Corrections to

Report S 242

Some approximations to the percentage points of the non-central t-distribution

by

Dr. Constance van Eeden

page	formula or line	reads	should read
17	(3.3;2)	$\frac{\tilde{\mu}_4}{\sigma^4}$	$\frac{\tilde{\mu}_4}{\sigma^4}$
28	5 from top	$3u_\alpha^2 - 24u_\alpha^3 + 29u_\alpha$	$3u_\alpha^5 - 24u_\alpha^3 + 29u_\alpha$
	6 from top	$(252u_\alpha^5-1688u_\alpha^3+1511$	$(252u_\alpha^5-1688u_\alpha^3+1511u_\alpha)+$
	(A.2;3)	$\mathcal{E}\chi^{-s}$	$\mathcal{E}\underline{\chi}^{-s}$
30	(A.2;15)	$\mathcal{E}(t-\mathcal{E}\underline{t})^3$	$\mathcal{E}(\underline{t}-\mathcal{E}\underline{t})^3$
32	last line	$\ldots - \frac{u_\alpha}{32f^2}\delta^3]$	$\ldots - \frac{u_\alpha}{32f^2}\delta^3]+O(\frac{1}{f^3})$.

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Report S 242

Some approximations to the percentage points of the non-central t-distribution

by

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0. Summary.

In this report the following approximations to the percentage points of the non-central t-distribution are considered.

I. Three closely related approximations based on a normal approximation to the distribution of $\underline{x} + k\sqrt{\underline{\chi}^2}$, where $\underline{x}$ possesses a normal distribution with zero mean and variance 1 and where $\underline{\chi}^2$ possesses a χ^2-distribution with f degrees of freedom ($\underline{x}$ and $\underline{\chi}^2$ are distributed independently). Two of these approximations were derived by W.J. JENNETT and B.L. WELCH (1939) and by N.L. JOHNSON and B.L. WELCH (1940) respectively. The third one was derived by W.A. HENDRICKS (1936) for the percentage points of the central t-distribution (STUDENT's distribution) and could be generalized for the non-central t-distribution.

II. Four closely related approximations based on a method used by H. GOLDBERG and H. LEVINE (1946) for the percentage points of the central t-distribution. In this case the percentage-points are expressed in a power-series in $\frac{1}{f}$, the coefficients being functions of a normal variable with zero mean and variance 1.

III. The approximation of M. MERRINGTON and E.S. PEARSON (1958) based on a Pearson type IV-approximation to the non-central t-distribution.

IV. The approximation of B.I. HARLEY (1957) based on an approximation of the distribution of a function of $\underline{t}$ by the distribution of the correlation coefficient in samples from a bivariate normal distribution.

V. An approximation mentioned in D.B. OWEN's table of factors for one-sided tolerance limits for a normal distribution (1958). This approximation is based on an approximate relation between one- and two-sided tolerance limits.

VI. An approximation based on a normal approximation to the non-central t-distribution.

These approximations are compared with the exact percentage points for several values of the non-centrality parameter δ, the probability α and the number of degrees of freedom f. For the

central t-distribution the approximations are especially important for large values of f, where no tables are available. Then one of the approximations II is the best one: it is identical with the exact percentage points in three decimal places for $0{,}50 \leq \alpha \leq 0{,}99$. For the non-central t-distribution we used f = 2(1)9. For these values of f the approximations I and II are the most important ones. In some cases III or IV may be better, but for these approximations more computational work is needed. The approximations V and VI are not recommended; VI only gives reasonable results for the central t-distribution.
Further the report contains a description of the exact tables of the non-central t-distribution.

1. Introduction.

This report contains some approximations to the percentage points of the non-central t-distribution. Most of these approximations are known in litterature; some of them however were only known for the special case of the central t-distribution (STUDENT's distribution), but could be generalized for the non-central t-distribution.

In section 2 a description will be given of the existing tables of the non-central t-distribution. The approximations to the percentage points will be given in section 3; in section 4 the approximations will be compared with the exact percentage points.

2. Description of the tables of the non-central t-distribution.

Consider two independent random variables $\underline{x}$ and $\underline{\chi}^2$, where $\underline{x}$ possesses a N(0,1)-distribution[1] and $\underline{\chi}^2$ a χ^2-distribution with f degrees of freedom. Then the random variable

$$(2;1) \qquad \underline{t} \overset{\text{def}}{=} \frac{\underline{x} + \delta}{\underline{\chi}} \sqrt{f}$$

possesses a non-central t-distribution with f degrees of freedom and non-centrality parameter δ.

1) "$\underline{x}$ possesses a $N(\mu,\sigma)$-distribution" is an abbreviation for : $\underline{x}$ possesses a normal distribution with mean μ and variance σ^2.

Let $t(f,\delta,\alpha)$ be defined by

(2;2) $P\left[\underline{t} \geqq t(f,\delta,\alpha) \mid f, \delta\right] = \alpha.$

Now we have

(2;3) $P\left[\underline{t} \geqq t \mid f,\delta\right] = P\left[\underline{t} \leqq -t \mid f, -\delta\right],$

consequently

(2;4) $t(f,\delta,\alpha) = -t(f,-\delta,1-\alpha).$

Tables of the non-central t-distribution are

I G.J. RESNIKOFF and G.J. LIEBERMAN (1957).

This table contains

a. A table of $\frac{t(f,\delta,\alpha)}{\sqrt{f}}$ as a function of f, α and

(2;5) $$p \overset{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{\frac{\delta}{\sqrt{f+1}}}^{\infty} e^{-\frac{1}{2}x^2}\,dx$$

for f = 2(1)24(5)49,

$$\alpha = \begin{cases} 0{,}995;0{,}99;0{,}95;0{,}90;0{,}75;0{,}50;0{,}25;0{,}10;0{,}05 \text{ for } f=2(1)4, \\ 0{,}995;0{,}99;0{,}95;0{,}90;0{,}75;0{,}50;0{,}25;0{,}10;0{,}05; 0{,}01;0{,}005 \text{ for } f=5(1)24(5)49 \end{cases}$$

and p = 0,25;0,15;0,10;0,065;0,04;0,025;0,01;0,004;0,0025; 0,001.

b. A table of

(2;6) $P\left[\frac{t}{\sqrt{f}} \leqq x \mid f,\delta\right]$

as a function of x, f and p, for the above mentioned values of f and p.
This table only contains values of $p < 0{,}5$ (so values of $\delta > 0$). For negative values of δ (2;3) and (2;4) may be used For $p = 0{,}5$ ($\delta = 0$) the non-central t-distribution reduces to STUDENT's distribution. In this case the exact percentage

points may e.g. be found in the tables of E.S. PEARSON and H.O. HARTLEY (1954) (p. 138). This table contains $t(f,0,\alpha)$ for

$$f = 1(1)30, 40, 60, 120$$

and $\alpha = 0{,}40; 0{,}25; 0{,}10; 0{,}05; 0{,}025; 0{,}01; 0{,}005; 0{,}0025; 0{,}001; 0{,}0005.$

The table of RESNIKOFF and LIEBERMAN will now be illustrated by means of an example[1]. Let $\underline{y}$ possess a $N(\mu,\sigma)$-distribution (μ and σ unknown); let $y_1,\ldots,y_n$ be n independent observations of $\underline{y}$. Using these observations the hypothesis

(2;7) $\quad H_o: \ P\left[\underline{y} \geq a\right] = p_o,$

will be tested, where a and p_o $(0 < p_o < 1)$ are given numbers. Then, if H_o is true, we have

(2;8) $\quad \dfrac{a-\mu}{\sigma} = u_{p_o},$

where u_ε is defined by

(2;9) $\quad P\left[\underline{u} \geq u_\varepsilon\right] = \varepsilon \quad$ ($\underline{u}$ possesses a $N(0,1)$-distribution).

The test statistic is

(2;10) $\quad t = \dfrac{\bar{y}-a}{s}\sqrt{n},$

where

(2;11) $$\left\{\begin{array}{l} \bar{y} \overset{\text{def}}{=} \dfrac{1}{n}\sum\limits_{i=1}^{n} y_i, \\[2ex] s^2 \overset{\text{def}}{=} \dfrac{1}{n-1}\sum\limits_{i=1}^{n} (y_i-\bar{y})^2. \end{array}\right.$$

Now we have

1) Further examples of the use of the non-central t-distribution are given by N.L. JOHNSON and B.L. WELCH (1940) and by G.J. RESNIKOFF and G.J. LIEBERMAN (1957).

$$(2;12) \qquad \underline{t} = \frac{\frac{\bar{\underline{y}}-\mu}{\sigma}\sqrt{n} - \frac{a-\mu}{\sigma}\sqrt{n}}{\frac{\underline{s}\sqrt{n-1}}{\sigma}} \sqrt{n-1},$$

where

$$(2;13) \qquad \begin{cases} \underline{x} \overset{\text{def}}{=} \frac{\bar{\underline{y}}-\mu}{\sigma}\sqrt{n} \text{ possesses a } N(0,1)\text{-distribution,} \\ \underline{\chi}^2 \overset{\text{def}}{=} \left(\frac{\underline{s}\sqrt{n-1}}{\sigma}\right)^2 \text{ possesses a } \chi^2\text{-distribution} \\ \qquad \text{with } n-1 \text{ degrees of freedom.} \end{cases}$$

Further $\underline{x}$ and $\underline{\chi}^2$ are distributed independently, consequently $\underline{t}$ possesses a non-central t-distribution with parameters $f = n-1$ and $\delta = -\frac{a-\mu}{\sigma}\sqrt{n}$. If the hypothesis H_o is true the non-centrality parameter is (cf.(2;8))

$$(2;14) \qquad \delta = -u_{p_o}\sqrt{n}.$$

If an upper critical region is chosen the tailprobability is

$$(2;15) \qquad P\left[\underline{t} \geq t \mid f = n-1,\ \delta = -u_{p_o}\sqrt{n}\right] =$$

$$= \begin{cases} P\left[\underline{t} \leq -t \mid f = n-1,\ \delta = u_{p_o}\sqrt{n}\right] \text{ if } p_o < 0{,}5 \\ 1 - P\left[\underline{t} \leq t \mid f = n-1,\ \delta = u_{1-p_o}\sqrt{n}\right] \text{ if } p_o > 0{,}5. \end{cases}$$

These probabilities may be found in table b mentioned above for $f = n-1$ and (cf. (2;5))

$$(2;16) \qquad \begin{cases} p = p_o & \text{if } p_o < 0{,}5 \\ p = 1 - p_o & \text{if } p_o > 0{,}5\ . \end{cases}$$

The critical values with level of significance α may be found in table a.
If e.g.

$$n = 10, \ p_o = 0{,}25, \ \alpha = 0{,}05$$

then we have (cf.(2;4))

$$(2;17) \qquad t(9;-u_{0,25}\sqrt{10};0{,}05) = -t(9;u_{0,25}\sqrt{10};0{,}95) =$$

$$= -0{,}164\sqrt{9} = -0{,}492.$$

With $\alpha = 0,10$ and the same values of n and p_o we have

$$(2;18)\quad t(9;-u_{0,25}\sqrt{10};0,10) = -t(9;u_{0,25}\sqrt{10};0,90) = = -0,283\sqrt{9} = -0,849.$$

If, however, $0,25 < p_o < 0,5$ or $0,5 < p_o < 0,75$ then the percentage points and tailprobabilities cannot be found in the table of RESNIKOFF and LIEBERMAN. In many cases the percentage points may then be found in the table of

II N.L. JOHNSON and B.L. WELCH (1940).

This table contains

a. A table (table V) of a quantity $\lambda(f,\delta,\alpha)$ as a function of f and of

$$(2;19)\qquad \eta' \overset{\text{def}}{=} \frac{\delta}{\sqrt{2f+\delta^2}}$$

for $\alpha = 0,05$
and $f = 4(1)9, 16, 36, 144, \infty$.
For other values of f $\lambda(f,\delta,\alpha)$ may be found by linear interpolation with respect to $\frac{12}{\sqrt{f}}$.
From f, δ and $\lambda(f,\delta,\alpha)$ then follows

$$(2;20)\quad t(f,\delta,\alpha) = \frac{\delta+\lambda(f,\delta,\alpha)\sqrt{1+\frac{1}{2f}\left[\delta^2-\{\lambda(f,\delta,\alpha)\}^2\right]}}{1-\frac{1}{2f}\left[\lambda(f,\delta,\alpha)\right]^2}.$$

By means of this table and the relation (2;4) the percentage points $t(f,\delta,\alpha)$ may be found for $\alpha=0,05$ and $\alpha=0,95$. For the example mentioned above ($f=9$, $\delta=-u_{0,25}\sqrt{10}$) we find

$$\delta = -2,1329, \qquad \eta' = -0,4492.$$

Linear interpolation in table V gives $\lambda(f,\delta,\alpha) = 1,6312$ and from (2;20) follows

$$(2;21)\qquad t(9;-u_{0,25}\sqrt{10};0,05) = -0,491.$$

b. For other values of α the following table (table IV) of JOHNSON and WELCH may be used.

Let $\delta(f,t_o,\alpha)$ be defined by

$$(2;22) \qquad P\left[\underline{t} \geqq t_o \mid f,\delta(f,t_o,\alpha)\right] = \alpha .$$

By means of table IV $\delta(f,t_o,\alpha)$ may be found. This table contains a quantity $\lambda(f,t_o,\alpha)$ as a function of f, α and

$$(2;23) \quad \begin{cases} y' = \dfrac{t_o}{\sqrt{2f + t_o^2}} & \text{if } |t_o| \leqq 0,75\sqrt{2f} \\[2ex] y = \sqrt{\dfrac{2f}{2f + t_o^2}} & \text{if } |t_o| \geqq 0,75\sqrt{2f} \end{cases}$$

for α = 0,50;0,40;0,30;0,20;0,10;0,05;0,025;0,01;0,005 and f = 4(1)9,16,36,144,∞.

For other values of f $\lambda(f,t_o,\alpha)$ may be found by linear interpolation with respect to $\frac{12}{\sqrt{f}}$.

From f, t_o and $\lambda(f,t_o,\alpha)$ then follows

$$(2;24) \qquad \delta(f,t_o,\alpha) = t_o - \lambda(f,t_o,\alpha)\sqrt{\frac{2f + t_o^2}{2f}} .$$

This table may also be used to calculate $t(f,\delta,\alpha)$. From (2;20) we see that $t(f,\delta,\alpha)$ may be found if $\lambda(f,\delta,\alpha)$ is known. In table IV λ is given as a function of f,t_o and α; consequently an iteration method must be used in order to find $t(f,\delta,\alpha)$. As a first approximation JOHNSON and WELCH give

$$(2;25) \qquad t(f,\delta,\alpha) \approx \frac{\delta + u_\alpha\sqrt{1 + \frac{1}{2f}(\delta^2 - u_\alpha^{\;2})}}{1 - \frac{1}{2f}u_\alpha^{\;2}} .$$

In the example mentioned above with f = 9, $\delta = -u_{0,25}\sqrt{10}$ and α=0,10 this approximation gives $t(f,\delta,\alpha)$ = -0,827; the exact value is (cf.(2;18)) -0,849.

If e.g. f=9, $\delta = -u_{0,30}\sqrt{10}$ and α=0,10 then $t(f,\delta,\alpha)$ cannot be found in the table of RESNIKOFF and LIEBERMAN, p being equal to 0,30. Also table V of JOHNSON and WELCH cannot be used, α being equal to 0,10. Consequently in this case $t(f,\delta,\alpha)$ can only be found by iteration, using table IV of JOHNSON and WELCH.

