STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

STATISTISCHE AFDELING

LEIDING: PROF. DR D. VAN DANTZIG
ADVISEUR VOOR STATISTISCHE CONSULTATIE: PROF. DR J. HEMELRIJK

Report S 221 (VP 13)

Machines served by a patrolling operator

by

J.Th. Runnenburg

(Prepublication)

July 1957

1. Introduction[1]

In industry[2] one often meets with a situation, in which one operator is charged with servicing a number of identical machines, laid out in a workshop. The type of machine we consider here is automatic. Once it has been started, it continues work indefinitely, till one of a limited number of special situations arises and then it stops automatically. On such an occasion it is the task of the operator to make some adjustment to the machine to put it in working condition again and to restart it doing its work.

To be more specific, let us suppose we are dealing with n machines, placed along a circular route, along which the operator walks in a fixed direction. The machines are numbered from 1 to n in such a way, that walking away from machine 1 the operator finds the other machines in natural order along his route. It seems reasonable to suppose, that the operator needs a fixed amount of time to walk from a specified machine to the next one, upon each occasion he performs that walk. Therefore let c_i be the (constant) walkingtime needed to pass from machine i to machine $i+1$ (taken modulo n) and

$$(1) \qquad c \stackrel{\text{def}}{=} \sum_{1}^{n} c_i$$

the total walkingtime needed to complete a full round.

The total time needed to adjust a stopped machine and restart it again (counted from the moment the operator starts doing work on that machine) we call the servicetime of that

---

1) In this paper proofs are given of some of the results discussed in a lecture in the series "Actualiteiten" of the Mathematical Centre, Amsterdam. Cf. Runnenburg [1957].

2) Questions arising in the textile industry gave rise to the present investigation.

machine (on that occasion). The servicetimes are independent nonnegative stochastic variables, denoted[3] by $\underline{s}$ (usually with a suffix), with a common distributionfunction $B(s)$, which is the same for all machines. The runningtime of a machine is the time from the moment the machine is restarted up to the next stoppage. All runningtimes are independent nonnegative stochastic variables denoted by $\underline{t}$ (usually with a suffix), all having the same distributionfunction

(2) $$A(t) \stackrel{\text{def}}{=} \begin{cases} 1 - e^{-\lambda t} & t \geq 0, \\ 0 & t < 0, \end{cases}$$

where λ is a positive constant. All servicetimes and runningtimes are independent stochastic variables, which do not depend on the history of the system under consideration.

At any time any individual machine is either in position 0 (working) or in position 1 (stopped). Let $q(x_1, \dots, x_n)$ be the probability, that at time 0 machine i is in position x_i $(i \in \{1, \dots, n\})$, where x_i is either 0 or 1 for all i. These probabilities are supposed to be given and satisfy[4]

(3) $$\sum_{(x_1, \dots, x_n)} q(x_1, \dots, x_n) = 1,$$

where the summation is over all possible combinations of 0 and 1 for each of the x_i.

The instructions of the operator are given by the following rules.

Rule 1) Start at time 0 in front of machine 1 and apply rule 2),

Rule 2) Notice whether the machine in front of you is in position 0 or 1 and apply rule 3),

---

3) Stochastic variables will be distinguished from numbers (e.g. from values they take in an experiment) by underlining their symbols.

4) $\sum_{(x_1, \dots, x_n)}$ means: sum over all integers x_i with $0 \leq x_i \leq i$ and $i \in \{1, \dots, n\}$.

Rule 3) If the machine in front of you is in position 0, walk along your route and stop in front of the next machine, then apply rule 2). If the machine in front of you is in position 1, put that machine in working order again and restart it, then walk to the next machine, stop in front of it and apply rule 2).

With these instructions the operator keeps doing his rounds indefinitely. There is however the possibility to let him rest between full rounds. One can for instance simply increase one of the c_i, say c_j, with a constant amount Δ, use $c_j + \Delta$ in the following calculations and instruct the operator to rest during a time Δ, when walking from machine j to machine $j+1$ (modulo n).

