda_k(1-X_k)),$$

for $k \in \{1,\ldots,r\}$, $n \in \{1,2,\ldots\}$ and for all X_k satisfying $0 \leq X_k \leq 1$. This may again be generalized by analytic continuation. Therefore (3.16) holds for all X_k satisfying $|X_k| \leq 1$.

We can now summarize our results. From (3.16) we have, that $\psi_{k,n}(\alpha)$ is a function of $f_{k,n}(X)$ and $\varphi_k(\alpha)$. The functions $f_{k,n}(X)$ are known to satisfy (3.12), but cannot be solved explicitly from those relations in terms of $g_0(X)$. However, as we are interested in the behaviour of the system in the long run, we will use (3.12) and (3.16) to find $\lim_{n\to\infty} \psi_{k,n}(\alpha)$.

The relations (3.12) and (3.16) can also be derived in a more formal way than it has been done here.

4. Convergence to a stationary distribution

Before making use of the relations (3.12) and (3.16) we shall prove some results connected with the convergence of the $p_{k,n}(a_1,\ldots,a_r)$ for $n \to \infty$, which justify the method of the next section.

Let us say that the system is in the state $(k; a_1,\ldots,a_r)$ at the departure of the n^{th} customer if the n^{th} departing customer is a k-customer and if he leaves for every $i \in \{1,\ldots,r\}$ a_i i-customers at the counter. Then all transition probabilities from a state at the n^{th} departure to any state at the $(n+1)^{st}$ departure are independent of n and can easily be calculated.

By considering only the moments, at which a customer leaves the system, we thus obtain a Markof chain, with a denumerable number of states. Let us denote this Markof chain by M. For every state there is a positive probability to reach in a finite number of steps a state where a departing customer leaves an empty counter, and from this situation any state can again be reached in any number of steps. We conclude that M is an irreducible and aperiodic Markof chain (cf. Feller [8] for the terminology and classification of states in Markof chains). From Corollary 1 in Feller [8] (p. 328) it follows immediately, that $\lim_{n\to\infty} p_{k,n}(a_1,\dots,a_r)$ exists and is independent of the initial distribution.

In the case of nonsaturation ($\sum_1^r \lambda_i \mu_i^{(1)} < 1$) all states are ergodic. To prove this, we need a theorem of Foster [9], which was given by Moustafa [12] in the following slightly generalized form:

Theorem 4.1. An irreducible, aperiodic Markof chain represented by the Markof matrix $\| p_{i,j} \|$ $(i,j = 1,2,\dots)$ is ergodic if for some $\varepsilon > 0$ and some integer i_0, there exists a non-negative solution $\{y_i\}$ of the inequalities

$$(4.1) \qquad \sum_1^\infty p_{i,j}\, y_j \leq y_i - \varepsilon \quad \text{for} \quad i > i_0 ,$$

$$(4.2) \qquad \sum_1^\infty p_{i,j}\, y_j < \infty \quad \text{for} \quad i \leq i_0 .$$

We note that $\sum_1^\infty p_{ij}\, y_j$ can be regarded as the expectation after one step, if we start in the i^{th} state, of a random variable [1] $\underline{y}$, taking values y_j with probabilities $p_{i,j}$.

Theorem 4.2. If $\sum_1^r \lambda_i \mu_i^{(1)} < 1$, all states in the Markof chain M are ergodic.

Proof. This theorem is an application of Th. 4.1. The states of M can be characterized by $(k; a_1, \dots, a_r)$, i.e. the priority number of the leaving customer and the number of customers of each priority left by him. With each state we associate a number y. By definition $y = \sum_1^r a_i \mu_i^{(1)}$ for the state $(k; a_1, \dots, a_r)$, i.e. y is the expectation of the time needed to serve the remaining customers and as such non-negative. If we start in the situation $(k; 0, \dots, 0, a_\ell, \dots, a_r)$ with $a_\ell > 0$ for an $\ell \leq r$, the next customer to be served is an ℓ-customer and the expectation of $\underline{y}$ after one step is then

$$\sum_1^{\ell-1} \lambda_i \mu_i^{(1)} \mu_\ell^{(1)} + (a_\ell + \lambda_\ell \mu_\ell^{(1)} - 1)\mu_\ell^{(1)} + \sum_{\ell+1}^r (a_i + \lambda_i \mu_\ell^{(1)})\mu_i^{(1)} =$$

$$= \sum_\ell^r a_i \mu_i^{(1)} + \mu_\ell^{(1)} \left\{ \sum_1^r \lambda_i \mu_i^{(1)} - 1 \right\} \leq \sum_\ell^r a_i \mu_i^{(1)} - \varepsilon ,$$

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1) Random variables are denoted by underlined symbols.

where

$$\varepsilon \stackrel{\text{def}}{=} \min_{1 \leq \ell \leq r} \mu_\ell^{(1)} \left\{ 1 - \sum_1^r \lambda_i \mu_i^{(1)} \right\}.$$

In fact the expected number of i-customers arriving during the servicetime of an ℓ-customer is $\lambda_i \mu_\ell^{(1)}$, and one ℓ-customer leaves the system at the end of this step. Therefore (4.1) is satisfied in this case. If we start in the state $(k; 0, \ldots, 0)$, the expectation of $\underline{y}$ after one step is finite, so (4.2) is satisfied for the r states with $a_1 = a_2 = \ldots = a_r = 0$.
Thus Th. 4.2 follows.

<u>Corollary</u>. If we define $p_k(a_1, \ldots, a_r) = \lim_{n \to \infty} p_{k,n}(a_1, \ldots, a_r)$ we have:

$$p_k(a_1, \ldots, a_r) > 0 \text{ for all } k \in \{1, \ldots, r\} \quad \text{and} \quad a_1 \geq 0, \ldots, a_r \geq 0,$$

$$\sum_1^r \sum [a_1 \geq 0, \ldots, a_r \geq 0] \, p_k(a_1, \ldots, a_r) = 1$$

and the $p_k(a_1, \ldots, a_r)$ form a stationary distribution for the Markof chain M. This is an immediate consequence of Th. 4.2 and Th. 2, p. 325 in Feller [8].

To prove also the convergence of $\sum [S] \, p_{k,n}(a_1, \ldots, a_r)$ where the summation is over an arbitrary set S of states, and the convergence of moments of the queue length, we need the following theorem.