Finally we mention the table of

III D.B. OWEN (1958)

This table contains

$$(2;26) \qquad \frac{t(f,\delta,\alpha)}{\sqrt{f+1}}$$

as a function of n, γ and P with

$$(2;27) \qquad \begin{cases} n = f+1 \\ \gamma = 1-\alpha \\ \delta = u_{1-P}\sqrt{n}. \end{cases}$$

The table is constructed for determining tolerance limits for a normal distribution. Let $\underline{y}$ possess a $N(\mu,\sigma)$-distribution (μ and σ unknown) and let $\bar{y}$ and s^2 be defined by (2;11). Let further (cf. (2;26) and (2;27))

$$(2;28) \qquad k(n,P,\gamma) \underline{\underline{\mathrm{def}}} \frac{t(f,\delta,\alpha)}{\sqrt{f+1}},$$

then $\bar{y} + k(n,P,\gamma)\, s$ is an upper tolerance limit with tolerance coefficient P and confidence coefficient γ, i.e. at least a proportion P of the normal distribution is less than $\bar{y} + k(n,P,\gamma)s$ with confidence γ. The quantity $\bar{y} - k(n,P,\gamma)s$ is a lower tolerance limit with tolerance coefficient P and confidence coefficient γ, i.e. at least a proportion P of the normal distribution is larger than $\bar{y} - k(n,P,\gamma)s$ with confidence γ.

The table consists of the following four parts:

a. A table of $k(n,P,\gamma)$ calculated by means of table V of JOHNSON and WELCH for the same values of f and α, consequently for

$$n = f+1 = 5(1)10, 17, 37, 145, \infty$$
$$\gamma = 1-\alpha = 0{,}95.$$

For P the following values are chosen

$P = 0{,}70(0{,}05)0{,}85;0{,}875;0{,}90;0{,}935;0{,}95;0{,}96;0{,}975;0{,}99;$
$0{,}995;0{,}996;0{,}9975;0{,}999;0{,}9995.$

b. A table of $k(n,P,\gamma)$ calculated by means of the above mentioned table of RESNIKOFF and LIEBERMAN (Ia). The table contains, for f and p = 1-P, the same values as the table of RESNIKOFF and LIEBERMAN; consequently

$$n = f+1 = 3(1)25(5)50$$
$$P = 1-p = 0{,}75;0{,}85;0{,}90;0{,}935;0{,}96;0{,}975;0{,}99;0{,}996;0{,}9975;0{,}999.$$

Further it contains $n=\infty$. For α however the table only contains values $< \frac{1}{2}$:

$$\gamma = 1-\alpha = \begin{cases} 0{,}75;0{,}90;0{,}95 \text{ for } n = 3(1)5 \\ 0{,}75;0{,}90;0{,}95;0{,}99;0{,}995 \text{ for } n = 6(1)25(5)50,\infty. \end{cases}$$

c. A table of the approximation to $k(n,P,\gamma)$ according to (2;25). This approximation is mentioned in the form

$$(2;29) \qquad k(n,P,\gamma) \approx \frac{-u_P + \sqrt{u_P^2 - AB}}{A} \text{ 1)}$$

where

$$(2;30) \qquad \begin{cases} A = 1 - \dfrac{u_\gamma^2}{2(n-1)} \\ B = u_P^2 - \dfrac{u_\gamma^2}{n}. \end{cases}$$

This table contains the following values of n, P and γ,

$$n = 2(1)200(5)400(25)1000,\infty$$
$$P = 0{,}70;0{,}80;0{,}90;0{,}95;0{,}99;0{,}999$$
$$\gamma = 0{,}70;0{,}80;0{,}90;0{,}95;0{,}99;0{,}999.$$

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1) In C. EISENHART, M.W. HASTAY and W.A. WALLIS (1947) this approximation is mentioned in the form (2;29) on p.19-20 and in the form (2;25) on p. 75.

d. A table of another approximation to $k(n,P,\gamma)$. This approximation will be described in section 3.5. The values of n, P and γ used in this table are

$$n = 2(1)102(2)180(5)300(10)400(25)750(50)1000,\infty$$
$$P = 0{,}875;0{,}95;0{,}975;0{,}995;0{,}9995$$
$$\gamma = 0{,}75;0{,}90;0{,}95;0{,}99.$$

By means of OWEN's tables $t(f,\delta,\alpha)$ may thus be found for positive values of δ and for $\alpha < \frac{1}{2}$ (cf. (2;27) and (2;28)).

From the foregoing we see

1. that the exact percentage points may easier be found from table a of RESNIKOFF and LIEBERMAN and the tables a and b of OWEN than from the table of JOHNSON and WELCH. Thus it is adviceable to use the table of JOHNSON and WELCH only if $t(f,\delta,\alpha)$ cannot be found in the other tables,
2. that the table of RESNIKOFF and LIEBERMAN does not only contain percentage points, but also tailprobabilities,
3. that an iteration method is necessary for the table of JOHNSON and WELCH if $\alpha \neq 0{,}05$ and $\neq 0{,}95$. In this case an accurate approximation is very useful,
4. that for some cases (e.g. for f=2 or f=3 and $0 < p < 0{,}50$ or $0{,}50 < p < 0{,}75$) the exact percentage points are not available. Thus also in these cases an accurate approximation will be very useful.

3. The approximations to $t(f,\delta,\alpha)$.

3.1. Approximations based on a normal approximation to the distribution of $\underline{\chi}$.

It follows from (2;1) that the inequality $\underline{t} \geq t(f,\delta,\alpha)$ may be written in the form

$$(3.1;1) \qquad \underline{x} - \frac{t(f,\delta,\alpha)}{\sqrt{f}}\,\underline{\chi} \geqq -\delta .$$

Now let

(3.1;2) $$\beta_f \overset{\text{def}}{=\!=} \frac{\sqrt{2}\,\Gamma(\frac{f+1}{2})}{\sqrt{f}\,\Gamma(\frac{f}{2})}$$

then we have

(3.1;3) $$\mathcal{E}\underline{\chi} = \beta_f \sqrt{f}$$

and

(3.1;4) $$\sigma^2(\underline{\chi}) = f(1 - \beta_f^2).$$

Further we have (see Appendix A.1)

(3.1;5) $$\begin{cases} 1.\ \lim\limits_{f\to\infty} \beta_f = 1, \\ 2.\ \lim\limits_{f\to\infty} f(1-\beta_f^2) = \frac{1}{2}, \\ 3.\ \frac{2f-1}{2f} < \beta_f^2 < \frac{2f}{2f+1} \quad \text{for } f = 1,2,\ldots \end{cases}$$

Further the random variable

$$\frac{\underline{\chi} - \mathcal{E}\underline{\chi}}{\sigma(\underline{\chi})}$$

is, for $f \to \infty$, asymptotically $N(0,1)$-distributed.
Therefore the distribution of $\underline{\chi}$ is, for finite f, approximated by a $N(\mathcal{E}\underline{\chi}, \sigma(\underline{\chi}))$-distribution. Thus the distribution of (cf.(3.1;1))

$$\underline{x} - \frac{t(f,\delta,\alpha)}{\sqrt{f}}\,\underline{\chi}$$

is approximated by a normal distribution with mean

$$-t(f,\delta,\alpha)\,\beta_f$$

and variance

$$1 + \{t(f,\delta,\alpha)\}^2(1-\beta_f^2).$$

This gives the approximation

(3.1;6) $$P\left[\underline{t} \geq t(f,\delta,\alpha)\,|\,f,\delta\right] \approx P\left[\underline{u} \geq \frac{-\delta + t(f,\delta,\alpha)\,\beta_f}{\sqrt{1+\{t(f,\delta,\alpha)\}^2(1-\beta_f^2)}}\right],$$

where $\underline{u}$ is N(0,1)-distributed.[1] An approximation to $t(f,\delta,\alpha)$ thus follows from (cf.(2;9))

$$(3.1;7) \qquad \frac{-\delta + t(f,\delta,\alpha)\,\beta_f}{\sqrt{1+\{t(f,\delta,\alpha)\}^2(1-\beta_f^2)}} \approx u_\alpha \quad .$$

and from (3.1;7) follows

$$\boxed{\text{Ia.} \qquad t(f,\delta,\alpha) \approx \frac{\delta\beta_f + u_\alpha\sqrt{\beta_f^2 + (1-\beta_f^2)(\delta^2-u_\alpha^2)}}{\beta_f^2 - u_\alpha^2(1-\beta_f^2)}}$$

This approximation was derived by W.J. JENNET and B.L. WELCH (1939).

A table of β_f may e.g. be found in the tables of E.S.PEARSON and H.O. HARTLEY (1954, p.184) for f = 1(1)20(5)50(10)100. A table of β_f, β_f^2 and $1-\beta_f^2$ for f = 1(1)20 may be found at the end of this report (table A). The paper of JENNET and WELCH (1939) contains a table of $\beta_f\sqrt{f}$ and $f\sqrt{2(1-\beta_f^2)}$.

For STUDENT's distribution ($\delta=0$) Ia reduces to

$$(3.1;8) \qquad t(f,\delta,\alpha) \approx \frac{u_\alpha}{\sqrt{\beta_f^2 - u_\alpha^2(1-\beta_f^2)}} \; .$$

In an analogous way W.A. HENDRICKS derived an approximation to the percentage points of STUDENT's distribution. He approximates the distribution of $\underline{\chi}$ by a N($\mathcal{E}\underline{\chi}$, $\sqrt{\tfrac{1}{2}}$)-distribution (cf.(3.1;4) and (3.1;5.2)). If this approximation is applied for the non-central t-distribution, we obtain

$$\boxed{\text{Ib.} \qquad t(f,\delta,\alpha) \approx \frac{\delta\beta_f + u_\alpha\sqrt{\beta_f^2 + \frac{1}{2f}(\delta^2-u_\alpha^2)}}{\beta_f^2 - \frac{1}{2f}u_\alpha^2}}$$

For $\delta=0$ Ib. reduces to (cf. HENDRICKS' paper (1936))

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1) If $t(f,\delta,\alpha)=0$ the equalitysign holds in (3.1;6).

$$(3.1;9) \qquad t(f,0,\alpha) \approx \frac{u_\alpha}{\sqrt{\beta_f^2 - \frac{1}{2f}\, u_\alpha^2}} \quad .$$

If we put $\beta_f=1$ in Ib (cf. (3.1;5.1)) we obtain

$$\boxed{\text{Ic.} \qquad t(f,\delta,\alpha) \approx \frac{\delta + u_\alpha \sqrt{1 + \frac{1}{2f}\,(\delta^2 - u_\alpha^2)}}{1 - \frac{1}{2f}\, u_\alpha^2}}$$

This is the approximation mentioned by JOHNSON and WELCH (1940) (cf. also (2;25)). In this case the distribution of $\underline{x}$ is approximated by a $N(\sqrt{f}, \sqrt{\frac{1}{2}})$-distribution.
For STUDENT's distribution Ic reduces to

$$(3.1;10) \qquad t(f,0,\alpha) \approx \frac{u_\alpha}{\sqrt{1 - \frac{1}{2f}\, u_\alpha^2}}$$

The approximations Ia, b and c can only be applied if respectively

$$(3.1;11) \qquad u_\alpha^2 \leqq \begin{cases} \delta^2 + \dfrac{\beta_f^2}{1 - \beta_f^2} \\ \delta^2 + 2f\,\beta_f^2 \\ \delta^2 + 2f \end{cases}$$

Now it follows from (3.1;5.3) that

$$(3.1;12) \qquad 2f\beta_f^2 < \frac{\beta_f^2}{1 - \beta_f^2} < 2f .$$

Consequently, for fixed f and δ, the approximation Ic can be applied for more values of α than Ia and Ia for more values of α than Ib.

If these approximations are applied to the examples of section 2 we obtain (cf. also (2;17) and (2;18))

Table 3.1;1

Comparison of the approximations Ia, b and c with the exact percentage points for f=9 and $\delta = -u_{0,25}\sqrt{10}$.

α	$t(f,\delta,\alpha)$ exact	approximations to $t(f,\delta,\alpha)$		
		Ia	Ib	Ic
0,05	-0,492	-0,491	-0,490	-0,478
0,10	-0,849	-0,850	-0,849	-0,827

From table 3.1;1 we see that the approximations Ia and b are in this case better than Ic.

3.2. The approximations of H. GOLDBERG and H. LEVINE (1946) and of A.M. PEISER (1943 and 1949).

H. GOLDBERG and H. LEVINE (1946) derived the following approximation for the percentage points of STUDENT's distribution

$$(3.2;1)\qquad t(f,0,\alpha) \approx u_\alpha + \frac{u_\alpha^3 + u_\alpha}{4f} + \frac{5u_\alpha^5 + 16u_\alpha^3 + 3u_\alpha}{96f^2} .$$

In order to simplify the computations they give a table of u_α, $\frac{u_\alpha^3 + u_\alpha}{4}$ and $\frac{5u_\alpha^5 + 16u_\alpha^3 + 3u_\alpha}{96}$ for α =0,25;0,10;0,05;0,025;0,01; 0,005;0,0025;0,001.

The approximation (3.2;1) is derived by means of a method given by E.A. CORNISH and R.A. FISHER (1937). By means of this method a random variable, whose distribution depends on a parameter f and possesses, for $f \to \infty$, asymptotically a N(0,1)-distribution, can be expressed as a function of a N(0,1)-variable in powers of $\frac{1}{f}$.

If this method is applied to the non-central t-distribution we obtain (cf. Appendix A.2)

$$\text{IIa.}\quad t(f,\delta,\alpha) \approx u_\alpha + \frac{u_\alpha^3 + u_\alpha}{4f} + \frac{5u_\alpha^5 + 16u_\alpha^3 + 3u_\alpha}{96f^2} + $$
$$+\delta\left[1+\frac{2u_\alpha^2+1}{4f}+\frac{u_\alpha}{4f}\delta+\frac{4u_\alpha^4+12u_\alpha^2+1}{32f^2}+\frac{u_\alpha^3+4u_\alpha}{16f^2}\delta-\frac{u_\alpha^2-1}{24f^2}\delta^2-\frac{u_\alpha}{32f^2}\delta^3\right]$$

If $t(f,0,\alpha)$ is substituted for the first three terms of IIa (cf.(3.2;1)) we obtain

$$\text{IIb.}\quad t(f,\delta,\alpha) \approx t(f,0,\alpha) +$$
$$+\delta\left[1+\frac{2u_\alpha^2+1}{4f}+\frac{u_\alpha}{4f}\delta+\frac{4u_\alpha^4+12u_\alpha^2+1}{32f^2}+\frac{u_\alpha^3+4u_\alpha}{16f^2}\delta-\frac{u_\alpha^2-1}{24f^2}\delta^2-\frac{u_\alpha}{32f^2}\delta^3\right]$$

In order to simplify the computations table B at the end of this report contains the coefficients

$$\frac{2u_\alpha^2+1}{4}\,,\quad \frac{u_\alpha}{4}\,,\quad \frac{4u_\alpha^4+12u_\alpha^2+1}{32}\,,\quad \frac{u_\alpha^3+4u_\alpha}{16}\,,\quad \frac{u_\alpha^2-1}{24} \quad\text{and}\quad \frac{u_\alpha}{32}$$

for α =0,50;0,30;0,25;0,10;0,05;0,025;0,01;0,005;0,0025;0,001.

Omitting in the approximations the terms of degree greater than 1 in $\frac{1}{f}$ we obtain

$$\text{IIc.}\quad t(f,\delta,\alpha) \approx u_\alpha + \frac{u_\alpha^3+u_\alpha}{4f} + \delta\left[1 + \frac{2u_\alpha^2+1}{4f} + \frac{u_\alpha}{4f}\delta\right]$$

and

$$\text{IId.}\quad t(f,\delta,\alpha) \approx t(f,0,\alpha) + \delta\left[1 + \frac{2u_\alpha^2+1}{4f} + \frac{u_\alpha}{4f}\delta\right]$$

For δ=0 IIc reduces to

(3.2;2) $$t(f,0,\alpha) \approx u_\alpha + \frac{u_\alpha^3 + u_\alpha}{4f} .$$

The approximation (3.2;2) was derived by A.M. PEISER (1943 and 1949).