In this paper we shall be concerned with $\underline{r}_m$ and $\underline{w}_m$, where m is the number of machines the operator passed since he started patrolling and

(4) $\underline{r}_m \stackrel{def}{=}$ duration of the complete round starting at the moment the operator leaves the m^{th} machine and finishing at the moment the operator leaves the $(m+n)^{th}$ machine,

(5) $\underline{w}_m \stackrel{def}{=}$ duration of the time the $(m+n)^{th}$ machine has been stopped when the operator reaches that machine.

In this paper we shall define stochastic variables $\underline{r}$ and $\underline{w}$ and their distributionfunctions and prove that [5]

(6) $$\lim_{m\to\infty} P\{\underline{r}_m \le r\} = P\{\underline{r} \le r\},$$

(7) $$\lim_{m\to\infty} \mathcal{E}\, \underline{r}_m^{\,k} = \mathcal{E}\, \underline{r}^{\,k},$$

(8) $$\lim_{m\to\infty} P\{\underline{w}_m \le w\} = P\{\underline{w} \le w\},$$

(9) $$\lim_{m\to\infty} \mathcal{E}\, \underline{w}_m^{\,k} = \mathcal{E}\, \underline{w}^{\,k}.$$

---

5) $P\{A\}$ stands for "probability of the event A", $P\{A|B\}$ for "probability of the event A given event B" and $\mathcal{E}$ is the symbol for expectation.

In a sequel to this paper we shall derive some asymptotic formulae for these quantities for $n \to \infty$.

2. Statistical properties of the system

Let the m^{th} inspection $(m \in \{1, 2, \ldots\})$, which is performed by the operator, consist in noting the positions of the first n machines he passes successively, at the moment he reaches them, starting at the m^{th} one. The result of this inspection may be denoted by a vector

$$(10) \qquad \underline{X}_m \stackrel{def}{=} (\underline{x}_m, \ldots, \underline{x}_{m+n-1}),$$

where $\underline{x}_{m+i} = 0$ if the $(m+i)^{th}$ machine is found to be working and $\underline{x}_{m+i} = 1$ if not, where $i \in \{1, \ldots, n\}$.

Upon carrying out an inspection, the operator may find any one of the 2^n vectors

$$(11) \qquad X \stackrel{def}{=} (x_1, \ldots, x_n),$$

where x_i is either 0 or 1 for all i. Each of these vectors specifies a possible state of the system of machines at the inspection. It follows from the definition, that the last $n-1$ components of the vector specifying the m^{th} state are the first $n-1$ components of the vector specifying the $(m+1)^{st}$ state.

One can quite easily compute the transitionprobabilities from one state to the next, i.e. from the state found at the m^{th} inspection to that found at the $(m+1)^{st}$. As we shall see presently, the last state given determines the transitionprobabilities to future states completely, so we have come across a Markov-chain. This shall lead us easily to the limits we are looking for. We refer to Feller [1950], Chapter 15, for the terminology and all results quoted on Markov-chains.

From the probabilities $q(x_1, \ldots, x_n)$ we can deduce the absolute probabilities $a_1(x_1, \ldots, x_n)$ for the first or initial state of the Markov-chain, where in general the m^{th} state of the chain is the state at the m^{th} inspection.

To simplify our considerations, we suppose that when the operator reaches the m^{th} machine $(m \in \{1,2,\dots\})$ on his patrol, a potential servicetime $\underline{s}_m$ is drawn from the distribution $B(s)$. Then the actual servicetime will be $\underline{x}_m \underline{s}_m$. As before the $\underline{s}_m$ are independent of the history of the system.