<u>Theorem 4.3</u>. Let an irreducible, aperiodic and ergodic Markof chain be represented by the Markof matrix $\| p_{i,j} \|$ $(i,j = 1, 2, \ldots)$. If $\pi_j \stackrel{\text{def}}{=} \lim_{n \to \infty} p_{i,j}^{(n)}$, where $p_{i,j}^{(n)}$ are the n step transition probabilities (these limits exist, are positive and independent of i (cf. Feller [8], p. 325) then we have for any non-negative state function F_j

$$\lim_{n \to \infty} \sum_1^\infty p_{i,j}^{(n)} F_j = \sum_1^\infty \pi_j F_j \quad \text{for every } i. \tag{4.3}$$

<u>Proof</u>. As $\lim_{n \to \infty} p_{i,j}^{(n)} = \pi_j$ and $F_j \geq 0$ we have for all positive integers i

$$\liminf_{n \to \infty} \sum_1^\infty p_{i,j}^{(n)} F_j \geq \sum_1^\infty \pi_j F_j, \tag{4.4}$$

because if $\varepsilon > 0$ and N is such that [1]

$$\sum_1^N \pi_j F_j \geq \sum_1^\infty \pi_j F_j - \varepsilon,$$

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1) If $\sum_1^\infty \pi_j F_j = \infty$, only some obvious changes are necessary.

we have

$$\liminf_{n\to\infty} \sum_1^\infty p_{i,j}^{(n)} F_j \geq \liminf_{n\to\infty} \sum_1^N p_{i,j}^{(n)} F_j = \sum_1^N \pi_j F_j \geq \sum_1^\infty \pi_j F_j - \varepsilon$$

for every $\varepsilon > 0$, whence (4.4) holds.

The proof of (4.3) is completed, if $\sum_1^\infty \pi_j F_j = \infty$.
If $\sum_1^\infty \pi_j F_j < \infty$ we proceed as follows. We know, that π_j is always positive, $\sum_1^\infty \pi_j = 1$ and $\pi_j = \sum_1^\infty \pi_\ell p_{\ell,j}^{(n)}$ for all positive integers n (cf. Feller [8], p. 325). Therefore we have for a fixed $N \geq i$ and every n

$$\sum_1^\infty \pi_j F_j = \sum_1^\infty \pi_\ell \sum_1^\infty p_{\ell,j}^{(n)} F_j \geq \sum [1 \leq \ell \leq N, \ell \neq i] \pi_\ell \sum_1^\infty p_{\ell,j}^{(n)} F_j + \pi_i \sum_1^\infty p_{i,j}^{(n)} F_j$$

so

$$\sum_1^\infty \pi_j F_j \geq \limsup_{n\to\infty} \left\{ \sum [1 \leq \ell \leq N, \ell \neq i] \pi_\ell \sum_1^\infty p_{\ell,j}^{(n)} F_j + \pi_i \sum_1^\infty p_{i,j}^{(n)} F_j \right\} \geq$$

$$\geq \sum [1 \leq \ell \leq N, \ell \neq i] \pi_\ell \liminf_{n\to\infty} \sum_1^\infty p_{\ell,j}^{(n)} F_j + \pi_i \limsup_{n\to\infty} \sum_1^\infty p_{i,j}^{(n)} F_j \geq$$

$$\geq \sum [1 \leq \ell \leq N, \ell \neq i] \pi_\ell \sum_1^\infty \pi_j F_j + \pi_i \limsup_{n\to\infty} \sum_1^\infty p_{i,j}^{(n)} F_j .$$

Now take $N \to \infty$

$$\sum_1^\infty \pi_j F_j \geq (1 - \pi_i) \sum_1^\infty \pi_j F_j + \pi_i \limsup_{n\to\infty} \sum_1^\infty p_{i,j}^{(n)} F_j .$$

As $\pi_i > 0$ this leads to

$$\limsup_{n\to\infty} \sum_1^\infty p_{i,j}^{(n)} F_j \leq \sum_1^\infty \pi_j F_j \tag{4.5}$$

for all i .
From (4.5) together with (4.4) we have (4.3).

Remark 1. The theorem remains true for arbitrary state functions F_j with $\sum_1^\infty \pi_j |F_j| < \infty$ as can be seen by writing

$$F_j = F_j^+ - F_j^- ,$$

where

$$F_j^+ \stackrel{\text{def}}{=} \frac{|F_j| + F_j}{2} ,$$

$$F_j^- \stackrel{\text{def}}{=} \frac{|F_j| - F_j}{2} .$$

Remark 2. If the Markof chain we consider has a probability $p_i^{(o)}$ of being in the state i in the initial situation ($p_i^{(o)} \geqq 0$ and $\sum_1^{\infty i} p_i^{(o)} = 1$), then by Th. 4.3

$$\lim_{n\to\infty} \sum_1^{\infty i} \sum_1^{\infty j} p_i^{(o)} p_{i,j}^{(n)} F_j = \sum_1^{\infty j} \pi_j F_j ,$$

provided F_j is bounded.

From the convergence of $p_{k,n}(a_1, \ldots, a_r)$ follows only the existence of $\lim_{n\to\infty} f_{k,n}(X)$, if $|X_i| < 1$ for $i \in \{1, \ldots, r\}$. We may now conclude, that even for $|X_i| \leqq 1$ for $i \in \{1, \ldots, r\}$

$$\lim_{n\to\infty} f_{k,n}(X) = \sum \lfloor a_1 \geqq 0, \ldots, a_r \geqq 0 \rfloor p_k(a_1, \ldots, a_r) X_1^{a_1} \ldots X_r^{a_r}.$$

This follows if we take the state function

$$F(i; a_1, \ldots, a_r) = \begin{cases} X_1^{a_1} \ldots X_r^{a_r} \text{ if } & i = k, \\ 0 \quad \text{ if } & i \neq k. \end{cases}$$

Thus $f_k(X) \overset{\text{def}}{=} \lim_{n\to\infty} f_{k,n}(X) = \sum \lfloor a_1 \geqq 0, \ldots, a_r \geqq 0 \rfloor p_k(a_1, \ldots, a_r) X_1^{a_1} \ldots X_r^{a_r}$

is a power series with positive coefficients, which converges if

$|X_i| \leqq 1$ for all $i \in \{1, \ldots, r\}$ and as $\sum_1^{r k} \sum \lfloor a_1 \geqq 0, \ldots, a_r \geqq 0 \rfloor p_k(a_1, \ldots, a_r) = 1$,

we conclude that

$$\text{(4.6)} \qquad \lim_{X_k \to 1} f_k(1^{k-1} X\, 1^{r-k}) = f_k(1^r) \qquad (|X_k| < 1).$$

Remark 3. From Th. 4.3 we also conclude that

$$\lim_{n\to\infty} \sum_1^{r k} \sum \lfloor a_1 \geqq 0, \ldots, a_r \geqq 0 \rfloor a_j\, p_{k,n}(a_1, \ldots, a_r) = \sum_1^{r k} \sum \lfloor a_1 \geqq 0, \ldots, a_r \geqq 0 \rfloor a_j\, p_k(a_1, \ldots, a_r)$$

i.e. the expected length of the queue of j-customers at the n^{th} departure tends to the expected length of the queue of j-customers derived from the stationary distribution, and analogously for the higher moments of the queue length, provided the initial state is fixed, i.e. $p_o(b_1, \ldots, b_r) = 1$ for a given initial state ($b_1, \ldots, b_r$).