If these approximations are applied to the examples of section 2 ($f=9$, $\delta=-u_{0,25}\sqrt{10}$) we obtain (cf. also table 3.1;1)

Table 3.2;1

Comparison of the approximations IIa, b, c and d with the exact percentage points for $f=9$ and $\delta=-u_{0,25}\sqrt{10}$.

α	$t(f,\delta,\alpha)$ exact	approximations to $t(f,\delta,\alpha)$			
		IIa	IIb	IIc	IId
0,05	-0,492	-0,491	-0,490	-0,491	-0,472
0,10	-0,849	-0,850	-0,850	-0,849	-0,842

Thus in this case the approximations IIa, b and c are better than IId.

3.3. The approximation of M. MERRINGTON and E.S. PEARSON (1958).

M. MERRINGTON and E.S. PEARSON (1958) give an approximation to the percentage points of the non-central t-distribution based on an approximation of the distribution of t by a Pearson type IV distribution.

This approximation is calculated as follows. The first four moments of t are

$$(3.3;1)\quad\begin{cases} \mu \overset{\text{def}}{=} \mathscr{E}(\underline{t}|f,\delta) = \frac{f}{f-1}\delta\beta_f, \\ \sigma^2 \overset{\text{def}}{=} \sigma^2(\underline{t}|f,\delta) = \frac{f}{f-2}(1+\delta^2)-\mu^2, \\ \tilde{\mu}_3 \overset{\text{def}}{=} \mathscr{E}\{(\underline{t}-\mu)^3|f,\delta\} = \mu\left\{\frac{f(2f-3+\delta^2)}{(f-2)(f-3)} - 2\sigma^2\right\}, \\ \tilde{\mu}_4 \overset{\text{def}}{=} \mathscr{E}\{(\underline{t}-\mu)^4|f,\delta\} = \\ = \frac{f^2}{(f-2)(f-4)}(3+6\delta^2+\delta^4)-\mu^2\left[\frac{f\{(f+1)\delta^2+3(3f-5)\}}{(f-2)(f-3)} - 3\sigma^2\right]. \end{cases}$$

Now let

$$(3.3;2)\qquad b_1 \overset{\text{def}}{=} \frac{\tilde{\mu}_3^2}{\sigma^6}\qquad\qquad b_2 \overset{\text{def}}{=} \frac{\tilde{\mu}_4}{\sigma^4}$$

and let $z(b_1,b_2,\alpha)$ be defined by

$$(3.3;3)\qquad P\left[\frac{\underline{z} - \mathscr{E}\underline{z}}{\sigma(\underline{z})} \geqq z(b_1,b_2,\alpha)\right] = \alpha,$$

where $\underline{z}$ possesses a Pearson-distribution with parameters b_1 and b_2. Then we have

$$\boxed{\text{III}\quad t(f,\delta,\alpha) \approx \mu + \sigma z(b_1,b_2,\alpha)}$$

A table of $z(b_1,b_2,\alpha)$ may e.g. be found in the tables E.S. PEARSON and H.O. HARTLEY (1954) (p.206) for

b_1 = 0,00;0,01;0,03;0,05(0,05)0,20(0,10)1,00,

b_2 = 1,8(0,2)5,0

and α = 0,995;0,99;0,975;0,95;0,05;0,025;0,01;0,005.

For STUDENT's distribution we obtain (cf.(3.3;1) and (3.3;2))

(3.3;4) $$b_1 = 0, \quad b_2 = \frac{3(f-2)}{f-4} \quad .$$

Consequently

(3.3;5) $$t(f,0,\alpha) \approx \sqrt{\frac{f}{f-2}}\, z(0, \frac{3(f-2)}{f-4}, \alpha).$$

The approximation III cannot be applied for all values of f, δ and α because

1. the fourth moment of $\underline{t}$ does not exist for $f \leqq 4$. Consequently the approximation can only be applied for $f \geqq 5$.

2. the above mentioned table of $z(b_1,b_2,\alpha)$ does not contain all values of b_1 and b_2 of the non-central t-distribution. If e.g. $\delta=0$ and $f=6$ then (cf.(3.3;4))

(3.3;6) $$b_1 = 0, \quad b_2 = 6,$$

but the table only contains values of $b_2 \leqq 5$. For $f=7$ and $\delta=0$ we have

(3.3;7) $$b_1 = 0, \quad b_2 = 5.$$

Consequently for STUDENT's distribution the approximation can only be used for $f \geqq 7$.
For a more detailed description of the relation between f, δ, b_1 and b_2 we refer to the paper of MERRINGTON and PEARSON.

3. Finally the approximation can only be applied for the values of α in the table of $z(b_1,b_2,\alpha)$.

3.4. The approximation of B.I. HARLEY (1957)

Let $\underline{r}$ denote the estimate of the correlation coefficient ρ of a two-dimensional normal distribution, calculated from n independent pairs of observations. Then, if $\rho=0$, the random variable

(3.4;1) $$\frac{\underline{r}}{\sqrt{1-\underline{r}^2}} \sqrt{n-2}$$

possesses a STUDENT-distribution with f=n-2 degrees of freedom. Thus if $r\,(n,\rho,\alpha)$ is defined by

(3.4;2) $$P\left[\underline{r} \geqq r(n,\rho,\alpha) \mid n,\rho\right] = \alpha,$$

we have

(3.4;3) $$t(f,0,\alpha) = \frac{r(f+2,0,\alpha)}{\sqrt{1-\{r(f+2,0,\alpha)\}^2}}\sqrt{f}.$$

B.I. HARLEY (1957) shows that, for $\rho \neq 0$, the random variable

(3.4;4) $$\frac{\underline{r}}{\sqrt{1-\underline{r}^2}}\sqrt{n-2}\sqrt{\frac{2(1-\rho^2)}{2-\rho^2}}$$

is approximately distributed as $\underline{t}$ with

(3.4;5) $$f = n-2, \qquad \delta = \rho\sqrt{\frac{2n-3}{2-\rho^2}}.$$

Consequently

$$\boxed{\text{IV} \quad t(f,\delta,\alpha) \approx \frac{r(f+2,\rho,\alpha)}{\sqrt{1-\{r(f+2,\rho,\alpha)\}^2}}\sqrt{\frac{f(2f+1)}{2f+1+\delta^2}} \quad \text{with } \rho = \delta\sqrt{\frac{2}{2f+1+\delta^2}} \neq 0}.$$

For $\rho \neq 0$ no tables of $r(n,\rho,\alpha)$ are available. The table of F.N. DAVID (1938) contains the exact distribution of $\underline{r}$ for $n=3(1)25,50,100,200,400$ and $\rho=0,0(0,1)0,9$. From this table the percentage points $r(n,\rho,\alpha)$ may be obtained by interpolation with respect to ρ and r, but this is not very accurate. Therefore the approximation can better be considered as an approximation to $r(n,\rho,\alpha)$ on the basis of the exact tables of $t(f,\delta,\alpha)$. We then obtain

(3.4;6) $$r(n,\rho,\alpha) \approx t(n-2,\delta,\alpha)\sqrt{\frac{2-\rho^2}{2(n-2)(1-\rho^2)+(2-\rho^2)\{t(n-2,\delta,\alpha)\}^2}}$$

with

(3.4;7) $$\delta = \rho\sqrt{\frac{2n-3}{2-\rho^2}}.$$

A comparison of the approximation (3.4;6) with the exact percentage

points $r(n,\rho,\alpha)$ is difficult, because of the lack of tables of $r(n,\rho,\alpha)$.
B.I. HARLEY gives a comparison of approximation IV with the exact values of $t(f,\delta,\alpha)$ for $f=7$ and 16. He chooses δ in such a way that ρ is a value occurring in the table of F.N. DAVID; then only an interpolation with respect to r is necessary to obtain $r(n,\rho,\alpha)$.

3.5. The approximation in the table of D.B. OWEN (1958).

Apart from the one-sided tolerance limits $\bar{y}+k(n,P,\gamma)s$ and $\bar{y}-k(n,P,\gamma)s$ mentioned in section 2, also two-sided tolerance limits may be obtained. Then a quantity $\lambda(n,\beta,\gamma)$ is determined in such a way that the proportion of the normal distribution between $\bar{y}-\lambda(n,\beta,\gamma)s$ and $\bar{y}+\lambda(n,\beta,\gamma)s$ is at least β with confidence γ.
A. WALD and J. WOLFOWITZ (1946) derived the following approximation to $\lambda(n,\beta,\gamma)$

$$(3.5;1) \qquad \lambda(n,\beta,\gamma) \approx r\sqrt{\frac{n-1}{\chi^2(n-1,\gamma)}},$$

where $\chi^2(n-1,\gamma)$ is defined by

$$(3.5;2) \qquad P\left[\underline{\chi}^2 \geqq \chi^2(n-1,\gamma)\,|\,n-1\right] = \gamma$$

and r by

$$(3.5;3) \qquad \frac{1}{\sqrt{2\pi}} \int_{\frac{1}{\sqrt{n}}-r}^{\frac{1}{\sqrt{n}}+r} e^{-\frac{1}{2}t^2}\,dt = \beta .$$

A table of $\lambda(n,\beta,\gamma)$ according to (3.5;1) may be found in C. EISENHART, N.W. HASTAY and W.A. WALLIS (1946, p.102-107) for

$n = 2(1)102(2)180(5)300(10)400(25)750(50)1000, \infty$
$\beta = 0,75;0,90;0,95;0,99;0,999,$
$\gamma = 0,75;0,90;0,95;0,99.$

According to McCLUNG (1955) we have

(3.5;4) $\qquad k(n,P,\gamma) \approx 1{,}05\ \lambda(n,2P-1,\gamma)$

and from (3.5;4) follows (cf. (2;27) and (2;28))

$$\boxed{\text{V} \quad t(f,\delta,\alpha) \approx 1{,}05\sqrt{f+1}\ \lambda(f+1,2P-1,1-\alpha) \text{ with } u_{1-P} = \frac{\delta}{\sqrt{f+1}} > 0}\ .$$

The table of D.B. OWEN (1958) contains this approximation to $\frac{t(f,\delta,\alpha)}{\sqrt{f+1}}$ for the abovementioned values of n, β and γ, thus for

$f = n-1 = 1(1)101(2)179(5)299(10)399(25)749(50)999,\infty,$

$P = \frac{\beta+1}{2} = 0{,}875;0{,}95;0{,}975;0{,}995;0{,}9995,$

$\alpha = 1-\gamma = 0{,}25;0{,}10,0{,}05;0{,}01.$

Now it may easily be seen that

(3.5;5) $\qquad \lim\limits_{n\to\infty} k(n,P,\gamma) = \lim\limits_{n\to\infty} \lambda(n,2P-1,\gamma) = u_{1-P}.$

Consequently

(3.5;5) $\qquad \lim\limits_{n\to\infty} \frac{k(n,P,\gamma)}{\lambda(n,2P-1,\gamma)} = 1,$

and from (3.5;5) it follows that, although the approximation may be accurate for small values of f, for large values of f, the accuracy decreases if f increases. Therefore we do not here consider this approximation any further.

3.6. <u>An approximation based on a normal approximation to the non-central t-distribution.</u>

The random variable

$$\frac{\underline{t} - \mathcal{E}\underline{t}}{\sigma(\underline{t})}\ ,$$

where (cf. (3.3;1))

$$(3.6;1)\qquad \begin{cases} \mathcal{E}\underline{t} = \frac{f}{f-1}\,\delta\beta_f. \\ \sigma^2(\underline{t}) = \frac{f}{f-2}\,(1+\delta^2) - \frac{f^2}{(f-1)^2}\,\delta^2\beta_f^2 \end{cases}$$

is, for $f \rightarrow \infty$, asymptotically N(0,1)-distributed. Consequently

$$\boxed{\text{VI}\quad t(f,\delta,\alpha) \approx \frac{f}{f-1}\,\delta\,\beta_f + u_\alpha\sqrt{\frac{f}{f-2}(1+\delta^2) - \frac{f^2}{(f-1)^2}\,\delta^2\beta_f^2}\ \text{ for } f > 2}.$$

For STUDENT's distribution this approximation reduces to

$$(3.6;2)\qquad t(f,0,\alpha) \approx u_\alpha\sqrt{\frac{f}{f-2}} \quad \text{for } f > 2.$$

The normal approximation to the distribution of $\frac{\underline{t} - \mathcal{E}\underline{t}}{\sigma(\underline{t})}$ may only be used for very small values of δ and large values of f. Therefore we only applied approximation VI for STUDENT's distribution.

4. Comparison of the approximations with the exact percentage points.

The approximations are compared with the exact percentage points for

α = 0,99;0,975;0,95;0,90;0,70;0,50;0,30;0,10;0,05;0,025;0,01,
p = 0,50;0,45;0,40;0,35;0,30;0,25;0,15;0,125

and for all values of $f \leq 9$, where exact percentage points are available. For p=0,50 we moreover used f = 30, 40 and 60.
The results are given in the tables C-M. In all tables the most accurate approximation is indicated as follows. Approximation III being applicable only in a few cases, we first indicated the most accurate approximation of all except III by underlining. If III is as accurate as the most accurate of all other approximations, it is underlined too; if III is the most accurate of all approximations it is underlined by --- . The most accurate approximation is defined as the one with the smallest difference with the exact value according to JOHNSON and WELCH; if this exact value is not available we compare the approximations with

the exact value according to RESNIKOFF and LIEBERMAN.

We first consider the approximations to STUDENT's distribution ($p = 0{,}50$; tables C and D). In this case the approximations

I a, b, c, II a, c, III and IV

may be applied.
The approximations II b and d reduce to the equality $t(f,0,\alpha) = t(f,0,\alpha)$ and approximation IV gives, for $p = 50$, the exact percentage points on the basis of tables of $r\ (n,0,\alpha)$(cf. section 3;4).
STUDENT's distribution being symmetric we only applied the approximations for values of $\alpha < \frac{1}{2}$.
From table C we see that, for small values of f, IIa is the most accurate approximation of all except III. In some cases, especially for small α, approximation III is better. However an approximation to $t\ (f,0,\alpha)$ for small values of f will only be necessary for values of α not occurring with existing tables. Then however III cannot be applied, the tables for this approximation only containing the values $\alpha = 0{,}05; 0{,}025; 0{,}01$ and $0{,}005$.
For large values of f (table D) IIa is the most accurate approximation; it is, in three decimal places, indentical with the exact value.

We now consider the non-central t-distribution. The exact values were calculated as follows:

For all values of α and p mentioned in the beginning of this section the exact percentage points were calculated from the tables of JOHNSON and WELCH for $f = 4(1)9$. A table of these exact values is given at the end of this report (table N,p.55,56). For $p = 0{,}25$ and $0{,}15$ they were also calculated from the table of RESNIKOFF and LIEBERMAN (cf. section 2, table Ia and Ib). For

$$\alpha = \begin{cases} 0{,}99; 0{,}95; 0{,}90; 0{,}50; 0{,}10; 0{,}05 & \text{and } f = 2(1)9 \\ 0{,}01 & \text{and } f = 5(1)9 \end{cases}$$

we used table Ia. For the other values of α, thus for

$$\alpha = \begin{cases} 0{,}975; 0{,}70; 0{,}30; 0{,}025 & \text{and } f = 2(1)9 \\ 0{,}01 & \text{and } f = 2(1)4 \end{cases}$$

the percentage points were found by linear interpolation in the table of the distribution function of t (cf. section 2, table Ib). The following approximations were applied

I a, b and c in all cases were (cf. section 3.1)

$$\beta_f^2 - (1-\beta_f^2)(\delta^2 - u_\alpha^2) \geqq 0 \qquad \text{for Ia}$$

$$\beta_f^2 - \frac{\delta^2 - u_\alpha^2}{2f} \geqq 0 \qquad \text{for Ib}$$

$$1 - \frac{\delta^2 - u_\alpha^2}{2f} \geqq 0 \qquad \text{for Ic.}$$

II a, b, c and d in all cases

III in all cases where it can be applied (cf. section 3.3), i.e. for

$p = 0{,}45$ and $f = 8{,}9$

$p = 0{,}40$ and $f = 8{,}9$

$p = 0{,}35$ and $f = 9$

and $\alpha = 0{,}99; 0{,}975; 0{,}95; 0{,}05; 0{,}025; 0{,}01$.