Let

$$(12) \qquad \rho(u_i, x_i) \overset{\text{def}}{=} x_i + (1-2x_i)(1-u_i)\, e^{-\lambda(c_1+\dots+c_{i-1}+x_1 s_1+\dots+x_{i-1}s_{i-1})}, \qquad \text{for} \quad i \geq 1,$$

where the exponent is 0 for $i=1$, then the probability of finding the positions $u_1,\dots,u_n$ at time 0 changed to positions $x_1,\dots,x_n$ at the first inspection, under the condition that s_i is the length of the potential servicetime of the i^{th} machine is given by

$$(13) \qquad \begin{aligned} &P\{\underline{x}_j = x_j \text{ for } 1 \leq j \leq n \mid \underline{u}_j = u_j \;\&\; \underline{s}_j = s_j \text{ for } 1 \leq j \leq n\} = \\ &= \prod_{i=1}^{n} P\{\underline{x}_i = x_i \mid \underline{u}_k = u_k \text{ for } 1 \leq k \leq i \;\&\; \underline{s}_\ell = s_\ell \text{ for } 1 \leq \ell \leq i-1\} = \\ &= \prod_{i=1}^{n} \rho(u_i, x_i). \end{aligned}$$

Therefore

$$(14) \qquad a_1(x_1,\dots,x_n) = \sum_{(u_1,\dots,u_n)} q(u_1,\dots,u_n) \int_{0-}^{\infty} \dots \int_{0-}^{\infty} \prod_{i=1}^{n} \rho(u_i, x_i)\, dB(s_1) \dots dB(s_n).$$

What do we know about the runningtime of a machine just after the operator has left that machine? There are two possibilities:

1) the operator had just served that machine, in which case the runningtime is independent of the time the operator will be under way before he reaches the machine again and has distributionfunction $A(t)$,
2) the operator did no work on that machine because it was still running. Going back to the last time he did serve that machine we see that the time the operator has spent since walking and servicing other machines is independent of, and smaller than the runningtime still going on.

Therefore the remaining runningtime of this machine has also $A(t)$ as distributionfunction (because $P\{\underline{t} \le t\} = A(t)$ implies $P\{\underline{t} - a \le t \mid \underline{t} > a\} = A(t)$ for $t \ge 0$ if a is a constant. This is the reason why we have to restrict our considerations to the particular distributionfunction for the runningtime, defined by (2)) and is also independent of the time the operator will be under way before he reaches this machine again, which means that for $m \ge 1$

$$
\begin{aligned}
&P\{\underline{x}_{m+n} = x_{m+n} \mid \underline{x}_1 = x_1 \,\&\, \dots \,\&\, \underline{x}_{m+n-1} = x_{m+n-1}\} = \\
(15)\quad &= P\{\underline{x}_{m+n} = x_{m+n} \mid \underline{x}_{m+1} = x_{m+1} \,\&\, \dots \,\&\, \underline{x}_{m+n-1} = x_{m+n-1}\},
\end{aligned}
$$

so

$$
\begin{aligned}
&P\{\underline{X}_{m+1} = (x_1, \dots, x_n) \mid \underline{X}_m = (y_1, \dots, y_n)\} = \\
(16)\quad &= P\{\underline{x}_{m+1} = x_1 \,\&\, \underline{x}_{m+2} = x_2 \,\&\, \dots \,\&\, \underline{x}_{m+n} = x_n \mid \underline{x}_m = y_1 \,\&\, \dots \,\&\, \underline{x}_{m+n-1} = y_n\} = \\
&= \begin{cases} P\{\underline{x}_{m+n} = x_{m+n} \mid \underline{x}_{m+1} = y_2 \,\&\, \dots \,\&\, \underline{x}_{m+n-1} = y_n\} & \text{if } x_i = y_{i+1} \text{ for } 1 \le i \le n-1, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

We note that this probability does <u>not</u> depend on m so our Markov-chain is stationary, and conclude from (15), that the last state given determines the transitionprobabilities to any future states.

Now for $m \ge 0$

$$
\begin{aligned}
&P\{\underline{x}_{m+n+1} = x_{m+n+1} \mid \underline{x}_{m+2} = x_{m+2}, \dots, \underline{x}_{m+n} = x_{m+n}\} = \\
(17)\quad &= \int_{0-}^{\infty} \dots \int_{0-}^{\infty} \Big\{(1 - x_{m+n+1})\, e^{-\lambda(c + x_{m+2} s_{m+2} + \dots + x_{m+n} s_{m+n})} + \\
&\qquad + x_{m+n+1}\left(1 - e^{-\lambda(c + x_{m+2} s_{m+2} + \dots + x_{m+n} s_{m+n})}\right)\Big\} dB(s_{m+2}) \dots dB(s_{m+n}) = \\
&= x_{m+n+1} + (1 - 2x_{m+n+1})\, e^{-\lambda c}\, \gamma^{x_{m+2} + \dots + x_{m+n}},
\end{aligned}
$$

where

$$
(18)\qquad \gamma \overset{\text{def}}{=} \int_{0-}^{\infty} e^{-\lambda s}\, dB(s).
$$