Theorem 4.4. If $\sum_1^{r i} \lambda_i \mu_i^{(1)} < 1$, the conditional distribution-functions of the waiting times $H_{k,n}(t)$ $(k \in \{1, \ldots, r\})$ converge to a non-degenerated distributionfunction $H_k(t)$ with

$$\psi_k(\alpha) \overset{\text{def}}{=} \int_{0-}^{\infty} e^{-\alpha t} d H_k(t)$$

satisfying

$$\text{(4.7)} \qquad f_k(1, \ldots, 1, 1 - \tfrac{\alpha}{\lambda_k}, 1, \ldots, 1) = f_k(1^r) \Psi_k(\alpha) \varphi_k(\alpha) \quad \text{for} \quad \left|1 - \tfrac{\alpha}{\lambda_k}\right| \leqq 1.$$

Proof. A distributionfunction of a non-negative random variable is uniquely determined if its Laplace transform is given on an interval which lies in the right half plane, because the Laplace transform of such a distributionfunction is analytic for all arguments with positive real part, and can thus be determined uniquely by analytic continuation, so that the uniqueness theorem for the inverse of a Laplace transform may be applied (cf. Widder [14] Th. 5A, p. 57 and Th. 6.3, p. 63).

From (3.16) follows the convergence of $\varphi_{k,n}(\alpha)$ for

$|1-\frac{\alpha}{\lambda_k}| \leqq 1$ as $\lim_{n\to\infty} f_{k,n}(1^r) > 0$.

We can now follow a standard method (compare e.g. Lévy [11], p. 49, proof of Th. 17[2]) to prove that $H_{k,n}(t)$ converges to a function $H_k(t)$ with $\varphi_k(\alpha) = \lim_{n\to\infty} \varphi_{k,n}(\alpha)$ satisfying (4.7).

$H_k(t)$ is a monotonic non-decreasing function, continuous from the right and satisfies $H_k(t) = 0$ for $t < 0$ and $\lim_{t\to\infty} H_k(t) = 1$, as from (4.7) $\lim_{\alpha\to 0} \varphi_k(\alpha) = 1$. This proves Th. 4.4.

All the foregoing theorems concerning the queuing problem are valid only if $\sum_1^r \lambda_i \mu_i^{(1)} < 1$. In the case of saturation $(\sum_1^r \lambda_i \mu_i^{(1)} \geqq 1)$ analogous theorems can be proved, although we did not succeed in finding simple proofs so far. In fact one can prove: If $\sum_1^s \lambda_i \mu_i^{(1)} < 1$ and $\sum_1^{s+1} \lambda_i \mu_i^{(1)} \geqq 1$ we have

$$\lim_{n\to\infty} \sum [a_{s+2} \geqq 0, \dots, a_r \geqq 0]\, p_{k,n}(a_1, \dots, a_r) = 0$$

and

$$\lim_{n\to\infty} \sum [a_{s+1} \geqq 0, \dots, a_r \geqq 0]\, p_{k,n}(a_1, \dots, a_r)$$

exists and is positive.

If we define

$$\bar{p}_k(a_1, \dots, a_s) \overset{\text{def}}{=} \lim_{n\to\infty} \sum [a_{s+1} \geqq 0, \dots, a_r \geqq 0]\, p_{k,n}(a_1, \dots, a_r)$$

we have for each $k \in \{1, \dots, r\}$

$$\lim_{n\to\infty} f_{k,n}(X\, 1^{r-s}) = \sum [a_1 \geqq 0, \dots, a_s \geqq 0]\, \bar{p}_k(a_1, \dots, a_s)\, X_1^{a_1} \dots X_s^{a_s},$$

whereas

$$\lim_{X\to 1} \sum_1^{s+1} {}_k\, f_{k,n}(X\, 1^{r-s}) = \sum_1^{s+1} {}_k \sum [a_1 \geqq 0, \dots, a_s \geqq 0]\, \bar{p}_k(a_1, \dots, a_s) = 1 \qquad (|X|<1)$$

$H_{k,n}(t)$ converges to a non-degenercte distributionfunction if $k \leqq s$ and $\lim_{n\to\infty} H_{k,n}(t) = 0$ for every finite t if $k \geqq s+1$. If $k \leqq s$ the moments of $H_{k,n}(t)$ do not necessarily converge to those of $H_k(t)$, i.e. we cannot conclude

$$(4.8) \qquad \lim_{n\to\infty} \int_0^\infty t^j\, dH_{k,n}(t) = \int_0^\infty t^j\, dH_k(t) \qquad \text{for } k \leqq s.$$

An example will show, that in some cases (4.8) does not hold. Take $s+2 \leq r$ and $\mu_{s+2}^{(1)} = \infty$. If we start from an initial situation with $a_1 = \dots = a_{s+1} = 0$, $a_{s+2} > 0$ it is clear that $\int_0^\infty t \, d H_{k,n}(t) = \infty \; (n \in \{1,2,\dots\})$, whereas $\int_0^\infty t \, d H_k(t)$ is not necessarily infinite for $k \leq s$.

5. The case of nonsaturation

In section 4 we proved that in the case of nonsaturation, i.e. if

(5.1) $$\sum_1^r \lambda_i \mu_i^{(1)} < 1 ,$$

for $k \in \{1, \dots, r\}$ and all X with $|X_1| \leq 1, \dots, |X_r| \leq 1$

$$f_k(X) \stackrel{\text{def}}{=} \lim_{n\to\infty} f_{k,n}(X)$$

exists.

According to Theorem 4.4 in this case the limits

$$H_k(t) \stackrel{\text{def}}{=} \lim_{n\to\infty} H_{k,n}(t)$$

for all real t and

$$\varphi_k(\alpha) \stackrel{\text{def}}{=} \lim_{n\to\infty} \varphi_{k,n}(\alpha)$$

for $\mathrm{Re}\,\alpha \geqq 0$ also exist and $\varphi_k(\alpha)$ satisfies

$$\varphi_k(\alpha) = \int_{0-}^\infty e^{-\alpha t} \, d H_k(t).$$

For $k \in \{1, \dots, r\}$ and $|X_1| \leq 1, \dots, |X_r| \leq 1$ we have from (3.12)

(5.2) $$X_k f_k(X) = \{ g(0^{k-1}X) - g(0^k X) + p_k X_k \, g(0^r) \} \varphi_k(\lambda(1-pX))$$

while (3.16) leads to

(5.3) $$f_k(1^{k-1} X \, 1^{r-k}) = f_k(1^r) \, \varphi_k(\lambda_k - \lambda_k X_k) \, \varphi_k(\lambda_k - \lambda_k X_k).$$

From (5.2) we conclude (for $|X_1| \leq 1, \dots, |X_r| \leq 1$ and arbitrary U_j satisfying $|U_1| \leq 1, \dots, |U_{k-1}| \leq 1$)

(5.4) $$\frac{f_k(X)}{\varphi_k(\lambda(1-pX))} = \frac{f_k(U^{(k)}X)}{\varphi_k(\lambda(1-p(U^{(k)}_{,}X)))},$$

for $X_k \neq 0$ (and by analytic continuation for $X_k = 0$ as well) and also

(5.5) $$\sum_1^r \frac{X_i f_i(X)}{\varphi_i(\lambda(1-pX))} = \sum_1^r \{ g(0^{i-1}X) - g(0^i X) + p X g(0^r) \} .$$