Approximation IV is not applied. As mentioned in section 3.4 it requires an interpolation in the tables of the distribution of r with respect to r and ρ; this will not give a very accurate result. Therefore this approximation is only compared with the exact values and with the other approximations for the same values of f, δ and α as used by HARLEY (cf. table M).

Approximation VI is not applied for the non-central t-distribution.

The tables E-L are summarized in fig. 1 (p. 57). In this figure we indicated, for all values of p and α used in the tables, the approximations which may be considered as good approximations to t (f, δ, α) for all values of $f \leqq 9$. However this figure may also be used for larger values of f; the only difference for larger values of f will be that more approximations can be used than indicated in fig. 1.

From fig. 1 we see that, among the approximations I, II and III, I and II are the most important ones and that Ia, b and IIa, b are the best ones.

Finally we consider approximation IV. From table M we see that this is a very good approximation. However, as mentioned earlier, it is difficult to apply it for those values of δ for which ρ does not occur in the tables of F.N. DAVID. But, of course, it may be used as an approximation to $r(\rho,n,\alpha)$ from the exact value $t(f,\delta,\alpha)$.

Appendix.

A.1. Proof of (3.1;5)

We first prove (3.1;5.3), thus

$$(A.1;1) \qquad \frac{2f-1}{2f} < \beta_f^2 < \frac{2f}{2f+1} \quad \text{for } f = 1,2,\ldots \ .$$

The inequality (A.1;1) is identical with (cf. (3.1;2))

$$(A.1;2) \qquad \frac{2f-1}{4} < \frac{\Gamma^2(\frac{f+1}{2})}{\Gamma^2(\frac{f}{2})} < \frac{f^2}{2f+1} \quad \text{for } f = 1,2,\ldots \ .$$

The second inequality in (A.1;2) has been proved by J. GURLAND (1956a, p. 645). The first inequality may be proved as follows. From the second inequality in (A.1;2) it follows that

$$(A.1;3) \qquad \frac{\Gamma^2(\frac{n}{2})}{\Gamma^2(\frac{n+1}{2})} > \frac{2n+1}{n^2} \qquad \text{for } n = 1,2,\ldots \ .$$

Substituting $f = n-1$ in (A.1;3) we obtain

$$(A.1;4) \qquad \frac{\Gamma^2(\frac{f+1}{2})}{\Gamma^2(\frac{f}{2}+1)} > \frac{2f+3}{(f+1)^2} \qquad \text{for } f = 0,1,\ldots \ .$$

Further we have $\Gamma(\frac{f}{2}+1) = \frac{f}{2}\Gamma(\frac{f}{2})$ for $f > 0$; consequently

$$(A.1;5) \qquad \frac{\Gamma^2(\frac{f+1}{2})}{\Gamma^2(\frac{f}{2})} > \frac{2f+3}{(f+1)^2} \cdot \frac{f^2}{4} > \frac{2f-1}{4} \quad \text{for } f = 1,2,\ldots \ .$$

Remarks

1. From (A.1;5) it follows that even the inequality

$$(A.1;6) \qquad \frac{2f-1}{2f} + \frac{1}{2f(f+1)^2} < \beta_f^2 < \frac{2f}{2f+1} \quad \text{for } f = 1,2,\ldots$$

holds. In section 3.1 we only need (A.1;1).

2. The second inequality in (A.1;2) is a special case of the inequality (cf. J. GURLAND (1956[b]))

(A.1;7) $$\frac{\Gamma^2(\lambda+\alpha)}{\Gamma(\lambda)\Gamma(\lambda+2\alpha)} \leqq \frac{\lambda}{\alpha^2+\lambda} \quad \text{for } \alpha+\lambda>0 \text{ and } \lambda>0.$$

The equality sign in (A.1;7) holds if and only if $\alpha=0$ and $\alpha=1$. Substituting $\alpha=\frac{1}{2}$ and $\lambda=\frac{f}{2}$ (A.1;7) reduces to the second inequality in (A.1;2). Consequently (A.1;1) and (A.1;6) hold for each $f>0$. In section 3.1 we only need (A.1;1) for $f = 1,2,\ldots$.

From (A.1;1) it follows that

(A.1;8) $$\tfrac{1}{2}\left\{1-\frac{1}{2f+1}\right\} < f(1-\beta_f^2) < \tfrac{1}{2},$$

which proves (3.1;5.2).
Finally (3.1;5.1) follows from (A.1;1).

A.2. Proof of the results of section 3.2.

Let K_r denote the r^{th} cumulant of the distribution of $\frac{t}{\sqrt{f}}$; let m denote the leading term from K_1 and v the leading term for K_2 expanded in powers of $\frac{1}{f}$. Let further

(A.2;1) $$\begin{cases} l_1 \overset{\text{def}}{=} \dfrac{K_1-m}{\sqrt{v}}, \\[2ex] l_2 \overset{\text{def}}{=} \dfrac{K_2-v}{v}, \\[2ex] l_r \overset{\text{def}}{=} \dfrac{K_r}{v^{\frac{1}{2}r}} \quad \text{for } r>2, \end{cases}$$

then, according to E.A. CORNISH and R.A. FISHER (1937).

$$(A.2;2)\quad t(f,\delta,\alpha) = u_\alpha + l_1 + \frac{1}{6} l_3 (u_\alpha^2 - 1) + \tfrac{1}{2} l_2 u_\alpha + \frac{1}{24} l_4 (u_\alpha^3 - 3u_\alpha) - \frac{1}{36} l_3^2 (2u_\alpha^3 - 5u_\alpha) +$$

$$-\frac{1}{6} l_2 l_3 (u_\alpha^2 - 1) + \frac{1}{120} l_5 (u_\alpha^4 - 6u_\alpha^2 + 3) - \frac{1}{24} l_3 l_4 (u_\alpha^4 - 5u_\alpha^2 + 2) +$$

$$+\frac{1}{324} l_3^3 (12u_\alpha^4 - 53u_\alpha^2 + 17) - \frac{1}{8} l_2^2 u_\alpha^2 - \frac{1}{16} l_2 l_4 (u_\alpha^3 - 3u_\alpha) +$$

$$+\frac{1}{720} l_6 (u_\alpha^5 - 10u_\alpha^3 + 15u_\alpha) + \frac{1}{72} l_2 l_3^2 (10u_\alpha^3 - 25u_\alpha) +$$

$$-\frac{1}{384} l_4^2 (3u_\alpha^2 - 24u_\alpha^3 + 29u_\alpha) - \frac{1}{180} l_3 l_5 (2u_\alpha^5 - 17u_\alpha^3 + 21u_\alpha)$$

$$+\frac{1}{288} l_3^2 l_4 (14u_\alpha^5 - 103u_\alpha^3 + 107u_\alpha) - \frac{1}{7776} l_3^4 (252u_\alpha^5 - 1688u_\alpha^3 + 1511$$

$$+ \; \ldots\ldots ,$$

where the terms with coefficients containing l_r for $r \geqq 7$ are omitted.

The cumulants of the distribution of $\underline{t}$ may be computed from the moments of $\underline{t}$, where (cf (2;1))

$$(A.2;3)\qquad \mathcal{E}\underline{t}^s = f^{\frac{1}{2}s}\, \mathcal{E}(\underline{z}+\delta)^s \;\; \mathcal{E}\chi^{-s},$$

with

$$(A.2;4)\qquad \mathcal{E}\underline{\chi}^{-s} = 2^{-\frac{1}{2}s} \frac{\Gamma(\frac{f-s}{2})}{\Gamma(\frac{f}{2})} \quad \text{for } f > s.$$

From (A.2;3) follows (cf. also (3.3;1))

$$(A.2;5)\quad K_2 = \frac{\sigma^2(\underline{t})}{f} = \frac{1}{f}\left[\frac{f}{f-2}(1+\delta^2) - \frac{f^2}{(f-1)^2}\,\delta^2 \beta_f^2\right]$$

where

$$(A.2;6)\qquad \beta_f^2 = 1 - \frac{1}{2f} + \frac{1}{8f^2} + O(\frac{1}{f^3}).$$

Consequently

(A.2;7) $$K_2 = \frac{1}{f}\left[\left(1+\frac{2}{f}+\frac{4}{f^2}+O\left(\frac{1}{f^3}\right)\right)(1+\delta^2)-\left(1+\frac{2}{f}+\frac{3}{f^2}+O\left(\frac{1}{f^3}\right)\right)\left(1-\frac{1}{2f}+\frac{1}{8f^2}+O\left(\frac{1}{f^3}\right)\right)\right]$$

$$= \frac{1}{f}\left[1 + \frac{4+\delta^2}{2f} + \frac{32+15\delta^2}{8f^2} + O\left(\frac{1}{f^3}\right)\right]$$

and from (A.2;7) follows

(A.2;8) $$v = \frac{1}{f} .$$

Thus

(A.2;9) $$l_2 = \frac{K_2-v}{v} = \frac{4+\delta^2}{2f} + \frac{32+15\delta^2}{8f^2} + O\left(\frac{1}{f^3}\right) .$$

Further

(A.2;10) $$K_1 = \frac{\mathcal{E}t}{\sqrt{f}} = \frac{1}{\sqrt{f}}\,\frac{f}{f-1}\,\delta\beta_f ,$$

where

(A.2;11) $$\beta_f = 1 - \frac{1}{4f} + \frac{1}{32f^2} + O\left(\frac{1}{f^3}\right) .$$

Consequently

(A.2;12) $$K_1 = \frac{\delta}{\sqrt{f}}\left[1 + \frac{1}{f} + \frac{1}{f^2} + O\left(\frac{1}{f^3}\right)\right]\left[1 - \frac{1}{4f} + \frac{1}{32f^2} + O\left(\frac{1}{f^3}\right)\right] =$$

$$= \frac{\delta}{\sqrt{f}}\left(1 + \frac{3}{4f} + \frac{25}{32f^2} + O\left(\frac{1}{f^3}\right)\right),$$

thus

(A.2;13) $$m = \frac{\delta}{\sqrt{f}}$$

and

(A.2;14) $$l_1 = \frac{K_1-m}{\sqrt{v}} = \delta\left(\frac{3}{4f} + \frac{25}{32f^2} + O\left(\frac{1}{f^3}\right)\right).$$

Further we have (cf. (3.3;1))

(A.2;15) $$l_3 = \mathcal{E}(t-\mathcal{E}\underline{t})^3 = \mathcal{E}\underline{t}\left[\frac{f(2f-3+\delta^2)}{(f-2)(f-3)} - 2\sigma^2(\underline{t})\right] =$$

$$= \delta\left[1 + \frac{3}{4f} + \frac{25}{32f^2} + o(\frac{1}{f^3})\right] \times$$

$$\times\left[2\left(1 + \frac{2}{f} + \frac{4}{f^2} + o(\frac{1}{f^3})\right)\left(1 + \frac{3}{f} + \frac{9}{f^2} + o(\frac{1}{f^3})\right)\left(1 - \frac{3-\delta^2}{2f}\right) + \right.$$

$$\left. - 2\left(1 + \frac{4+\delta^2}{2f} + \frac{32+15\delta^2}{8f^2} + o(\frac{1}{f^3})\right)\right] =$$

$$= \delta\left[\frac{3}{f} + \frac{69+5\delta^2}{4f^2} + o(\frac{1}{f^3})\right] .$$

In the same way we find

(A.2;16) $$l_4 = \frac{6}{f} + \frac{18\delta^2+48}{f^2} + o(\frac{1}{f^3})$$

(A.2;17) $$l_5 = \frac{105}{f^2}\delta + o(\frac{1}{f^3})$$

(A.2;18) $$l_6 = \frac{240}{f^2} + o(\frac{1}{f^3})$$

and

(A.2;19) $$l_r = o(\frac{1}{f^3}) \qquad \text{for} \quad r \geqq 7.$$

Further we have

$$
(A.2;20)\quad
\begin{cases}
l_3^2 = \dfrac{9}{f^2}\,\delta^2 + O(\dfrac{1}{f^3}) \\[2ex]
l_2 l_3 = \delta\,\dfrac{3(4+\delta^2)}{2f^2} + O(\dfrac{1}{f^3}) \\[2ex]
l_3 l_4 = \delta\,\dfrac{18}{f^2} + O(\dfrac{1}{f^3}) \\[2ex]
l_3^3 = O(\dfrac{1}{f^3}) \\[2ex]
l_2^2 = \dfrac{16+8\delta^2+\delta^4}{4f^2} + O(\dfrac{1}{f^3}) \\[2ex]
l_2 l_4 = \dfrac{3(4+\delta^2)}{f^2} + O(\dfrac{1}{f^3}) \\[2ex]
l_2 l_3^2 = O(\dfrac{1}{f^3}) \\[2ex]
l_4^2 = \dfrac{36}{f^2} + O(\dfrac{1}{f^3}) \\[2ex]
l_3 l_5 = O(\dfrac{1}{f^3}) \\[2ex]
l_3^2 l_4 = O(\dfrac{1}{f^3}) \\[2ex]
l_3^4 = O(\dfrac{1}{f^4})
\end{cases}
$$

Substituting $l_1, l_2, \ldots$ in (A.2;2) we obtain

$$
(A.2;21)\quad t(f,\delta,\alpha) = u_\alpha + \delta(\frac{3}{4f} + \frac{25}{32f^2}) + \frac{u_\alpha^2-1}{6}\,\delta(\frac{3}{f} + \frac{69+5\delta^2}{4f^2}) +
$$

$$
+ \frac{u_\alpha}{2}(\frac{4+\delta^2}{2f} + \frac{32+15\delta^2}{8f^2}) + \frac{u_\alpha^3-3u_\alpha}{24}(\frac{6}{f} + \frac{18\delta^2+48}{f^2}) +
$$

$$
- \frac{2u_\alpha^3-5u_\alpha}{36}\;\frac{9}{f^2}\,\delta^2 - \frac{u_\alpha^2-1}{6}\,\delta\,\frac{3(4+\delta^2)}{2f^2} +
$$

$$+ \frac{u_\alpha^4-6u_\alpha^2+3}{120} \frac{105}{f^2} \delta - \frac{u_\alpha^4-5u_\alpha^2+2}{24} \frac{18}{f^2} \delta - \frac{1}{8} \frac{16+8\delta^2+\delta^4}{4f^2} u_\alpha^2$$

$$- \frac{u_\alpha^3-3u_\alpha}{16} \frac{3(4+\delta^2)}{f^2} + \frac{u_\alpha^5-10u_\alpha^3+15u_\alpha}{720} \frac{240}{f^2}$$

$$- \frac{3u_\alpha^5-24u_\alpha^3+29u_\alpha}{384} \frac{36}{f^2} + 0(\frac{1}{f^3})$$

$$= u_\alpha + \frac{u_\alpha^3+u_\alpha}{4f} + \frac{5u_\alpha^5+16u_\alpha^3+3u_\alpha}{96f^2} +$$

$$+ \delta \left[1 + \frac{2u_\alpha^2+1}{4f} + \frac{u_\alpha}{4f} \delta + \frac{4u_\alpha^4+12u_\alpha^2+1}{32f^2} + \frac{u_\alpha^3+4u_\alpha}{16f^2} \delta - \frac{u_\alpha^2-1}{24f^2} \delta^2 - \frac{u_\alpha}{32f^2} \delta^3\right].$$

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Table A

β_f, β_f^2 and $1-\beta_f^2$ for $f = 1(1)20$.