Our Markov-chain is irreducible and aperiodic, for every state can be reached from every other state in n steps as well as in $n+1$ steps with positive probability. Therefore all states belong to the same class and because there are only a finite number of states they must all be ergodic. This means that the absolute probabilities $a^{(m)}(x_1, \ldots, x_n)$ of finding the system in state $(x_1, \ldots, x_n)$ at the m^{th} inspection have a (positive) limit for $m \to \infty$. If

(19) $$a(x_1, \ldots, x_n) \overset{\text{def}}{=} \lim_{m \to \infty} a^{(m)}(x_1, \ldots, x_n),$$

then the $a(x_1, \ldots, x_n)$ are the unique solution of the system of equations

(20) $$a(x_1, \ldots, x_n) = \sum_{(y_1, \ldots, y_n)} a(y_1, \ldots, y_n) P\{x_1, \ldots, x_n | y_1, \ldots, y_n\},$$

where $P\{x_1, \ldots, x_n | y_1, \ldots, y_n\}$ denotes the transitionprobability from the state $(y_1, \ldots, y_n)$ to the next state $(x_1, \ldots, x_n)$ and which satisfies

(21) $$\sum_{(x_1, \ldots, x_n)} a(x_1, \ldots, x_n) = 1.$$

Substituting (17) in (20) we have

(22) $$a(x_1, \ldots, x_n) = \{a(0, x_1, \ldots, x_{n-1}) + a(1, x_1, \ldots, x_{n-1})\}\{x_n + (1-2x_n)\, e^{-\lambda c}\, \gamma^{x_1 + \cdots + x_{n-1}}\}$$

which is satisfied by

(23) $$a(x_1, \ldots, x_n) = A \prod_{j=0}^{x-1} (-1 + e^{\lambda c} \gamma^{-j}),$$

where

(24) $$x \overset{\text{def}}{=} \sum_1^n x_i$$

and A is a constant.

From (21) we have

$$A^{-1} = \sum_{(x_1,\dots,x_n)} \prod_{j=0}^{x-1} (1+e^{\lambda c}\gamma^{-j}) = \tag{25}$$
$$= \sum_{x=0}^{n} \binom{n}{x} \prod_{j=0}^{x-1} (1+e^{\lambda c}\gamma^{-j}) .$$

We note that $a(x_1, \dots, x_n)$ is a symmetric function of $x_1, \dots, x_n$.

3. <u>Calculation of</u> $P\{\underline{r} \leq r\}$ <u>and</u> $P\{\underline{w} \leq w\}$.

Let

$$\underline{z}_m \overset{def}{=} c + \sum_{i=1}^{n-1} \underline{x}_{m+i}\,\underline{s}_{m+i} . \tag{26}$$

From the definition of $\underline{r}_m$ and $\underline{w}_m$ we have

$$\underline{r}_m = c + \sum_{i=1}^{n} \underline{x}_{m+i}\,\underline{s}_{m+i} = \underline{z}_m + \underline{x}_{m+n}\,\underline{s}_{m+n} \tag{27}$$

and

$$\underline{w}_m = \begin{cases} \underline{z}_m - \underline{t} & \text{if } \underline{z}_m - \underline{t} > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{28}$$

Therefore

$$P\{\underline{r}_m \leq r\} = P\{c + \sum_{i=1}^{n} \underline{x}_{m+i}\,\underline{s}_{m+i} \leq r\} = \tag{29}$$
$$= \sum_{k=0}^{n} P\{c + \sum_{i=1}^{n} \underline{x}_{m+i}\,\underline{s}_{m+i} \leq r \mid \sum_{i=1}^{n} \underline{x}_{m+i} = k\} . P\{\sum_{i=1}^{n} \underline{x}_{m+i} = k\} .$$