Formula (5.5) simplifies to

$$(5.6) \qquad \sum_1^r f_i(X) \frac{X_i - \varphi_i(\lambda(1-pX))}{\varphi_i(\lambda(1-pX))} = g(0^r)(pX-1).$$

To determine $f_k(X)$ we introduce $y_{k,1}, \ldots, y_{k,k-1}$, defined (for $k \in \{2, \ldots, r\}$) by

$$(5.7) \qquad y_{k,i} - \varphi_i\Big(\lambda\big(1 - \sum_1^{k-1}{}_j\, p_j y_{k,j} - \sum_k^r{}_j\, p_j X_j\big)\Big) = 0$$

for $i \in \{1, \ldots, k-1\}$. The $y_{k,i}$ are thus functions of $X_k, \ldots, X_r$. We shall prove (always assuming (5.1)):

Lemma 5.1. Equations (5.7) have for every set of complex numbers $X_k, \ldots, X_r$, satisfying $\sum_k^r{}_i\, \lambda_i \operatorname{Re} X_i < \sum_k^r{}_i\, \lambda_i$ exactly one solution for $y_{k,1}, \ldots, y_{k,k-1}$, with $|y_{k,1}| < 1, \ldots, |y_{k,k-1}| < 1$.

Proof: Consider the equation

$$(5.8) \qquad z - \sum_1^{k-1}{}_i\, \lambda_i \varphi_i\Big(\lambda - z - \sum_k^r{}_j\, \lambda_j X_j\Big) = 0.$$

By Rouché's Theorem (see Titchmarsh [13], p. 116): "If $p(z)$ and $q(z)$ are analytic inside and on a closed contour C, and $|q(z)| < |p(z)|$ on C, then $p(z)$ and $p(z) + q(z)$ have the same number of zeros inside C", taking $p(z) \overset{def}{=} z$, $q(z) \overset{def}{=} -\sum_1^{k-1}{}_i\, \lambda_i \varphi_i(\lambda - z - \sum_k^r{}_j\, \lambda_j X_j)$ and for C the circle $|z| = \sum_1^{k-1}{}_i\, \lambda_i$ we have that

$$z - \sum_1^{k-1}{}_i\, \lambda_i \varphi_i\Big(\lambda - z - \sum_k^r{}_j\, \lambda_j X_j\Big)$$

has exactly one zero $z_k = z_k(X_k, \ldots, X_r)$ with $|z_k| < \sum_1^{k-1}{}_i\, \lambda_i$ for a fixed set of complex numbers $X_k, \ldots, X_r$, satisfying

$$\sum_k^r{}_i\, \lambda_i \operatorname{Re} X_i < \sum_k^r{}_i\, \lambda_i .$$

If we now take

$$y_{k,i} = \varphi_i\Big(\lambda - z_k - \sum_k^r{}_j\, \lambda_j X_j\Big)$$

equations (5.7) are solved and

$$|y_{k,i}| < 1$$

because $\operatorname{Re}(\lambda - z_k - \sum_k^r{}_j\, \lambda_j X_j) > 0$ and $|\varphi_i(\alpha)| < 1$ for $\operatorname{Re} \alpha > 0$. A second solution $y^*_{k,i}$ leads to $z^*_k \overset{def}{=} \sum_1^{k-1}{}_i\, \lambda_i y^*_{k,i}$ where z^*_k satisfies (5.8) and $|z^*_k| < \sum_1^{k-1}{}_i\, \lambda_i$. But then $z^*_k = z_k$ and therefore $y^*_{k,i} = y_{k,i}$.

Lemma 5.2. The solution z_k of (5.8) is an analytic function of the variables $X_k, \ldots, X_r$ for all $X_k, \ldots, X_r$ satisfying

$$\sum_k^r{}_i\, \lambda_i \operatorname{Re} X_i < \sum_k^r{}_i\, \lambda_i .$$

Remark (cf Bochner and Martin [1], p. 30). A function $f(z_1,\dots,z_\ell)$ is an analytic function of the ℓ complex variables $z_1,\dots,z_\ell$ in a certain region, if in some neighbourhood of every point $(z_1^\circ,\dots,z_\ell^\circ)$ of that region it is the sum of an absolutely convergent powerseries in $z_1-z_1^\circ,\dots,z_\ell-z_\ell^\circ$.

Proof: Consider the point $X_k=c_k,\dots,X_r=c_r$, with $\sum_k^r \lambda_i \operatorname{Re} c_i < \sum_k^r \lambda_i$. By using theorem 9 from Bochner and Martin [1], p. 39 (for the special case $k=1$ in their notation), we prove that z_k is analytic in the point $(c_k,\dots,c_r)$. The theorem reads in our notation:

If the function $F(z,X_k,\dots,X_r)$ is an analytic function of $r-k+1$ (complex) variables in the neighbourhood of the point $(a,c_k,\dots,c_r)$, if $F(a,c_k,\dots,c_r)=0$ and if $\frac{\partial F}{\partial z}\neq 0$ for $z=a, X_k=c_k,\dots,X_r=c_r$ then the equation

$$F(z,X_k,\dots,X_r)=0$$

has a unique solution

$$z_k=z_k(X_k,\dots,X_r)$$

equal to a for $X_k=c_k,\dots,X_r=c_r$ and analytic in the neighbourhood of the point $(c_k,\dots,c_r)$.

We take

$$F(z,X_k,\dots,X_r)\stackrel{\text{def}}{=} z-\sum_1^{k-1}\lambda_i\varphi_i(\lambda-z-\sum_k^r\lambda_j X_j)$$

with $z,X_k,\dots,X_r$ as (functionally) independent complex variables. This function is analytic in the neighbourhood of $(a,c_k,\dots,c_r)$, if $\operatorname{Re} a+\sum_k^r\lambda_j\operatorname{Re} c_j<\lambda$. This holds in particular for $a=z_k(c_k,\dots,c_r)$, where z_k is the only zero of $F(z,X_k,\dots,X_r)$ with $|z_k|<\sum_1^{k-1}\lambda_i$ (see proof of lemma 5.1), because $\operatorname{Re} z_k+\sum_k^r\lambda_j\operatorname{Re} c_j<\sum_1^{k-1}\lambda_i+\sum_k^r\lambda_j=\lambda$. Furthermore $F(z_k(c_k,\dots,c_r),c_k,\dots,c_r)=0$ as z_k satisfies (5.11) and

$$\left|\frac{\partial F}{\partial z}\right|=\left|1-\sum_1^{k-1}\lambda_i\frac{\partial\varphi_i}{\partial z}\right|\geq 1-\sum_1^{k-1}\lambda_i\mu_i^{(1)}>0,$$

for $z=a, X_k=c_k,\dots,X_r=c_r$, because

$$\left|\frac{\partial\varphi_i}{\partial z}\right|=\left|\int_0^\infty t\exp\{-t(\lambda-z-\sum_k^r\lambda_j X_j)\}dF_i(t)\right|\leq\int_0^\infty t\,dF_i(t)=\mu_i$$

and (5.1) holds. Therefore the equation (5.8) has a unique solution $z_k=z_k(X_k,\dots,X_r)$ equal to $z_k(c_k,\dots,c_r)$ for $X_k=c_k,\dots,X_r=c_r$, which is analytic (only this is new) in the neighbourhood of $(c_k,\dots,c_r)$.