f	β_f	β_f^2	$1-\beta_f^2$
1	0,797885	0,636620	0,363380
2	0,886227	0,785398	0,214602
3	0,921318	0,848827	0,151173
4	0,939986	0,883574	0,116426
5	0,951533	0,905415	0,094585
6	0,959369	0,920389	0,079611
7	0,965030	0,931283	0,068717
8	0,969311	0,939564	0,060436
9	0,972659	0,946066	0,053934
10	0,975350	0,951308	0,048692
11	0,977559	0,955622	0,044378
12	0,979406	0,959236	0,040764
13	0,980971	0,962304	0,037696
14	0,982316	0,964945	0,035055
15	0,983484	0,967241	0,032759
16	0,984506	0,969252	0,030748
17	0,985410	0,971033	0,028967
18	0,986214	0,972618	0,027382
19	0,986934	0,974039	0,025961
20	0,987583	0,975320	0,024680

Table B

$\frac{2u_\alpha^2+1}{4}$, $\frac{u_\alpha}{4}$, $\frac{4u_\alpha^4+12u_\alpha^2+1}{32}$, $\frac{u_\alpha^3+4u_\alpha}{16}$, $\frac{u_\alpha^2-1}{24}$ and $\frac{u_\alpha}{32}$ for

$\alpha = 0{,}001;\ 0{,}0025;\ 0{,}005;\ 0{,}01;\ 0{,}025;\ 0{,}05;\ 0{,}10;\ 0{,}25;\ 0{,}30;\ 0{,}50.$

α	$\frac{2u_\alpha^2+1}{4}$	$\frac{u_\alpha}{4}$	$\frac{4u_\alpha^4+12u_\alpha^2+1}{32}$	$\frac{u_\alpha^3+4u_\alpha}{16}$	$\frac{u_\alpha^2-1}{24}$	$\frac{u_\alpha}{32}$
0,001	5,0248	0,7726	15,0115	2,6170	0,3562	0,0966
0,0025	4,1897	0,7018	10,7467	2,0841	0,2866	0,0877
0,005	3,5674	0,6440	8,0221	1,7121	0,2348	0,0805
0,01	2,9559	0,5816	5,7218	1,3685	0,1838	0,0727
0,025	2,1707	0,4900	3,3164	0,9606	0,1184	0,0612
0,05	1,6028	0,4112	1,9608	0,6894	0,0711	0,0514
0,10	1,0712	0,3204	0,9843	0,4519	0,0268	0,0400
0,25	0,4775	0,1686	0,2277	0,1878	-0,0227	0,0211
0,30	0,3875	0,1311	0,1438	0,1401	-0,0302	0,0164
0,50	0,25	0	0,0313	0	-0,0417	0

Table C

Comparison of the approximations with the exact percentage points for STUDENT's distribution (f=2(1)9)

α	f	$t(f,0,\alpha)$ exact	approximations to $t(f,0,\alpha)$ I a	I b	I c	II a	II c	III	VI
0,30	2	0,617	0,615	0,619	0,543	0,619	0,608	-	-
	3	0,584	0,584	0,585	0,537	0,585	0,580	-	0,908
	4	0,569	0,568	0,569	0,534	0,569	0,566	-	0,742
	5	0,559	0,559	0,560	0,532	0,560	0,558	-	0,677
	6	0,553	0,553	0,554	0,531	0,553	0,552	-	0,642
	7	0,549	0,549	0,549	0,530	0,549	0,548	-	0,620
	8	0,546	0,546	0,546	0,529	0,546	0,545	-	0,606
	9	0,543	0,543	0,544	0,528	0,544	0,543	-	0,595
0,10	2	1,886	1,948	2,093	1,669	1,848	1,705	-	-
	3	1,638	1,654	1,690	1,504	1,627	1,564	-	2,220
	4	1,533	1,540	1,556	1,438	1,529	1,493	-	1,812
	5	1,476	1,480	1,489	1,402	1,474	1,451	-	1,654
	6	1,440	1,442	1,448	1,379	1,439	1,423	-	1,570
	7	1,415	1,417	1,421	1,364	1,414	1,402	-	1,516
	8	1,397	1,398	1,401	1,353	1,396	1,387	-	1,480
	9	1,383	1,384	1,386	1,344	1,383	1,376	-	1,453
0,05	2	2,920	3,635	4,983	2,892	2,762	2,407	-	-
	3	2,353	2,480	2,608	2,220	2,311	2,153	-	2,849
	4	2,132	2,181	2,227	2,022	2,115	2,026	-	2,326
	5	2,015	2,041	2,064	1,926	2,006	1,950	-	2,123
	6	1,943	1,959	1,973	1,869	1,938	1,899	-	2,015
	7	1,895	1,905	1,915	1,831	1,892	1,863	1,893	1,946
	8	1,860	1,867	1,874	1,805	1,858	1,835	1,865	1,899
	9	1,833	1,839	1,844	1,784	1,832	1,814	1,837	1,865
0,025	2	4,303	-	-	9,849	3,852	3,146	-	-
	3	3,182	3,785	4,293	3,268	3,064	2,751	-	3,395
	4	2,776	2,967	3,086	2,719	2,729	2,553	-	2,772
	5	2,571	2,662	2,715	2,498	2,547	2,434	-	2,530
	6	2,447	2,500	2,530	2,377	2,434	2,355	-	2,400
	7	2,365	2,399	2,418	2,301	2,356	2,299	2,366	2,319
	8	2,306	2,330	2,343	2,248	2,301	2,256	2,309	2,263
	9	2,262	2,280	2,290	2,210	2,258	2,224	2,268	2,222
0,01	2	6,965	-	-	-	5,621	4,191	-	-
	3	4,541	13,278	-	7,432	4,205	3,569	-	4,029
	4	3,747	4,620	5,112	4,090	3,616	3,259	-	3,290
	5	3,365	3,708	3,855	3,435	3,301	3,072	-	3,003
	6	3,143	3,325	3,395	3,140	3,107	2,948	-	2,849
	7	2,998	3,110	3,152	2,970	2,976	2,859	2,993	2,753
	8	2,896	2,973	3,000	2,860	2,882	2,792	2,893	2,686
	9	2,821	2,876	2,896	2,782	2,811	2,741	2,823	2,638

Table D

Comparison of the approximations with the exact percentage points for STUDENT's distribution (f=30,40,60)

α	f	t(f,0,α) exact	approximations to t(f,0,α)						
			I			II		III	VI
			a	b	c	a	c		
0,30	30	0,530	0,530	0,530	0,526	0,530	0,530	-	0,543
	40	0,529	0,529	0,529	0,525	0,529	0,529	-	0,538
	60	0,527	0,527	0,527	0,525	0,527	0,527	-	0,533
0,10	30	1,310	1,311	1,311	1,300	1,310	1,310	-	1,327
	40	1,303	1,303	1,303	1,295	1,303	1,303	-	1,315
	60	1,296	1,296	1,297	1,290	1,296	1,296	-	1,304
0,05	30	1,697	1,698	1,698	1,683	1,697	1,696	1,698	1,703
	40	1,684	1,684	1,684	1,673	1,684	1,683	1,683	1,688
	60	1,671	1,671	1,671	1,664	1,671	1,670	1,668	1,673
0,025	30	2,042	2,043	2,044	2,026	2,042	2,039	2,041	2,029
	40	2,021	2,022	2,022	2,009	2,021	2,019	2,019	2,011
	60	2,000	2,001	2,001	1,992	2,000	1,999	2,000	1,994
0,01	30	2,457	2,460	2,461	2,439	2,457	2,451	2,458	2,048
	40	2,423	2,425	[illegible]	[illegible]	2,423	2,420	2,425	2,387
	60	2,390	2,391	2,391	2,381	2,390	2,388	2,392	2,366

Table E

Comparison of the approximations with the exact percentage points for p = 0,45

α	f	t(f,δ,α) exact	approximations to t(f,δ,α): I a	I b	I c	II a	II b	II c	II d	III
0,99	4	-3,13	-3,66	-3,96	-3,28	-3,05	-3,18	-2,78	-3,27	-
	5	-2,80	-3,01	-3,10	-2,80	-2,76	-2,82	-2,59	-2,89	-
	6	-2,59	-2,70	-2,75	-2,56	-2,57	-2,61	-2,46	-2,66	-
	7	-2,46	-2,52	-2,55	-2,41	-2,44	-2,47	-2,36	-2,50	-
	8	-2,35	-2,40	-2,41	-2,31	-2,35	-2,36	-2,29	-2,39	-2,35
	9	-2,27	-2,30	-2,32	-2,23	-2,27	-2,28	-2,22	-2,30	-2,28
0,975	4	-2,279	-2,393	-2,469	-2,204	-2,252	-2,299	-2,129	-2,352	-
	5	-2,092	-2,144	-2,177	-2,017	-2,078	-2,102	-2,002	-2,139	-
	6	-1,969	-1,999	-2,017	-1,904	-1,963	-1,976	-1,911	-2,003	-
	7	-1,882	-1,901	-1,912	-1,825	-1,878	-1,887	-1,842	-1,908	-
	8	-1,815	-1,828	-1,836	-1,765	-1,813	-1,818	-1,785	-1,835	-1,816
	9	-1,760	-1,770	-1,776	-1,717	-1,759	-1,763	-1,739	-1,777	-1,762
0,95	4	-1,707	-1,734	-1,763	-1,612	-1,698	-1,715	-1,640	-1,746	-
	5	-1,589	-1,604	-1,618	-1,516	-1,586	-1,595	-1,551	-1,616	-
	6	-1,510	-1,519	-1,527	-1,451	-1,508	-1,513	-1,485	-1,529	-
	7	-1,451	-1,456	-1,462	-1,401	-1,450	-1,453	-1,434	-1,466	-
	8	-1,403	-1,407	-1,410	-1,360	-1,403	-1,405	-1,390	-1,415	-1,408
	9	-1,363	-1,366	-1,368	-1,326	-1,363	-1,364	-1,353	-1,372	-1,363
0,90	4	-1,166	-1,169	-1,178	-1,093	-1,164	-1,168	-1,143	-1,183	-
	5	-1,096	-1,098	-1,103	-1,041	-1,096	-1,098	-1,083	-1,108	-
	6	-1,045	-1,046	-1,049	-1,002	-1,045	-1,046	-1,037	-1,054	-
	7	-1,004	-1,005	-1,007	-0,968	-1,004	-1,005	-0,998	-1,011	-
	8	-0,970	-0,970	-0,972	-0,939	-0,969	-0,970	-0,965	-0,975	-
	9	-0,940	-0,940	-0,941	-0,914	-0,940	-0,940	-0,937	-0,944	-
0,70	4	-0,261	-0,261	-0,261	-0,245	-0,261	-0,261	-0,260	-0,263	-
	5	-0,229	-0,229	-0,229	-0,218	-0,229	-0,228	-0,229	-0,230	-
	6	-0,200	-0,201	-0,201	-0,193	-0,200	-0,200	-0,200	-0,201	-
	7	-0,175	-0,176	-0,176	-0,169	-0,175	-0,175	-0,175	-0,176	-
	8	-0,152	-0,152	-0,152	-0,148	-0,152	-0,152	-0,152	-0,153	-
	9	-0,131	-0,131	-0,131	-0,127	-0,131	-0,130	-0,131	-0,131	-
0,50	4	0,299	0,299	0,299	0,281	0,299	0,299	0,299	0,299	-
	5	0,323	0,324	0,324	0,308	0,324	0,324	0,323	0,323	-
	6	0,347	0,347	0,347	0,333	0,347	0,347	0,346	0,346	-
	7	0,368	0,368	0,368	0,356	0,368	0,368	0,368	0,368	-
	8	0,389	0,389	0,389	0,377	0,389	0,389	0,389	0,389	-
	9	0,409	0,409	0,409	0,398	0,409	0,409	0,409	0,409	-

Table E (continued)

α	f	$t(f,\delta,\alpha)$ exact	approximations to $t(f,\delta,\alpha)$							
			I			II				III
			a	b	c	a	b	c	d	
0,30	4	0,883	0,882	0,884	0,827	0,883	0,883	0,877	0,880	-
	5	0,896	0,895	0,896	0,851	0,897	0,896	0,892	0,893	-
	6	0,912	0,911	0,912	0,873	0,911	0,911	0,909	0,910	-
	7	0,928	0,928	0,928	0,895	0,928	0,928	0,926	0,927	-
	8	0,945	0,944	0,945	0,915	0,945	0,945	0,943	0,944	-
	9	0,961	0,961	0,961	0,934	0,962	0,961	0,960	0,960	-
0,10	4	1,918	1,932	1,957	1,800	1,911	1,915	1,856	1,896	-
	5	1,872	1,879	1,893	1,778	1,868	1,870	1,831	1,856	-
	6	1,850	1,854	1,864	1,772	1,847	1,848	1,821	1,838	-
	7	1,839	1,843	1,850	1,774	1,838	1,839	1,818	1,831	-
	8	1,838	1,840	1,845	1,780	1,836	1,837	1,820	1,830	-
	9	1,840	1,842	1,846	1,788	1,839	1,839	1,826	1,833	-
0,05	4	2,579	2,664	2,732	2,462	2,555	2,572	2,428	2,534	-
	5	2,461	2,506	2,542	2,361	2,447	2,456	2,364	2,429	-
	6	2,396	2,424	2,445	2,310	2,387	2,392	2,328	2,372	-
	7	2,356	2,377	2,391	2,282	2,352	2,355	2,307	2,339	-
	8	2,333	2,348	2,359	2,268	2,331	2,333	2,295	2,320	2,332
	9	2,320	2,332	2,340	2,262	2,318	2,319	2,290	2,309	2,319
0,025	4	3,303	3,604	3,779	3,285	3,235	3,282	2,996	3,219	-
	5	3,077	3,225	3,302	3,017	3,042	3,066	2,885	3,022	-
	6	2,947	3,037	3,081	2,882	2,929	2,942	2,817	2,909	-
	7	2,870	2,929	2,957	2,805	2,857	2,866	2,774	2,840	-
	8	2,819	2,861	2,881	2,758	2,811	2,816	2,744	2,794	2,817
	9	2,783	2,816	2,831	2,728	2,778	2,782	2,726	2,764	2,779
0,01	4	4,41	5,75	6,51	5,02	4,22	4,35	3,76	4,25	-
	5	3,96	4,50	4,71	4,14	3,88	3,94	3,57	3,87	-
	6	3,73	4,01	4,11	3,77	3,67	3,71	3,46	3,65	-
	7	3,57	3,75	3,81	3,57	3,54	3,56	3,38	3,51	-
	8	3,46	3,59	3,63	3,45	3,45	3,46	3,32	3,42	3,46
	9	3,39	3,49	3,51	3,37	3,38	3,39	3,28	3,36	3,39