Because the $\underline{s}_{m+i}$ are independent of the $\underline{x}_{m+i}$, we have

$$P\{c + \sum_{i=1}^{n} \underline{x}_{m+i}\,\underline{s}_{m+i} \leq r \mid \sum_{i=1}^{n} \underline{x}_{m+i} = k\} =$$
$$= P\{c + \sum_{i=1}^{k} \underline{s}_i \leq r \mid \sum_{i=1}^{n} \underline{x}_{m+i} = k\} = \tag{30}$$
$$= P\{c + \sum_{i=1}^{k} \underline{s}_i \leq r\} ,$$

where $\underline{s}_1, \ldots, \underline{s}_k$ are independently distributed, each with distributionfunction $B(s)$. Therefore

(31) $$P\{\underline{r}_m \leq r\} = \sum_{0}^{n} {}_k\, P\{c + \sum_{1}^{k} {}_i\, \underline{s}_i \leq r\} \,.\, P\{\sum_{1}^{n} {}_i\, \underline{x}_{m+i} = k\}$$

and

(32) $$\lim_{m\to\infty} P\{\underline{r}_m \leq r\} = \sum_{0}^{n} {}_k\, P\{c + \sum_{1}^{k} {}_i\, \underline{s}_i \leq r\} \lim_{m\to\infty} P\{\sum_{1}^{n} {}_i\, \underline{x}_{m+i} = k\} =$$
$$= \sum_{0}^{n} {}_k\, P\{c + \sum_{1}^{k} {}_i\, \underline{s}_i \leq r\} \binom{n}{k} A \prod_{j=0}^{k-1} (-1 + e^{\lambda c} \gamma^{-j}) .$$

If we introduce

(33) $$\underline{r} \stackrel{def}{=} c + \sum_{1}^{n} {}_i\, \underline{v}_i\, \underline{s}_i$$

and

(34) $$\underline{z} \stackrel{def}{=} c + \sum_{1}^{n-1} {}_i\, \underline{v}_i\, \underline{s}_i$$

where all $\underline{v}_i$ are either 0 or 1 and

(35) $$P\{\underline{v}_1 = v_1, \ldots, \underline{v}_n = v_n\} \stackrel{def}{=} a(v_1, \ldots, v_n)$$

and the $\underline{s}_1, \ldots, \underline{s}_n$ are independent stochastic variables, each with distributionfunction $B(s)$ and each of them independent of $\underline{v}_1, \ldots, \underline{v}_n$, then

(36) $$\lim_{m\to\infty} P\{\underline{r}_m \leq r\} = P\{\underline{r} \leq r\} .$$

From (31) and (36) we have at once for every nonnegative k

(37) $$\lim_{m\to\infty} \mathcal{E}\, \underline{r}_m^{\,k} = \mathcal{E}\, \underline{r}^{\,k} .$$

In much the same way one can obtain the limiting distribution of $\underline{w}_m$. For $w \geq 0$ we have

$$(38) \quad P\{\underline{w}_m \leq w\} = P\{\underline{w}_m = 0\} + P\{0 < \underline{w}_m \leq w\} =$$

$$= P\{\underline{z}_m - \underline{t} \leq 0\} + P\{0 < \underline{z}_m - \underline{t} \leq w\} =$$

$$= P\{\underline{z}_m - \underline{t} \leq w\}$$

and so

$$(39) \quad \lim_{m \to \infty} P\{\underline{w}_m \leq w\} = P\{\underline{z} - \underline{t} \leq w\} = P\{\underline{w} \leq w\}$$

if

$$(40) \quad \underline{w} \stackrel{def}{=} \begin{cases} \underline{z} - \underline{t} & \text{if} \quad \underline{z} - \underline{t} > 0 \\ 0 & \text{otherwise} \end{cases}$$

and for every nonnegative k

$$(41) \quad \lim_{m \to \infty} \mathcal{E}\, \underline{w}_m^{\,k} = \mathcal{E}\, \underline{w}^{\,k}.$$