Lemma 5.3. If we keep $\sum_k^r\lambda_i\operatorname{Re} X_i<\sum_k^r\lambda_i$, we have

(5.9) $$\lim z_k(X_k,\dots,X_r)=1 \quad\text{for}\quad \sum_k^r\lambda_i(X_i-1)\to 0.$$

Proof: Again we apply Rouché's Theorem (see proof of lemma 5.1), this time with

$$p(x) \overset{\text{def}}{=} x - \sum_1^{k-1} \lambda_i + \varepsilon$$

$$q(x) \overset{\text{def}}{=} - \sum_1^{k-1} \lambda_i \left\{ \varphi_i \left(\lambda - x - \sum_k^r \lambda_j X_j\right) - 1 \right\} - \varepsilon$$

where ε is a (small) positive number. For C we take the circle with radius ε and centre $\sum_1^{k-1} \lambda_i - \varepsilon$. If we take $\sum_k^r \lambda_i X_i$ sufficiently close to $\sum_k^r \lambda_i$, it turns out that $|p(x)| = \varepsilon$ and $|q(x)| < \varepsilon$ for all points of C . Thus

$$p(x) + q(x) = x - \sum_1^{k-1} \lambda_i \varphi_i \left(\lambda - x - \sum_k^r \lambda_j X_j\right)$$

has a zero inside C . As x_k is the only zero of this function for $|x| < \sum_1^{k-1} \lambda_i$, the zero we have found must be x_k , i.e. given $\varepsilon > 0$ we have proved

$$\left| x_k - \sum_1^{k-1} \lambda_i + \varepsilon \right| < \varepsilon$$

and thus

$$\left| x_k - \sum_1^{k-1} \lambda_i \right| < 2\varepsilon$$

if $\sum_k^r \lambda_i (X_i - 1)$ is sufficiently near 0, provided $\sum_k^r \lambda_i (\mathrm{Re}\, X_i - 1) < 0$

Remark. We also have $\mathrm{Re}\left(\lambda - x_k - \sum_k^r \lambda_j X_j\right) > 0$.

Lemma 5.4.

$$\lim \left(\frac{\partial}{\partial X_k}\right)^{\ell} x_k (X_k, \ldots, X_r) \quad \text{for} \quad \sum_k^r \lambda_i (X_i - 1) \to 0$$

exists, if we keep $\sum_k^r \lambda_i \mathrm{Re}\, X_i < \sum_k^r \lambda_i$, for every $\ell \in \{0, 1, \ldots, m\}$, if $\mu_i^{(m)} < \infty$ for every $i, k \in \{1, \ldots, r\}$ and $m \geq 1$.

Proof: $\left(\frac{\partial}{\partial X_k}\right) x_k (X_k, \ldots, X_r)$ can be obtained by partial differentiation of

$$x_k = \sum_1^{k-1} \lambda_i \varphi_i \left(\lambda - x_k - \sum_k^r \lambda_j X_j\right)$$

with respect to X_k , for $\sum_k^r \lambda_i \mathrm{Re}\, X_i < \sum_k^r \lambda_i$ and solving for $\frac{\partial x_k}{\partial X_k}$. We obtain a fraction, from which we find the higher partial derivatives by ordinary partial differentiation, applying the chain rule and substituting for those derivatives already obtained. Remembering that $\mu_i^{(m)} < \infty$ implies that $\varphi_i(\alpha)$ is an m times continuously differentiable function for $\mathrm{Re}\, \alpha \geq 0$, which may be differentiated under the integralsign, that (5.1) holds and lemma 5.3, we can easily verify the statement of the lemma.

If

$$A_{k\ell} \overset{\text{abb}}{=} \lim \left(\frac{\partial}{\partial X_k}\right)^{\ell} z_k(X_k, \ldots, X_r) \quad \text{for} \quad \sum_k^r \lambda_i (X_i - 1) \to 0,$$

we find

$$(5.10) \qquad A_{k1} = \frac{\lambda_k \sum_1^{k-1} \lambda_i \mu_i^{(1)}}{1 - \sum_1^{k-1} \lambda_i \mu_i^{(1)}},$$

$$(5.11) \qquad A_{k2} = \frac{\lambda_k^2 \sum_1^{k-1} \lambda_i \mu_i^{(2)}}{\left(1 - \sum_1^{k-1} \lambda_i \mu_i^{(1)}\right)^3},$$

$$(5.12) \qquad A_{k3} = \frac{\lambda_k^3 \sum_1^{k-1} \lambda_i \mu_i^{(3)}}{\left(1 - \sum_1^{k-1} \lambda_i \mu_i^{(1)}\right)^4} + \frac{3 \lambda_k^3 \left(\sum_1^{k-1} \lambda_i \mu_i^{(2)}\right)^2}{\left(1 - \sum_1^{k-1} \lambda_i \mu_i^{(1)}\right)^5}.$$

From (5.6) we obtain, substituting $X_i = y_{k,i}$ for $i \in \{1, \ldots, k-1\}$

$$(5.13) \qquad \frac{X_k - \varphi_k(\lambda(1 - p(y_{(k)}, X)))}{\varphi_k(\lambda(1 - p(y_{(k)}, X)))} f_k(y_{(k)}, X) =$$
$$= - \sum_{k+1}^{r} \frac{X_i - \varphi_i(\lambda(1 - p(y_{(k)}, X)))}{\varphi_i(\lambda(1 - p(y_{(k)}, X)))} f_i(y_{(k)}, X) + (p(y_{(k)}, X) - 1) g(0^r)$$

and by using (5.4) we have

$$(5.14) \qquad \frac{X_k - \varphi_k(\lambda(1 - p(y_{(k)}, X)))}{\varphi_k(\lambda(1 - pX))} f_k(X) =$$
$$= - \sum_{k+1}^{r} \frac{X_i - \varphi_i(\lambda(1 - p(y_{(k)}, X)))}{\varphi_i(\lambda(1 - pX))} f_i(X) + (p(y_{(k)}, X) - 1) g(0^r)$$

for all X_j satisfying $|X_j| \leq 1$ for $j \neq k$ and $|X_k| < 1$.

We have

$$X_k = \varphi_k(\lambda(1 - p(y_{(k)}, X)))$$

only for

$$X_k = y_{k+1,k}(X_{k+1}, \ldots, X_r)$$

and therefore the $f_k(X)$ can be obtained successively for all $X_1, \ldots, X_r$ satisfying $|X_j| \leq 1$ for $j \neq k$ and $|X_k| < 1$ (either directly or else by analytic continuation) from (5.14), starting with $f_r(X)$ if $g(0^r)$ is known. We shall not try to obtain the $f_k(X)$ explicitly, but use (5.14) in the sequel.