Table F

Comparison of the approximations with the exact percent[illegible] = 0,40

α	f	t(f,δ,α) exact	approximations to t(f,δ,α) I a	I b	I c	II a	II b	II c	II d	III
0,99	4	-2,56	-2,85	-3,01	-2,50	-2,50	-2,64	-2,32	-2,81	-
	5	-2,26	-2,30	-2,43	-2,22	-2,24	-2,30	-2,13	-2,42	-
	6	-2,08	-2,13	-2,15	-2,02	-2,05	-2,10	-1,99	-2,19	-
	7	-1,94	-1,97	-1,99	[illegible]	-1,93	-1,95	-1,88	-2,02	-
	8	-1,83	[illegible]	-1,86	-1,79	-1,85	-1,84	-1,79	-1,90	-1,83
	9	-1,75	-1,76	-1,77	-1,71	-1,74	-1,75	-1,72	-1,80	-1,74
0,975	4	-1,812	-1,872	-1,916	-1,732	-1,797	-1,884	-1,719	-1,942	-
	5	-1,635	-1,661	-1,681	-1,567	-1,628	-1,652	-1,582	-1,719	-
	6	-1,512	-1,526	-1,535	-1,456	-1,509	-1,522	-1,479	-1,571	-
	7	-1,417	-1,425	-1,432	-1,371	-1,415	-1,424	-1,396	-1,462	-
	8	-1,341	-1,346	-1,350	-1,301	-1,340	-1,345	-1,325	-1,375	-1,345
	9	-1,275	-1,279	-1,281	-1,244	-1,274	-1,278	-1,265	-1,303	-1,274
0,95	4	-1,303	-1,316	-1,332	-1,227	-1,299	-1,316	-1,266	-1,372	-
	5	-1,182	-1,188	-1,196	-1,125	-1,181	-1,190	-1,162	-1,227	-
	6	-1,092	-1,096	-1,100	-1,048	-1,092	-1,097	-1,031	-1,125	-
	7	-1,020	-1,022	-1,025	-0,985	-1,020	-1,023	-1,013	-1,045	-
	8	-0,958	-0,960	-0,961	-0,929	-0,959	-0,961	-0,953	-0,978	-0,958
	9	-0,903	-0,905	-0,906	-0,879	-0,904	-0,905	-0,900	-0,919	-0,910
0,90	4	-0,810	-0,812	-0,817	-0,761	-0,811	-0,815	-0,801	-0,841	-
	5	-0,727	-0,728	-0,730	-0,691	-0,728	-0,730	-0,722	-0,747	-
	6	-0,660	-0,660	-0,661	-0,632	-0,660	-0,661	-0,657	-0,674	-
	7	-0,601	-0,602	-0,603	-0,581	-0,602	-0,603	-0,599	-0,612	-
	8	-0,550	-0,550	-0,551	-0,533	-0,550	-0,551	-0,548	-0,558	-
	9	-0,503	-0,503	-0,503	-0,489	-0,503	-0,503	-0,503	-0,510	-
0,70	4	0,045	0,045	0,045	0,042	0,044	0,044	0,045	0,042	-
	5	0,101	0,101	0,101	0,096	0,100	0,101	0,100	0,099	-
	6	0,152	0,151	0,152	0,145	0,152	0,152	0,152	0,151	-
	7	0,198	0,198	0,198	0,191	0,198	0,198	0,198	0,197	-
	8	0,242	0,242	0,242	0,235	0,242	0,242	0,242	0,241	-
	9	0,283	0,283	0,283	0,276	0,283	0,284	0,283	0,283	-
0,50	4	0,603	0,603	0,603	0,566	0,603	0,603	0,602	0,602	-
	5	0,653	0,652	0,652	0,621	0,653	0,653	0,652	0,652	-
	6	0,699	0,699	0,699	0,670	0,699	0,699	0,698	0,698	-
	7	0,743	0,742	0,742	0,716	0,743	0,743	0,742	0,742	-
	8	0,784	0,784	0,784	0,760	0,784	0,784	0,784	0,784	-
	9	0,824	0,824	0,824	0,801	0,824	0,824	0,823	0,823	-

Table F (continued)

α	f	t(f,δ,α) exact	approximations to t(f, δ, α)							
			I			II				III
			a	b	c	a	b	c	d	
0,30	4	1,209	1,206	1,209	1,131	1,209	1,209	1,198	1,201	-
	5	1,244	1,242	1,244	1,180	1,245	1,244	1,237	1,238	-
	6	1,282	1,280	1,281	1,226	1,281	1,281	1,275	1,276	-
	7	1,319	1,317	1,318	1,270	1,318	1,318	1,314	1,315	-
	8	1,355	1,354	1,355	1,312	1,355	1,355	1,351	1,352	-
	9	1,391	1,390	1,391	1,351	1,392	1,391	1,388	1,388	-
0,10	4	2,328	2,350	2,386	2,186	2,316	2,320	2,237	2,277	-
	5	2,290	2,302	2,323	2,176	2,283	2,285	2,229	2,254	-
	6	2,281	2,289	2,303	2,185	2,276	2,277	2,237	2,254	-
	7	2,285	2,292	2,301	2,204	2,282	2,283	2,252	2,265	-
	8	2,298	2,303	2,311	2,227	2,296	2,297	2,272	2,282	-
	9	2,316	2,320	2,326	2,252	2,315	2,315	2,295	2,302	-
0,05	4	3,062	3,188	3,284	2,938	3,023	3,040	2,852	2,958	-
	5	2,936	3,006	3,056	2,827	2,915	2,924	2,801	2,866	-
	6	2,874	2,920	2,951	2,779	2,862	2,867	2,779	2,823	-
	7	2,844	2,877	2,899	2,761	2,838	2,841	2,774	2,806	-
	8	2,833	2,858	2,873	2,758	2,829	2,831	2,777	2,802	2,833
	9	2,832	2,852	2,864	2,764	2,829	2,830	2,787	2,806	2,831
0,025	4	3,873	4,312	4,556	3,911	3,777	3,824	3,466	3,689	-
	5	3,620	3,839	3,946	3,582	3,570	3,594	3,362	3,499	-
	6	3,483	3,618	3,679	3,427	3,456	3,469	3,304	3,396	-
	7	3,406	3,493	3,537	3,346	3,388	3,397	3,273	3,339	-
	8	3,361	3,428	3,456	3,301	3,349	3,354	3,257	3,307	3,360
	9	3,336	3,387	3,408	3,278	3,326	3,330	3,253	3,291	3,336
0,01	4	5,12	7,05	8,16	6,09	4,88	5,01	4,29	4,78	-
	5	4,61	5,38	5,67	4,93	4,49	4,56	4,10	4,40	-
	6	4,34	4,76	4,90	4,47	4,27	4,31	3,99	4,19	-
	7	4,18	4,44	4,53	4,23	4,13	4,16	3,92	4,06	-
	8	4,07	4,26	4,31	4,08	4,04	4,06	3,88	3,98	4,08
	9	4,00	4,14	4,18	4,00	3,98	3,99	3,85	3,93	4,00

Table G

Comparison of the approximations with the exact percentage points for p = 0,35

α	f	$t(f,\delta,\alpha)$ exact	approximations to $t(f,\delta,\alpha)$							
			I			II				III
			a	b	c	a	b	c	d	
0,99	4	-2,02	-2,16	-2,24	-1,98	-1,99	-2,12	-1,87	-2,36	-
	5	-1,76	-1,80	-1,83	-1,70	-1,74	-1,80	-1,67	-1,97	-
	6	-1,57	-1,60	-1,61	-1,52	-1,57	-1,60	-1,53	-1,72	-
	7	-1,44	-1,45	-1,46	-1,39	-1,43	-1,46	-1,41	-1,55	-
	8	-1,32	-1,33	-1,34	-1,29	-1,32	-1,34	-1,31	-1,41	-
	9	-1,23	-1,24	-1,24	-1,20	-1,23	-1,24	-1,22	-1,30	-1,23
0,975	4	-1,366	-1,392	-1,415	-1,294	-1,359	-1,406	-1,315	-1,538	-
	5	-1,195	-1,205	-1,215	-1,140	-1,192	-1,216	-1,168	-1,305	-
	6	-1,067	-1,072	-1,077	-1,025	-1,066	-1,079	-1,052	-1,144	-
	7	-0,963	-0,966	-0,969	-0,930	-0,962	-0,971	-0,954	-1,020	-
	8	-0,874	-0,876	-0,878	-0,848	-0,875	-0,880	-0,868	-0,918	-
	9	-0,795	-0,797	-0,798	-0,774	-0,795	-0,799	-0,793	-0,831	-0,801
0,95	4	-0,913	-0,917	-0,924	-0,857	-0,912	-0,929	-0,895	-1,001	-
	5	-0,785	-0,787	-0,790	-0,746	-0,784	-0,793	-0,777	-0,842	-
	6	-0,682	-0,683	-0,685	-0,655	-0,682	-0,687	-0,679	-0,723	-
	7	-0,595	-0,596	-0,597	-0,574	-0,596	-0,599	-0,593	-0,625	-
	8	-0,518	-0,518	-0,519	-0,502	-0,519	-0,521	-0,516	-0,541	-
	9	-0,447	-0,448	-0,448	-0,435	-0,448	-0,449	-0,446	-0,465	-0,451
0,90	4	-0,463	-0,464	-0,465	-0,435	-0,464	-0,468	-0,460	-0,500	-
	5	-0,364	-0,363	-0,364	-0,345	-0,364	-0,366	-0,362	-0,387	-
	6	-0,277	-0,277	-0,278	-0,266	-0,278	-0,279	-0,277	-0,294	-
	7	-0,200	-0,201	-0,201	-0,193	-0,200	-0,201	-0,200	-0,213	-
	8	-0,131	-0,130	-0,130	-0,126	-0,130	-0,131	-0,130	-0,140	-
	9	-0,065	-0,065	-0,065	-0,063	-0,065	-0,065	-0,065	-0,072	-
0,70	4	0,355	0,355	0,354	0,334	0,355	0,355	0,355	0,352	-
	5	0,436	0,436	0,436	0,415	0,436	0,437	0,436	0,435	-
	6	0,510	0,510	0,510	0,490	0,511	0,511	0,511	0,510	-
	7	0,580	0,580	0,579	0,560	0,580	0,580	0,580	0,579	-
	8	0,645	0,645	0,644	0,625	0,645	0,645	0,645	0,644	-
	9	0,706	0,706	0,706	0,687	0,706	0,707	0,706	0,706	-
0,50	4	0,919	0,917	0,917	0,862	0,919	0,919	0,915	0,915	-
	5	0,993	0,992	0,992	0,944	0,994	0,994	0,991	0,991	-
	6	1,064	1,063	1,063	1,019	1,064	1,064	1,062	1,062	-
	7	1,131	1,129	1,129	1,090	1,131	1,131	1,129	1,129	-
	8	1,194	1,192	1,192	1,156	1,194	1,194	1,192	1,192	-
	9	1,254	1,253	1,253	1,218	1,254	1,254	1,252	1,252	-

Table G (continued)

α	f	t(f, δ, α) exact	approximations to t(f, δ, α)							
			I			II				III
			a	b	c	a	b	c	d	
0,30	4	1,554	1,547	1,553	1,451	1,553	1,553	1,535	1,538	-
	5	1,611	1,607	1,610	1,526	1,611	1,610	1,598	1,599	-
	6	1,670	1,667	1,669	1,597	1,670	1,670	1,660	1,661	-
	7	1,728	1,726	1,728	1,664	1,728	1,728	1,720	1,721	-
	8	1,786	1,783	1,785	1,727	1,785	1,785	1,779	1,780	-
	9	1,841	1,839	1,840	1,788	1,842	1,841	1,836	1,836	-
0,10	4	2,771	2,803	2,853	2,603	2,752	2,756	2,645	2,685	-
	5	2,737	2,758	2,787	2,604	2,728	2,730	2,654	2,679	-
	6	2,741	2,755	2,774	2,628	2,735	2,736	2,680	2,697	-
	7	2,761	2,771	2,785	2,663	2,756	2,757	2,713	2,726	-
	8	2,789	2,797	2,808	2,703	2,786	2,787	2,751	2,761	-
	9	2,822	2,829	2,838	2,745	2,821	2,821	2,792	2,799	-
0,05	4	3,586	3,766	3,894	3,461	3,531	3,548	3,309	3,415	-
	5	3,450	3,552	3,619	3,334	3,420	3,429	3,270	3,335	-
	6	3,390	3,458	3,500	3,287	3,373	3,378	3,262	3,306	-
	7	3,368	3,418	3,447	3,276	3,358	3,361	3,272	3,304	-
	8	3,368	3,405	3,427	3,284	3,361	3,363	3,291	3,316	-
	9	3,379	3,410	3,426	3,303	3,374	3,375	3,317	3,336	3,376
0,025	4	4,494	5,104	5,430	4,609	4,366	4,413	3,973	4,196	-
	5	4,208	4,519	4,661	4,205	4,141	4,165	3,875	4,012	-
	6	4,062	4,254	4,335	4,023	4,023	4,036	3,828	3,920	-
	7	3,985	4,118	4,171	3,934	3,959	3,968	3,810	3,876	-
	8	3,944	4,043	4,081	3,890	3,928	3,933	3,807	3,857	-
	9	3,928	4,004	4,033	3,872	3,914	3,918	3,817	3,855	3,925
0,01	4	5,89	8,54	10,06	7,30	5,58	5,72	4,87	5,35	-
	5	5,31	6,37	6,77	5,81	5,16	5,23	4,68	4,97	-
	6	5,01	5,59	5,78	5,23	4,92	4,96	4,57	4,77	-
	7	4,84	5,21	5,32	4,94	4,78	4,80	4,51	4,65	-
	8	4,73	4,99	5,06	4,78	4,69	4,70	4,47	4,58	-
	9	4,66	4,86	4,91	4,68	4,63	4,64	4,46	4,54	4,66

Table H

Comparison of the approximations with the exact percentage points for p = 0,30

α	f	t(f,δ,α) exact	approximations to t(f,δ,α)						
			I			II			
			a	b	c	a	b	c	d
0,99	4	-1,50	-1,55	-1,59	-1,44	-1,49	-1,62	-1,42	-1,91
	5	-1,25	-1,28	-1,29	-1,21	-1,25	-1,32	-1,22	-1,51
	6	-1,09	-1,09	-1,10	-1,04	-1,08	-1,12	-1,06	-1,26
	7	-0,95	-0,95	-0,95	-0,91	-0,94	-0,97	-0,93	-1,07
	8	-0,83	-0,83	-0,83	-0,80	-0,82	-0,84	-0,82	-0,92
	9	-0,72	-0,72	-0,72	-0,70	-0,72	-0,73	-0,72	-0,80
0,975	4	-0,935	-0,943	-0,953	-0,880	-0,933	-0,980	-0,912	-1,135
	5	-0,763	-0,766	-0,770	-0,727	-0,763	-0,787	-0,754	-0,891
	6	-0,627	-0,629	-0,631	-0,602	-0,628	-0,641	-0,623	-0,715
	7	-0,512	-0,512	-0,513	-0,494	-0,511	-0,520	-0,510	-0,576
	8	-0,409	-0,409	-0,410	-0,396	-0,409	-0,414	-0,408	-0,458
	9	-0,315	-0,315	-0,316	-0,307	-0,315	-0,319	-0,315	-0,353
0,95	4	-0,529	-0,531	-0,533	-0,498	-0,531	-0,548	-0,525	-0,631
	5	-0,390	-0,392	-0,392	-0,372	-0,391	-0,400	-0,389	-0,454
	6	-0,272	-0,273	-0,274	-0,262	-0,273	-0,278	-0,273	-0,317
	7	-0,168	-0,169	-0,169	-0,163	-0,170	-0,173	-0,169	-0,201
	8	-0,074	-0,074	-0,074	-0,072	-0,075	-0,077	-0,074	-0,099
	9	0,014	0,014	0,014	0,013	0,014	0,013	0,014	-0,005
0,90	4	-0,117	-0,117	-0,117	-0,110	-0,117	-0,121	-0,116	-0,156
	5	0,003	0,003	0,003	0,003	0,003	0,001	0,003	-0,022
	6	0,110	0,110	0,110	0,105	0,109	0,108	0,109	0,092
	7	0,207	0,207	0,207	0,200	0,207	0,206	0,207	0,194
	8	0,297	0,297	0,297	0,288	0,298	0,297	0,298	0,288
	9	0,382	0,382	0,382	0,372	0,382	0,382	0,382	0,375
0,70	4	0,676	0,675	0,674	0,635	0,676	0,676	0,675	0,672
	5	0,784	0,783	0,782	0,746	0,783	0,784	0,783	0,782
	6	0,883	0,883	0,882	0,848	0,884	0,884	0,883	0,882
	7	0,977	0,976	0,975	0,942	0,977	0,977	0,976	0,975
	8	1,064	1,064	1,063	1,032	1,064	1,064	1,064	1,063
	9	1,148	1,147	1,146	1,116	1,147	1,148	1,147	1,147
0,50	4	1,252	1,247	1,247	1,173	1,252	1,252	1,246	1,246
	5	1,354	1,350	1,350	1,285	1,354	1,354	1,349	1,349
	6	1,450	1,446	1,446	1,387	1,450	1,450	1,445	1,445
	7	1,540	1,537	1,537	1,483	1,540	1,540	1,536	1,536
	8	1,626	1,623	1,623	1,573	1,626	1,626	1,622	1,622
	9	1,707	1,705	1,705	1,658	1,707	1,707	1,704	1,704