Of course $\underline{w}$ and $\underline{z}$ are closely connected. In fact

$$(42) \quad P\{\underline{w} \leq w\} = \int_0^\infty P\{\underline{z} \leq t + w\} \lambda e^{-\lambda t} dt$$

from which we have

$$(43) \quad \mathcal{E}\, \underline{w} = \mathcal{E}\, \underline{z} - \frac{1}{\lambda} + \frac{1}{\lambda} \int_0^\infty e^{-\lambda x} dP\{\underline{z} \leq x\}.$$

4. Approximation for large n

If we want to compute $\mathcal{E}\, \underline{w}$ for large n , we can do so by making use of (43). First we introduce

$$(44) \quad p \stackrel{def}{=} \mathcal{E}\, \underline{v}_1 = \ldots = \mathcal{E}\, \underline{v}_n$$

from which by (26)

$$(45) \quad \mathcal{E}\, \underline{z} = c + (n-1)\, p\, \mathcal{E}\, \underline{s} \, ,$$

and after some calculations

$$(46) \qquad \int_0^\infty e^{-\lambda z} \, dP\{\underline{z} \le z\} = 1 - p.$$

All we need to find $\mathcal{E}\,\underline{w}$ from (43) is p, where from (44)

$$(47) \qquad \begin{aligned} p &= \sum_{(v_1,\dots,v_n)} v_1 \, a(v_1,\dots,v_n) = \\ &= A \sum_{k=0}^{n-1} \binom{n-1}{k} \prod_{j=0}^{k} \left(-1 + e^{\lambda c} \gamma^{-j}\right) = \\ &= \frac{A}{n} \sum_{k=0}^{n} k \binom{n}{k} \prod_{j=0}^{k-1} \left(-1 + e^{\lambda c} \gamma^{-j}\right) \end{aligned}$$

which is quite a job if n is not small.

In a second paper we shall prove, that if we replace our original $\underline{s}$ by $\underline{s}_n$, where

$$(48) \qquad P\{\underline{s}_n \le s\} = B'(ns)$$

and $B'(s)$ has finite first and second moment, then

$$(49) \qquad \lim_{n\to\infty} p = 1 - e^{-\lambda\eta}$$

where η is the only positive root of

$$(50) \qquad \eta = c + b'(1 - e^{-\lambda\eta})$$

and

$$(51) \qquad b' \overset{\text{def}}{=} \mathcal{E}\,\underline{s}'.$$

The truth of this statement can be gathered from the following considerations. From the above conditions it follows that $\underline{z}$ converges almost surely to a constant if n tends to infinity, so let

$$(52) \qquad \eta \overset{\text{def}}{=} \text{a.s.} \lim_{n\to\infty} \underline{z}.$$

Now $\underline{\chi}$ is the time needed to return to a machine, in the stationary situation. If we may regard this $\underline{\chi}$ as a constant η, then the probability of finding a machine stopped upon returning to it, is given by

(53) $$1 - e^{-\lambda\eta}$$

so we may expect

(54) $$\lim_{n\to\infty} p = 1 - e^{-\lambda\eta}$$

and thus from (45)

(55) $$\eta = c + b'(1 - e^{-\lambda\eta}),$$

which is equation (50).

(56) If $$\underline{k}_n = \sum_1^n \underline{v}_i ,$$

then we have for n tending to infinity

(57) $$P\{\underline{k}_n \leq k\} \sim \int_{-\infty}^{k} \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}\, dx$$

if

(58) $$\tau = \frac{\eta - c}{b'} ,$$

(59) $$\mu = n\tau ,$$

(60) $$\sigma^2 = \frac{n\tau(1-\tau)}{1-\lambda b'(1-\tau)}$$

and

(61) $$|k - n\tau| < n^a$$

where

(62) $$\tfrac{1}{2} < a < \tfrac{2}{3} .$$

Literature

W. Feller [1950], Probability theory and its applications, Wiley, New York, 1950 .

J.Th. Runnenburg [1957], Een wachttijdprobleem bij machines, Voordracht in de serie "Actualiteiten", Mathematisch Centrum, ZW 1957-007, 30 Maart 1957.