The constant $g(0^r)$ is determined by the condition

$$(5.15) \qquad g(1^r) = 1.$$

If we take $X_i = 1$ in (5.14) for $i \neq k$ and, keeping $|X_k| < 1$, take $X_k \to 1$, we have from lemma 5.3. that both sides of (5.14) tend to zero. It can be seen, that $X_k \neq \varphi_k(\lambda(1-p(y_{(k)}, 1^{k-1}X_k 1^{r-k})))$ for $X_k \neq 1$, therefore, always keeping $|X_k| < 1$ and using l'Hopitals' rule

(5.16) $$f_k(1^r) = \lim_{X_k \to 1} f_k(1^{k-1}X 1^{r-k}) =$$

$$= -\sum_{k+1}^{r} f_i(1^r) \lim_{X_k \to 1} \frac{1-\varphi_i(\lambda(1-p(y_{(k)}, 1^{k-1}X 1^{r-k})))}{X_k - \varphi_k(\lambda(1-p(y_{(k)}, 1^{k-1}X 1^{r-k})))} +$$

$$+g(0^r) \lim_{X_k \to 1} \frac{p(y_{(k)}, 1^{k-1}X 1^{r-k}) - 1}{X_k - \varphi_k(\lambda(1-p(y_{(k)}, 1^{k-1}X 1^{r-k})))} = \sum_{k+1}^{r} f_i(1^r) \frac{\mu_i^{(1)}(\lambda_k + A_{k,1})}{1-\mu_k^{(1)}(\lambda_k + A_{k,1})} + \frac{g_0^{(r)}}{\lambda} \frac{\lambda_k + A_{k,1}}{1-\mu_k^{(1)}(\lambda_k + A_{k,1})}$$

or with (5.10)

(5.17) $$f_k(1^r) = \frac{\sum_{k+1}^{r} f_i(1^r)\mu_i^{(1)}\lambda_k + g(0^r)\,p_k}{1-\sum_{1}^{k}\lambda_i\mu_i^{(1)}} .$$

Solving (5.17) for $f_k(1^r)$ leads to

(5.18) $$f_k(1^r) = \frac{p_k\, g(0^r)}{1-\sum_{1}^{r}\lambda_i\mu_i^{(1)}} \qquad \text{for} \qquad k \in \{1,\dots,r\}.$$

Because

(5.19) $$g(1^r) = 1 ,$$

we finally have

(5.20) $$f_k(1^r) = p_k$$

and

(5.21) $$g(0^r) = 1 - \sum_{1}^{r}\lambda_i\mu_i^{(1)} .$$

We thus proved

Theorem 5.1. The functions $f_k(X)$ satisfy the equations

(5.22) $$f_k(X) = \frac{\varphi_k(\lambda(1-pX))}{X_k - \varphi_k(\lambda(1-p(y_{(k)}, X)))} \cdot$$

$$\cdot \left\{ -\sum_{k+1}^{r} f_i(1^{i-1}X) \frac{X_i - \varphi_i(\lambda(1-p(y_{(k)}, X)))}{\varphi_i(\lambda(1-p(1^{i-1}, X)))} + \left(1-\sum_{1}^{r}\lambda_i\mu_i^{(1)}\right)(p(y_{(k)}, X) - 1) \right\}$$

for $|X_i| \leq 1$ $(i \neq k)$, $|X_k| < 1$, $X_k \neq y_{k+1,k}(X_{k+1}, \dots, X_r)$ and all $k \in \{1,\dots,r\}$.

They can be obtained successively from these equations starting from $k = r$.

The derivation of (5.20) and (5.21) here given is unnecessarily long and complicated, but the same method leads us to the moments of the waiting time distribution as we shall now show.

In section 4 we proved that in the nonsaturated case the $f_k(X)$ are powerseries with non-negative coefficients, absolutely convergent for $|X_1| \leqq 1, \dots, |X_r| \leqq 1$ and $k \in \{1, \dots, r\}$. If we differentiate a function of this kind n times ($n \in \{0, 1, \dots\}$) with respect to one of its arguments and take the limits (in any order) $X_1 \to 1, \dots, X_r \to 1$ keeping $|X_i| \leqq 1$ for all $i \in \{1, \dots, r\}$ then either the resulting expression is finite and the powerseries for this derivative converges for $|X_1| \leqq 1, \dots, |X_r| \leqq 1$ or the limit is $+\infty$. Moreover in all cases we have

$$(5.23) \qquad \left\{ \left(\frac{\partial}{\partial X_k} \right)^n f_k(X) \right\}_{X_1 = \dots = X_r = 1} = \lim_{X \to 1} \left(\frac{\partial}{\partial X_k} \right)^n f_k(X) \qquad (|X| < 1).$$

From (5.22) we see, that

$$(5.24) \qquad \lim_{X \to 1} \left(\frac{\partial}{\partial X_k} \right)^n f_k(X) \qquad (|X| < 1)$$

exists if $\varphi_j(\lambda(1 - pX))$ is $(n+1)$ -times differentiable with respect to X_k for $j \in \{1, \dots, r\}$ and $k \in \{1, \dots, r\}$. This is certainly the case if the $(n+1)^{\text{st}}$ moments of all $F_\ell(x)$ $(\ell \in \{1, \dots, r\})$ exist. If (as the only alternative) at least one of these moments is $+\infty$, then we find from (5.22)

$$(5.25) \qquad \lim_{X \to 1} \left(\frac{\partial}{\partial X_k} \right)^n f_k(X) = +\infty \qquad (|X| < 1).$$

If we take $X_i = 1$ for $i \neq k$ in (5.22), differentiate with respect to X_k, then let $X_k \to 1$ and use (5.10), (5.11) and (5.20) the result is

$$(5.26) \qquad \left(\frac{\partial f_k}{\partial X_k} \right)_{X_1 = \dots = X_r = 1} = \frac{\lambda_k^2 \mu_k^{(1)}}{\lambda} + \frac{\lambda_k^2 \sum_1^r \lambda_i \mu_i^{(2)}}{2\lambda \left(1 - \sum_1^{k-1} \lambda_i \mu_i^{(1)}\right)\left(1 - \sum_1^{k} \lambda_i \mu_i^{(1)}\right)},$$

whilst we find in the same way from the second partial derivative of (5.22) with respect to X_k

$$(5.27)\quad \left(\frac{\partial^2 f_K}{\partial X_K^2}\right)_{X_1=\dots=X_r=1} = \frac{\lambda_K^3\,\mu_K^{(2)}}{\lambda} + \frac{\lambda_K^3\,\mu_K^{(1)}\sum_1^r \lambda_i\mu_i^{(2)}}{\lambda\left(1-\sum_1^{K-1}\lambda_i\mu_i^{(1)}\right)\left(1-\sum_1^{K}\lambda_i\mu_i^{(1)}\right)} +$$

$$+ \frac{\lambda_K^3\sum_1^r \lambda_i\mu_i^{(3)}}{3\lambda\left(1-\sum_1^{K-1}\lambda_i\mu_i^{(1)}\right)^2\left(1-\sum_1^{K}\lambda_i\mu_i^{(1)}\right)} + \frac{\lambda_K^3\sum_1^r \lambda_i\mu_i^{(2)}\sum_1^K\lambda_j\mu_j^{(2)}}{2\lambda\left(1-\sum_1^{K-1}\lambda_i\mu_i^{(1)}\right)^2\left(1-\sum_1^{K}\lambda_i\mu_i^{(1)}\right)^2} +$$

$$+ \frac{\lambda_K^3\sum_1^r \lambda_i\mu_i^{(2)}\sum_1^{K-1}\lambda_j\mu_j^{(2)}}{2\lambda\left(1-\sum_1^{K-1}\lambda_i\mu_i^{(1)}\right)^3\left(1-\sum_1^{K}\lambda_i\mu_i^{(1)}\right)}.$$