Table H (continued)

α	f	t(f,δ,α) exact	approximations to t(f,δ,α)						
			I			II			
			a	b	c	a	b	c	d
0,30	4	1,926	1,914	1,922	1,794	1,924	1,924	1,897	1,900
	5	2,005	1,997	2,002	1,896	2,005	2,004	1,985	1,986
	6	2,086	2,080	2,084	1,992	2,086	2,086	2,071	2,072
	7	2,166	2,161	2,165	2,083	2,167	2,167	2,155	2,156
	8	2,245	2,241	2,243	2,170	2,245	2,245	2,235	2,236
	9	2,321	2,317	2,319	2,252	2,322	2,321	2,313	2,313
0,10	4	3,255	3,301	3,367	3,061	3,229	3.233	3,090	3,130
	5	3,228	3,256	3,295	3,071	3,213	3,215	3,116	3,141
	6	3,240	3,262	3,283	3,110	3,233	3,234	3,161	3,178
	7	3,275	3,291	3,310	3,161	[illegible]	3,270	3,213	3,226
	8	3,320	3,332	3,347	3,218	3,315	3,316	3,270	3,280
	9	3,369	3,380	3,392	3,278	3,367	3.367	3,329	3,336
0,05	4	4,163	4,408	4,575	4,041	4,089	4.106	3,810	3,916
	5	4,012	4,155	4,243	3,894	3,973	3,982	3,782	3,847
	6	3,954	4,049	4,104	3,845	3,930	3,935	3,789	3,833
	7	3,940	4,010	4,048	3,840	3,925	3,928	3,815	3,847
	8	3,950	4,004	4,032	3,859	3,939	3,941	3,851	3,876
	9	3,973	4,018	4,040	3,890	3,966	3,967	3,893	3,912
0,025	4	5,181	5,995	6,418	5,392	5,013	5,060	4,530	4,753
	5	4,856	5,276	5,460	4.898	4,768	4,792	4,438	4,575
	6	4,698	4,960	5,065	4,683	4,645	4,658	4,402	4,494
	7	4,618	4,802	4,871	4,582	4,583	4,592	4,396	4,462
	8	4,581	4,721	4,770	4,537	4,558	4,563	4,408	4,458
	9	4,573	4,681	4,719	4,523	4,554	4,558	4,432	4,470
0,01	4	6,74	10,25	12,24	8,69	6,37	6,50	5,50	5,99
	5	6,09	7,49	8,00	6,80	5,90	5,96	5,31	5,60
	6	5,76	6,53	6,77	6,10	5,64	5,67	5,21	5,40
	7	5,56	6,06	6,21	5,75	5,48	5,51	5,15	5,29
	8	5,46	5,81	5,90	5,55	5,39	5,41	5,13	5,23
	9	5,38	5,65	5,72	5,44	5,34	5,35	5,12	5,20

Table J

Comparison of the approximations with the exact percentage points for $p = 0{,}25$

α	f	$t(f,\delta,\alpha)$ exact I[1]	$t(f,\delta,\alpha)$ exact II[2]	approximations to $t(f,\delta,\alpha)$: I a	I b	I c	II a	II b	II c	II d
0,99	2	-2,085	-	-	-	-	-1,958	-3,302	-1,693	-4,467
	3	-1,341	-	-1,420	-1,493	-1,272	-1,322	-1,658	-1,244	-2,216
	4	-1,000	-0,99	-1,014	-1,029	-0,944	-0,995	-1,126	-0,967	-1,455
	5	-0,774	-0,77	-0,777	-0,782	-0,736	-0,772	-0,836	-0,761	-1,054
	6	-0,600	-0,60	-0,599	-0,601	-0,573	-0,598	-0,634	-0,593	-0,788
	7	-0,450	-0,45	-0,450	-0,451	-0,434	-0,450	-0,472	-0,448	-0,587
	8	-0,320	-0,32	-0,320	-0,320	-0,310	-0,320	-0,334	-0,318	-0,422
	9	-0,204	-0,20	-0,202	-0,201	-0,196	-0,201	-0,211	-0,201	-0,281
0,975	2	-1,146	-	-1,224	-1,350	-1,042	-1,128	-1,579	-1,044	-2,201
	3	-0,747	-	-0,752	-0,764	-0,687	-0,743	-0,861	-0,723	-1,154
	4	-0,512	-0,511	-0,512	-0,515	-0,480	-0,511	-0,558	-0,505	-0,728
	5	-0,333	-0,334	-0,334	-0,335	-0,318	-0,334	-0,358	-0,332	-0,469
	6	-0,191	-0,186	-0,186	-0,186	-0,178	-0,186	-0,199	-0,185	-0,277
	7	-0,058	-0,054	-0,054	-0,053	-0,052	-0,054	-0,063	-0,054	-0,120
	8	0,060	0,065	0,065	0,065	0,063	0,065	0,060	0,066	0,016
	9	0,174	0,176	0,176	0,176	0,171	0,177	0,173	0,176	0,138
0,95	2	-0,605	-	-0,611	-0,626	-0,534	-0,605	-0,763	-0,583	-1,096
	3	-0,336	-	-0,336	-0,338	-0,309	-0,337	-0,379	-0,333	-0,533
	4	-0,148	-0,147	-0,148	-0,148	-0,139	-0,148	-0,165	-0,147	-0,253
	5	0,007	0,008	0,008	0,008	0,007	0,008	-0,001	0,007	-0,058
	6	0,145	0,145	0,144	0,144	0,138	0,144	0,139	0,144	0,100
	7	0,267	0,268	0,268	0,268	0,259	0,268	0,265	0,268	0,236
	8	0,382	0,383	0,383	0,383	0,371	0,383	0,381	0,383	0,358
	9	0,492	0,491	0,491	0,490	0,478	0,491	0,490	0,491	0,472
0,90	2	-0,130	-	-0,131	-0,131	-0,115	-0,131	-0,169	-0,130	-0,311
	3	0,073	-	0,073	0,073	0,067	0,073	0,062	0,072	-0,002
	4	0,236	0,237	0,237	0,236	0,223	0,237	0,233	0,237	0,197
	5	0,380	0,380	0,380	0,380	0,362	0,380	0,378	0,380	0,355
	6	0,509	0,510	0,510	0,510	0,490	0,510	0,509	0,510	0,493
	7	0,630	0,631	0,631	0,630	0,609	0,631	0,630	0,631	0,618
	8	0,744	0,744	0,743	0,743	0,721	0,744	0,743	0,744	0,734
	9	0,849	0,850	0,850	0,849	0,827	0,850	0,850	0,849	0,842

1) Exact percentage points according to RESNIKOFF and LIEBERMAN (1957),
2) Exact percentage points according to JOHNSON and WELCH (1940).

Table J (continued)

α	f	t(f, δ, α) exact		approximations to t(f, δ, α)						
		I	II	I			II			
				a	b	c	a	b	c	d
0,70	2	0,699	-	0,697	0,692	0,619	0,700	0,702	0,697	0,688
	3	0,866	-	0,864	0,861	0,797	0,866	0,867	0,864	0,860
	4	1,016	1,016	1,014	1,012	0,955	1,016	1,016	1,014	1,011
	5	1,153	1,154	1,152	1,150	1,097	1,153	1,154	1,151	1,150
	6	1,280	1,281	1,279	1,278	1,228	1,281	1,281	1,278	1,277
	7	1,399	1,400	1,398	1,397	1,350	1,400	1,400	1,397	1,396
	8	1,511	1,512	1,510	1,509	1,465	1,512	1,512	1,509	1,508
	9	1,618	1,619	1,617	1,616	1,574	1,618	1,619	1,616	1,616
0,50	2	1,339	-	1,318	1,318	1,168	1,340	1,340	1,314	1,314
	3	1,477	-	1,464	1,464	1,349	1,477	1,477	1,461	1,461
	4	1,614	1,614	1,604	1,604	1,508	1,614	1,614	1,602	1,602
	5	1,744	1,744	1,736	1,736	1,652	1,745	1,745	1,735	1,735
	6	1,867	1,867	1,860	1,860	1,785	1,868	1,868	1,859	1,859
	7	1,982	1,982	1,977	1,977	1,908	1,983	1,983	1,976	1,976
	8	2,093	2,093	2,088	2,088	2,023	2,093	2,093	2,087	2,087
	9	2,196	2,198	2,193	2,193	2,133	2,198	2,198	2,192	2,192
0,30	2	2,211	-	2,154	2,197	1,890	2,197	2,195	2,092	2,101
	3	2,245	-	2,216	2,235	2,030	2,240	2,239	2,183	2,187
	4	2,335	2,334	2,315	2,327	2,169	2,333	2,333	2,295	2,298
	5	2,438	2,437	2,424	2,431	2,301	2,437	2,436	2,410	2,411
	6	2,544	2,543	2,532	2,538	2,425	2,542	2,542	2,521	2,522
	7	2,647	2,647	2,638	2,642	2,542	2,646	2,646	2,630	2,631
	8	2,749	2,748	2,740	2,743	2,653	2,748	2,748	2,734	2,735
	9	2,846	2,846	2,839	2,842	2,759	2,846	2,845	2,834	2,834
0,10	2	4,507	-	4,914	5,656	4,080	4,273	4,311	3,718	3,899
	3	3,944	-	4,067	4,250	3,648	3,869	3,880	3,589	3,663
	4	3,796	3,796	3,859	3,944	3,573	3,762	3,766	3,587	3,627
	5	3,772	3,772	3,812	3,862	3,592	3,753	3,755	3,632	3,657
	6	3,797	3,797	3,826	3,860	3,645	3,785	3,786	3,696	3,713
	7	3,847	3,847	3,868	3,893	3,713	3,838	3,839	3,768	3,781
	8	3,906	3,906	3,925	3,944	3,789	3,900	3,901	3,845	3,855
	9	3,975	3,973	3,989	4,005	3,867	3,970	3,970	3,925	3,932
0,05	2	6,589	-	10,724	19,628	7,755	5,903	6,061	4,792	5,305
	3	5,238	-	5,988	6,585	5,223	5,025	5,067	4,472	4,672
	4	4,806	4,809	5,135	5,347	4,697	4,713	4,730	4,372	4,478
	5	4,642	4,642	4,833	4,945	4,523	4,589	4,598	4,356	4,421
	6	4,583	4,584	4,713	4,783	4,470	4,550	4,555	4,379	4,423
	7	4,575	4,576	4,672	4,721	4,471	4,554	4,557	4,422	4,454
	8	4,596	4,596	4,672	4,709	4,499	4,581	4,583	4,474	4,499
	9	4,632	4,633	4,695	4,724	4,543	4,621	4,622	4,535	4,554

Table J (continued)

α	f	t(f,δ,α) exact I	t(f,δ,α) exact II	approximations to t(f,δ,α) I a	I b	I c	II a	II b	II c	II d
0,025	2	9,475	-	-51,848	-10,482	60,045	7,844	8,295	5,917	7,074
	3	6,798	-	10,024	12,683	8,187	6,323	6,441	5,373	5,804
	4	5,950	5,954	7,011	7,544	6,283	5,739	5,786	5,158	5,381
	5	5,579	5,584	6,135	6,367	5,683	5,469	5,493	5,071	5,208
	6	5,408	5,408	5,757	5,890	5,428	5,338	5,351	5,045	5,137
	7	5,329	5,325	5,572	5,660	5,311	5,277	5,286	5,053	5,119
	8	5,295	5,293	5,480	5,543	5,263	5,259	5,264	5,079	5,129
	9	5,293	5,290	5,440	5,487	5,252	5,265	5,269	5,119	5,157
0,01	2	15,132	-	-	-	-	10,944	12,288	7,483	10,257
	3	9,440	-	82,384	-45,229	28,803	8,293	8,629	6,600	7,572
	4	7,725	7,73	12,199	14,719	10,269	7,241	7,372	6,213	6,701
	5	6,979	6,96	8,768	9,415	7,939	6,720	6,784	6,019	6,312
	6	6,587	6,59	7,593	7,895	7,074	6,435	6,471	5,921	6,116
	7	6,371	6,38	7,033	7,211	6,655	6,271	6,293	5,875	6,014
	8	6,254	6,26	6,725	6,843	6,425	6,176	6,190	5,861	5,965
	9	6,180	6,18	6,544	6,630	6,296	6,125	6,135	5,868	5,948

Table K

Comparison of the approximations with the exact percentage points for p = 0,15

α	f	t(f, δ, α) exact I[1]	t(f, δ, α) exact II[2]	approximations to t(f, δ, α) I a	I b	I c	II a	II b	II c	II d
0,99	2	-0,728	-	-0,756	-0,803	-0,652	-0,722	-2,066	-0,680	-3,454
	3	-0,291	-	-0,291	-0,293	-0,267	-0,291	-0,627	-0,287	-1,259
	4	-0,008	-0,01	-0,009	-0,009	-0,009	-0,009	-0,140	-0,010	-0,498
	5	0,217	0,22	0,218	0,218	0,207	0,218	0,154	0,218	-0,075
	6	0,416	0,42	0,417	0,416	0,400	0,416	0,380	0,416	0,221
	7	0,593	0,60	0,598	0,596	0,578	0,597	0,575	0,596	0,457
	8	0,764	0,77	0,766	0,764	0,743	0,764	0,750	0,763	0,659
	9	0,921	0,92	0,924	0,922	0,899	0,922	0,912	0,919	0,839
0,975	2	-0,196	-	-0,195	-0,197	-0,169	-0,196	-0,647	-0,192	-1,349
	3	0,118	-	0,120	0,120	0,111	0,120	0,002	0,120	-0,311
	4	0,362	0,364	0,364	0,363	0,343	0,364	0,317	0,364	0,141
	5	0,574	0,576	0,576	0,575	0,550	0,576	0,552	0,575	0,438
	6	0,764	0,767	0,768	0,766	0,738	0,767	0,754	0,765	0,673
	7	0,942	0,943	0,945	0,943	0,914	0,944	0,935	0,940	0,874
	8	1,106	1,108	1,112	1,109	1,079	1,109	1,104	1,104	1,055
	9	1,261	1,266	1,269	1,266	1,236	1,268	1,264	1,25[illegible]	1,221
0,95	2	0,164	-	0,165	0,164	0,146	0,164	0,006	0,164	-0,349
	3	0,438	-	0,439	0,437	0,405	0,438	0,396	0,438	0,238
	4	0,670	0,669	0,670	0,668	0,632	0,669	0,652	0,668	0,562
	5	0,877	0,876	0,878	0,875	0,837	0,877	0,868	0,873	0,808
	6	1,066	1,065	1,068	1,064	1,026	1,067	1,062	1,060	1,016
	7	1,244	1,242	1,245	1,242	1,203	1,243	1,240	1,235	1,203
	8	1,409	1,409	1,412	1,408	1,370	1,410	1,408	1,400	1,375
	9	1,566	1,567	1,569	1,567	1,529	1,569	1,568	1,556	1,537
0,90	2	0,535	-	0,536	0,529	0,477	0,535	0,497	0,535	0,354
	3	0,793	-	0,794	0,788	0,735	0,794	0,783	0,790	0,716
	4	1,020	1,020	1,022	1,017	0,964	1,021	1,017	1,015	0,975
	5	1,228	1,227	1,229	1,224	1,172	1,228	1,226	1,219	1,194
	6	1,418	1,417	1,419	1,415	1,365	1,419	1,418	1,407	1,390
	7	1,595	1,596	1,598	1,594	1,545	1,599	1,598	1,585	1,572
	8	1,765	1,764	1,766	1,763	1,715	1,768	1,767	1,751	1,741
	9	1,926	1,924	1,926	1,923	1,876	1,928	1,927	1,909	1,902

1) Exact percentage points according to RESNIKOFF and LIEBERMAN (1957).