From (5.3) we have by differentiating with respect to X_K

$$(5.28)\quad \left(\frac{\partial f_K}{\partial X_K}\right)_{X_1=\dots=X_r=1} = f_K(1^r)\left(\frac{d}{dX_K}\left\{\varphi_K(\lambda_K(1-X_K))\,\psi_K(\lambda_K(1-X_K))\right\}\right)_{X_K=1} =$$

$$= f_K(1^r)\left\{\lambda_K\mu_K^{(1)} + \lambda_K\,\mathcal{E}\,\underline{w}_K\right\},$$

$$(5.29)\quad \left(\frac{\partial^2 f_K}{\partial X_K^2}\right)_{X_1=\dots=X_r=1} = f_K(1^r)\left\{\lambda_K^2\mu_K^{(2)} + 2\lambda_K^2\mu_K^{(1)}\,\mathcal{E}\,\underline{w}_K + \lambda_K^2\,\mathcal{E}\,\underline{w}_K^2\right\},$$

if $\mathcal{E}\,\underline{w}_K$ and $\mathcal{E}\,\underline{w}_K^2$ are the first and second moment of the stationary waiting time distribution $H_K(t)$ respectively.

On combining (5.26), (5.27), (5.28) and (5.29) we obtain:

Theorem 5.2. The first and second moment of the stationary waiting time distribution $H_K(t)$, for $K\in\{1,\dots,r\}$, are respectively

$$(5.30)\quad \mathcal{E}\,\underline{w}_K = \frac{\sum_1^r \lambda_i\mu_i^{(2)}}{2\left(1-\sum_1^{K-1}\lambda_i\mu_i^{(1)}\right)\left(1-\sum_1^{K}\lambda_i\mu_i^{(1)}\right)}$$

and

$$(5.31)\quad \mathcal{E}\,\underline{w}_K^2 = \frac{\sum_1^r \lambda_i\mu_i^{(3)}}{3\left(1-\sum_1^{K-1}\lambda_i\mu_i^{(1)}\right)^2\left(1-\sum_1^{K}\lambda_i\mu_i^{(1)}\right)} +$$

$$+ \frac{\sum_1^r \lambda_i\mu_i^{(2)}\sum_1^K\lambda_j\mu_j^{(2)}}{2\left(1-\sum_1^{K-1}\lambda_i\mu_i^{(1)}\right)^2\left(1-\sum_1^{K}\lambda_i\mu_i^{(1)}\right)^2} + \frac{\sum_1^r \lambda_i\mu_i^{(2)}\sum_1^{K-1}\lambda_j\mu_j^{(2)}}{2\left(1-\sum_1^{K-1}\lambda_i\mu_i^{(1)}\right)^3\left(1-\sum_1^{K}\lambda_i\mu_i^{(1)}\right)}.$$

Our (5.30) is Cobham's formula (3) (see [2]).

The function $\psi_K(\alpha)$ can be found from (5.3), at least for $\left|1-\frac{\alpha}{\lambda_K}\right| \leq 1$,

$$(5.32)\quad \psi_K(\alpha) = \frac{\lambda f_K\left(1,\dots,1,1-\frac{\alpha}{\lambda_K},1,\dots,1\right)}{\lambda_K\,\varphi_K(\alpha)},$$

which, if combined with (5.22), leads to

$$(5.33) \quad \psi_k(\alpha) = \frac{\left(1-\sum_1^r \lambda_i \mu_i^{(1)}\right)\left(-\sum_1^{k-1} \lambda_j + z_k^* - \alpha\right) - \sum_{k+1}^r \lambda_i \left(1-\varphi_i\left(\sum_1^{k-1} \lambda_j - z_k^* + \alpha\right)\right)}{\lambda_k - \alpha - \lambda_k \varphi_k\left(\sum_1^{k-1} \lambda_j - z_k^* + \alpha\right)},$$

where $z_1^* = 0$ and $z_k^* = z_k^*(\alpha)$ satisfies (5.8) for $X_k = 1 - \frac{\alpha}{\lambda_k}$, $X_i = 1$ ($i \neq k$ and $k \geq 2$) i.e.

$$(5.34) \qquad z_k^* - \sum_1^{k-1} \lambda_i \varphi_i\left(\sum_1^{k-1} \lambda_j - z_k^* + \alpha\right) = 0.$$

<u>Therefore $\psi_1(\alpha)$ is explicitly given by</u>

$$(5.35) \qquad \psi_1(\alpha) = -\frac{\left(1-\sum_1^r \lambda_i \mu_i^{(1)}\right)\alpha + \sum_2^r \lambda_i (1-\varphi_i(\alpha))}{\lambda_1 - \alpha - \lambda_1 \varphi_1(\alpha)},$$

while $\psi_k(\alpha)$ for $k \in \{2, \ldots, r\}$ contains the z_k^*.

As an illustration we give the following example:

Take $r = 2$, $F_1(x) = F_2(x) = 1 - \exp\left(-\frac{x}{\mu}\right)$, then

$$(5.36) \qquad \varphi_1(\alpha) = \varphi_2(\alpha) = \frac{1}{\alpha\mu + 1},$$

$$(5.37) \qquad \psi_1(\alpha) = \frac{1 - \lambda_1\mu + \alpha\mu + \alpha\lambda\mu^2}{1 - \lambda_1\mu + \alpha\mu},$$

$$(5.38) \qquad \psi_2(\alpha) = \frac{(1-\lambda\mu)(-\lambda_1 + z_2^* - \alpha)\{(\lambda_1 - z_2^* + \alpha)\mu + 1\}}{(\lambda_2 - \alpha)\{(\lambda_1 - z_2^* + \alpha)\mu + 1\} - \lambda_2},$$

which leads to the following waiting time distributions ($t \geq 0$)

$$(5.39) \qquad H_1(t) = 1 - \lambda\mu \exp\left\{-\frac{(1-\lambda_1\mu)t}{\mu}\right\},$$

$$(5.40) \qquad H_2(t) = 1 - \lambda\mu + \frac{\lambda^2\mu}{\lambda_2}\left(1 - \exp\left\{-\frac{\lambda_2(1-\lambda\mu)t}{\lambda\mu}\right\}\right) +$$

$$- 2\lambda_1(1-\lambda\mu)\int_0^t ds \int_0^s \frac{I_1\left(2u\sqrt{\frac{\lambda_1}{\mu}}\right)}{2u\sqrt{\lambda_1\mu}} \exp\left\{-\frac{\lambda_1 + \lambda^2\mu}{\lambda\mu}u\right\} du,$$

where $I_1(x)$ is the modified Besselfunction of the first order and of the first kind.

The result (5.40) contradicts equation (27) as given by R.E. Cox [4].

6. The case of saturation

If (5.1) is not satisfied, we can find a positive integer s, $0 \leqq s < r$, such that

$$(6.1) \qquad \sum_1^s \lambda_i \mu_i^{(1)} < 1 \ , \quad \sum_1^{s+1} \lambda_i \mu_i^{(1)} \geqq 1 \ .$$

In section 4 we stated already without proof, that

$$(6.2) \qquad f_k(X) \stackrel{\mathrm{def}}{=} \lim_{n \to \infty} f_{k,n}(X)$$

exists for $k \in \{1, \ldots, r\}$ and that

$$(6.3) \qquad f_k(X) = 0$$

if at least one X_j satisfies $|X_j| < 1$ for $j \in \{s+1, \ldots, r\}$.

As a consequence of (6.3), it cannot be true that

$$(6.4) \qquad f_k(1^r) = \lim_{X \to 1} f_k(X) ,$$

as the right hand side in (6.4) equals 0 and

$$(6.5) \qquad g(1^r) = 1 .$$

The functions $f_k(X)$ thus cannot be powerseries with positive coefficients and the method of section 4 cannot be applied.

But if instead of $f_k(X)$ only $f_k(X 1^{r-s})$ is considered, we can repeat the argument of section 5 with some alterations.

From (5.2) and (6.3) we have at once

$$(6.6) \qquad f_k(X) = 0$$

for $k \in \{s+2, \ldots, r\}$ and $|X_i| \leqq 1$, for all $i \in \{1, \ldots, r\}$.

If for $k \in \{1, \ldots, r\}$

$$(6.7) \qquad \bar{f}_k(X) \stackrel{\mathrm{def}}{=} f_k(X 1^{r-s})$$

one can prove that for $k \in \{1, \ldots, s+1\}$, $\bar{f}_k(X)$ again is a powerseries with non-negative coefficients, absolutely convergent for $|X_1| \leqq 1, \ldots, |X_s| \leqq 1$ and satisfying

$$\sum_1^{s+1} \bar{f}_k(1^r) = 1 .$$

From (5.2) we have for $\kappa \in \{1, \dots, s\}$ and $X_{s+1} = \dots = X_r = 1$

$$(6.8) \qquad X_\kappa \bar{F}_\kappa(X) = \sum_1^{s+1} \left\{ \bar{F}_i(0^{\kappa-1} X 1^{r-s}) - \bar{F}_i(0^\kappa X 1^{r-s}) \right\} \cdot$$

$$\cdot \varphi_\kappa\left(\sum_1^s \lambda_i (1 - X_i)\right)$$

and for $\kappa = s+1$

$$(6.9) \qquad \bar{F}_{s+1}(X) = \sum_1^{s+1} \bar{F}_i(0^r)\, \varphi_{s+1}\left(\sum_1^s \lambda_i (1 - X_i)\right).$$

From (6.8) and (6.9)

$$(6.10) \qquad X_\kappa \bar{F}_\kappa(X) = \sum_1^s \left\{ \bar{F}_i(0^{\kappa-1} X 1^{r-s}) - \bar{F}_i(0^\kappa X 1^{r-s}) \right\} +$$

$$+ \sum_1^{s+1} \bar{F}_i(0^r) \left\{ \varphi_{s+1}\left(\sum_1^s \lambda_j - \sum_\kappa^s \lambda_j X_j\right) - \varphi_{s+1}\left(\sum_1^s \lambda_j - \sum_{\kappa+1}^s \lambda_j X_j\right) \right\} \varphi_\kappa\left(\sum_1^s \lambda_i (1 - X_i)\right).$$

Equation (6.10) is the analogue of (5.2), while the analogue of (5.4) is (for $|X_i| \leqq 1$, $i \in \{1, \dots, s\}$ and $|U_j| \leqq 1$, $j \in \{1, \dots, \kappa-1\}$ and $\kappa \in \{2, \dots, s\}$)

$$(6.11) \qquad \frac{\bar{F}_\kappa(X 1^{r-s})}{\varphi_\kappa(\lambda(1 - p(X, 1^{r-s})))} = \frac{\bar{F}_\kappa(U^{(\kappa)} X 1^{r-s})}{\varphi_\kappa(\lambda(1 - p(U^{(\kappa)}, X, 1^{r-s})))}$$

and (5.3) can be written

$$(6.12) \qquad \bar{F}_\kappa(1^{\kappa-1} X 1^{r-\kappa}) = \bar{F}_\kappa(1^r)\, \psi_\kappa(\lambda_\kappa(1 - X_\kappa))\, \varphi_\kappa(\lambda_\kappa(1 - X_\kappa))$$

for $\kappa \in \{1, \dots, s\}$. Therefore the moments of the waiting time distribution can be found as in section 5 for $\kappa \in \{1, \dots, s\}$. One obtains

$$(6.13) \qquad \bar{F}_\kappa(1^r) = \frac{\lambda_\kappa \mu_{s+1}^{(1)}}{1 - \sum_1^s \lambda_i \mu_i^{(1)} + \mu_{s+1}^{(1)} \sum_1^s \lambda_i} \qquad \text{for } \kappa \in \{1, \dots, s\},$$

$$(6.14) \qquad \bar{F}_{s+1}(1^r) = \frac{1 - \sum_1^s \lambda_i \mu_i^{(1)}}{1 - \sum_1^s \lambda_i \mu_i^{(1)} + \mu_{s+1} \sum_1^s \lambda_i} ,$$

(6.15) $$\sum_{1}^{s+1} \bar{f}_i(o^r) = \frac{1-\sum_{1}^{s} \lambda_i \mu_i^{(1)}}{1-\sum_{1}^{s} \lambda_i \mu_i^{(1)} + \mu_{s+1} \sum_{1}^{s} \lambda_i} ,$$

(6.16) $$\mathcal{E}\,\underline{w}_k = \frac{\sum_{1}^{s} \lambda_i \mu_i^{(2)} + \frac{\mu_{s+1}^{(2)}}{\mu_{s+1}^{(1)}}\left(1-\sum_{1}^{s} \lambda_i \mu_i^{(1)}\right)}{2\left(1-\sum_{1}^{k-1} \lambda_i \mu_i^{(1)}\right)\left(1-\sum_{1}^{k} \lambda_i \mu_i^{(1)}\right)} \quad \text{for} \quad k \in \{1,\dots,s\}$$

which is Cobham's formula (cf [3]).
In addition one can prove, that

$$\mathcal{E}\,\underline{w}_k = \infty \qquad \text{for} \qquad k \in \{s+1, \dots, r\}.$$

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