2) Exact percentage points according to JOHNSON and WELCH (1940).

Table K (continued)

α	f	t(f,δ,α) exact I	t(f,δ,α) exact II	approximations to t(f,δ,α) I a	I b	I c	II a	II b	II c	II d
0,70	2	1,344	-	1,330	1,317	1,186	1,351	1,353	1,324	1,315
	3	1,592	-	1,582	1,573	1,462	1,597	1,598	1,573	1,569
	4	1,818	1,819	1,810	1,804	1,705	1,824	1,824	1,800	1,797
	5	2,026	2,027	2,019	2,015	1,925	2,032	2,033	2,008	2,007
	6	2,220	2,221	2,214	2,210	2,127	2,227	2,227	2,203	2,202
	7	2,402	2,402	2,396	2,392	2,315	2,408	2,408	2,385	2,384
	8	2,573	2,574	2,568	2,565	2,491	2,579	2,579	2,556	2,555
	9	2,736	2,737	2,731	2,729	2,659	2,742	2,743	2,719	2,719
0,50	2	2,085	-	2,026	2,026	1,795	2,094	2,094	2,019	2,019
	3	2,286	-	2,250	2,250	2,073	2,294	2,294	2,245	2,245
	4	2,492	2,491	2,465	2,465	2,318	2,499	2,499	2,462	2,462
	5	2,690	2,688	2,668	2,668	2,539	2,696	2,696	2,666	2,666
	6	2,876	2,875	2,858	2,858	2,742	2,883	2,883	2,856	2,856
	7	3,053	3,053	3,038	3,038	2,931	3,060	3,060	3,036	3,036
	8	3,222	3,222	3,208	3,208	3,109	3,227	3,227	3,206	3,206
	9	3,384	3,382	3,369	3,369	3,277	3,388	3,388	3,369	3,369
0,30	2	3,171	-	3,050	3,123	2,670	3,152	3,150	2,962	2,971
	3	3,213	-	3,150	3,183	2,883	3,210	3,209	3,108	3,112
	4	3,349	3,348	3,306	3,327	3,095	3,349	3,349	3,284	3,287
	5	3,507	3,505	3,475	3,489	3,296	3,508	3,507	3,463	3,464
	6	3,668	3,667	3,642	3,652	3,486	3,668	3,668	3,635	3,636
	7	3,827	3,827	3,805	3,813	3,665	3,827	3,827	3,803	3,804
	8	3,982	3,982	3,963	3,970	3,836	3,983	3,983	3,963	3,964
	9	4,133	4,131	4,115	4,121	3,998	4,133	4,132	4,118	4,118
0,10	2	6,182	-	6,814	7,960	5,614	5,784	5,822	4,978	5,159
	3	5,368	-	5,566	5,853	4,973	5,233	5,244	4,836	4,910
	4	5,166	5,167	5,271	5,407	4,869	5,098	5,102	4,861	4,901
	5	5,143	5,144	5,213	5,295	4,903	5,102	5,104	4,947	4,972
	6	5,193	5,192	5,244	5,300	4,989	5,163	5,164	5,056	5,073
	7	5,276	5,275	5,315	5,357	5,097	5,251	5,252	5,175	5,188
	8	5,374	5,374	5,407	5,440	5,215	5,354	5,355	5,300	5,310
	9	5,481	5,481	5,509	5,536	5,337	5,464	5,464	5,426	5,433
0,05	2	8,962	-	15,372	29,027	10,949	7,857	8,015	6,303	6,816
	3	7,039	-	8,245	9,162	7,145	6,685	6,727	5,922	6,122
	4	6,452	6,458	6,992	7,321	6,372	6,281	6,298	5,824	5,930
	5	6,239	6,238	6,561	6,736	6,124	6,134	6,142	5,832	5,897
	6	6,173	6,173	6,395	6,507	6,054	6,100	6,105	5,889	5,933
	7	6,178	6,176	6,346	6,425	6,064	6,123	6,126	5,970	6,002
	8	6,220	6,220	6,355	6,415	6,114	6,178	6,180	6,064	6,089
	9	6,285	6,286	6,398	6,445	6,185	6,251	6,252	6,166	6,185

Table K (continued)

α	f	t(f, δ, α) exact		approximations to t (f,δ,α)						
		I	II	I			II			
				a	b	c	a	b	c	d
0,025	2	12,822	-	-81,418	-17,985	90,756	10,317	10,768	7,679	8,836
	3	9,081	-	14,125	18,199	11,413	8,318	8,436	7,025	7,456
	4	7,918	7,934	9,621	10,437	8,574	7,563	7,610	6,786	7,009
	5	7,432	7,436	8,337	8,694	7,695	7,225	7,249	6,706	6,843
	6	7,208	7,210	7,793	7,999	7,328	7,071	7,084	6,703	6,795
	7	7,112	7,112	7,533	7,670	7,166	7,011	7,019	6,741	6,807
	8	7,087	7,085	7,409	7,508	7,104	7,007	7,012	6,801	6,851
	9	7,098	7,100	7,360	7,435	7,097	7,034	7,038	6,877	6,915
0,01	2	20,435	-	-7,706	-4,803	-9,520	14,222	15,566	9,576	12,350
	3	12,557	-	124,726	-71,500	42,569	10,793	11,129	8,517	9,489
	4	10,200	10,25	17,196	21,047	14,336	9,441	9,572	8,070	8,558
	5	9,210	9,22	12,060	13,048	10,859	8,783	8,847	7,861	8,154
	6	8,713	8,70	10,330	10,793	9,589	8,431	8,467	7,770	7,965
	7	8,435	8,43	9,516	9,790	8,980	8,238	8,260	7,742	7,881
	8	8,287	8,29	9,075	9,260	8,653	8,135	8,149	7,753	7,857
	9	8,211	8,21	8,822	8,957	8,473	8,089	8,098	7,789	7,869

Table L

Comparison of the approximations with the exact percentage points for $p = 0{,}125$

α	f	$t(f,\delta,\alpha)$ exact	approximations to $t(f,\delta,\alpha)$						
			I			II			
			a	b	c	a	b	c	d
0,99	4	0,25	0,25	0,25	0,24	0,25	0,12	0,25	-0,24
	5	0,49	0,49	0,49	0,47	0,49	0,42	0,49	0,20
	6	0,70	0,70	0,70	0,67	0,70	0,66	0,70	0,50
	7	0,89	0,90	0,89	0,87	0,89	0,87	0,89	0,75
	8	1,08	1,08	1,07	1,05	1,08	1,06	1,07	0,96
	9	1,25	1,25	1,25	1,22	1,25	1,24	1,24	1,16
0,975	4	0,605	0,607	0,604	0,572	0,606	0,559	0,605	0,382
	5	0,832	0,835	0,831	0,796	0,833	0,809	0,829	0,692
	6	1,039	1,043	1,039	1,003	1,041	1,028	1,033	0,941
	7	1,232	1,236	1,231	1,196	1,235	1,226	1,223	1,157
	8	1,414	1,419	1,415	1,378	1,417	1,412	1,402	1,352
	9	1,587	1,592	1,588	1,551	1,591	1,587	1,572	1,533
0,95	4	0,902	0,905	0,900	0,854	0,903	0,886	0,897	0,791
	5	1,127	1,131	1,126	1,080	1,130	1,121	1,118	1,053
	6	1,335	1,339	1,335	1,289	1,339	1,334	1,322	1,278
	7	1,530	1,534	1,530	1,484	1,534	1,531	1,514	1,482
	8	1,714	1,718	1,714	1,669	1,718	1,716	1,695	1,670
	9	1,888	1,893	1,889	1,844	1,893	1,892	1,867	1,848
0,90	4	1,251	1,254	1,246	1,183	1,254	1,250	1,238	1,198
	5	1,478	1,481	1,475	1,414	1,483	1,481	1,462	1,437
	6	1,689	1,692	1,686	1,627	1,694	1,693	1,669	1,652
	7	1,887	1,889	1,884	1,827	1,893	1,892	1,865	1,852
	8	2,073	2,076	2,071	2,016	2,098	2,097	2,049	2,039
	9	2,250	2,253	2,249	2,195	2,258	2,258	2,223	2,216
0,70	4	2,066	2,055	2,048	1,937	2,078	2,078	2,039	2,036
	5	2,297	2,287	2,282	2,181	2,308	2,309	2,270	2,269
	6	2,512	2,503	2,498	2,405	2,523	2,523	2,486	2,485
	7	2,713	2,705	2,701	2,614	2,724	2,724	2,687	2,686
	8	2,904	2,896	2,892	2,810	2,914	2,914	2,878	2,877
	9	3,084	3,077	3,074	2,996	3,094	3,095	3,059	3,059
0,50	4	2,769	2,736	2,736	2,572	2,782	2,782	2,733	2,733
	5	2,987	2,961	2,961	2,818	2,999	2,999	2,958	2,958
	6	3,194	3,172	3,172	3,043	3,206	3,206	3,170	3,170
	7	3,391	3,371	3,371	3,254	3,401	3,401	3,370	3,370
	8	3,578	3,560	3,560	3,451	3,587	3,587	3,559	3,559
	9	3,756	3,740	3,740	3,638	3,765	3,765	3,739	3,739

Table L (continued)

α	f	$t(f,\delta,\alpha)$ exact	approximations to $t(f,\delta,\alpha)$						
			I			II			
			a	b	c	a	b	c	d
0,30	4	3,674	3,623	3,646	3,391	3,676	3,676	3,604	3,607
	5	3,847	3,810	3,826	3,614	3,850	3,849	3,802	3,803
	6	4,026	3,996	4,008	3,824	4,028	4,028	3,994	3,995
	7	4,203	4,177	4,187	4,023	4,204	4,204	4,180	4,181
	8	4,375	4,352	4,360	4,213	4,376	4,376	4,358	4,359
	9	4,541	4,522	4,528	4,393	4,543	4,542	4,530	4,530
0,10	4	5,613	5,730	5,883	5,290	5,527	5,531	5,284	5,324
	5	5,590	5,668	5,759	5,329	5,534	5,536	5,381	5,406
	6	5,644	5,703	5,767	5,425	5,603	5,604	5,504	5,521
	7	5,737	5,783	5,830	5,545	5,702	5,703	5,638	5,651
	8	5,847	5,886	5,923	5,676	5,817	5,818	5,777	5,787
	9	5,967	5,999	6,030	5,811	5,940	5,940	5,918	5,925
0,05	4	7,001	7,599	7,966	6,919	6,782	6,799	6,309	6,415
	5	6,758	7,125	7,320	6,646	6,627	6,636	6,324	6,389
	6	6,689	6,944	7,068	6,570	6,594	6,599	6,390	6,434
	7	6,696	6,890	6,979	6,582	6,624	6,627	6,483	6,515
	8	6,746	6,902	6,969	6,638	6,687	6,689	6,589	6,614
	9	6,821	6,951	7,003	6,718	6,769	6,770	6,704	6,723
0,025	4	8,579	10,476	11,383	9,324	8,145	8,192	7,331	7,554
	5	8,042	9,058	9,455	8,354	7,784	7,808	7,253	7,390
	6	7,797	8,459	8,688	7,950	7,624	7,637	7,256	7,348
	7	7,694	8,174	8,326	7,772	7,564	7,573	7,302	7,368
	8	7,669	8,038	8,149	7,705	7,564	7,569	7,373	7,423
	9	7,687	7,985	8,070	7,698	7,597	7,601	7,459	7,497
0,01	4	11,08	18,82	23,10	15,66	10,14	10,27	8,69	9,18
	5	9,95	13,14	14,24	11,82	9,44	9,50	8,48	8,77
	6	9,41	11,23	11,74	10,41	9,07	9,10	8,39	8,58
	7	9,10	10,33	10,63	9,74	8,86	8,89	8,37	8,50
	8	8,94	9,84	10,05	9,38	8,76	8,77	8,38	8,49
	9	8,87	9,57	9,72	9,19	8,71	8,72	8,43	8,51

Table M

Comparison of approximation IV with the exact percentage points and the other approximations

f=7

δ	p	α	$t(f,\delta,\alpha)$ exact	approximations to $t(f,\delta,\alpha)$								
				I			II				III	IV
				a	b	c	a	b	c	d		
0,5533	0,42	0,4	0,843	0,842	0,842	0,813	0,843	0,843	0,841	0,841	-	0,843
		0,2	1,512	1,511	1,514	1,457	1,512	1,512	1,503	1,506	-	1,513
		0,05	2,621	2,648	2,667	2,542	2,616	2,619	2,561	2,593	-	2,623
		0,01	3,90	4,12	4,19	3,93	3,86	3,88	3,67	3,81	-	3,90
1,8146	0,26	0,4	2,190	2,183	2,184	2,105	2,190	2,190	2,180	2,180	-	2,188
		0,2	2,992	2,988	2,998	2,876	2,992	2,992	2,963	2,966	-	2,992
		0,05	4,434	4,524	4,572	4,331	4,414	4,417	4,286	4,318	-	4,436
		0,01	6,20	6,82	6,99	6,45	6,10	6,12	5,71	5,85	-	6,20
2,6568	0,17	0,4	3,105	3,091	3,094	2,931	3,110	3,110	3,091	3,091	-	3,100
		0,2	4,030	4,020	4,036	3,867	4,027	4,027	3,991	3,994	-	4,026
		0,05	5,739	5,887	5,958	5,628	5,697	5,701	5,542	5,574	-	5,741
		0,01	7,87	8,83	9,08	8,34	7,70	7,73	7,22	7,36	-	7,89
3,1953	0,13	0,4	3,699	3,675	3,679	3,544	3,706	3,706	3,679	3,679	-	3,688
		0,2	4,708	4,694	4,714	4,512	4,701	4,701	4,671	4,674	-	4,701
		0,05	6,603	6,791	6,878	6,488	6,533	6,536	6,389	6,421	-	6,608
		0,01	8,99	10,18	10,48	9,60	8,75	8,77	8,25	8,39	-	9,01
f=16												
0,8207	0,42	0,4	1,096	1,096	1,096	1,079	1,096	1,096	1,096	1,096	-	1,096
		0,2	1,727	1,727	1,727	1,699	1,727	1,727	1,725	1,726	-	1,727
		0,05	2,674	2,680	2,683	2,635	2,674	2,674	2,660	2,666	2,673	2,674
		0,01	3,60	3,64	3,64	3,57	3,60	3,60	3,56	3,58	3,60	3,60
2,6914	0,26	0,4	3,029	3,025	3,025	2,978	3,029	3,029	3,026	3,026	-	3,027
		0,2	3,760	3,757	3,760	3,697	3,760	3,760	3,752	3,753	-	3,758
		0,05	4,920	4,945	4,955	4,860	4,917	4,917	4,887	4,893	4,919	4,922
		0,01	6,12	6,23	6,26	6,12	6,10	6,10	6,01	6,03	6,11	6,12
3,9407	0,17	0,4	4,334	4,326	4,326	4,258	4,336	4,336	4,330	4,330	-	4,330
		0,2	5,159	5,155	5,159	5,071	5,157	5,157	5,158	5,159	-	5,156
		0,05	6,502	6,546	6,562	6,431	6,487	6,487	6,475	6,480	6,502	6,509
		0,01	7,91	8,11	8,14	7,95	7,87	7,87	7,79	7,82	7,92	7,94
4,7394	0,13	0,4	5,173	5,161	5,162	5,080	5,177	5,177	5,170	5,170	-	5,165
		0,2	6,068	6,061	6,067	5,962	6,060	6,060	6,078	6,079	-	6,063
		0,05	7,538	7,596	7,616	7,462	7,503	7,504	7,532	7,537	7,543	7,546
		0,01	9,10	9,35	9,39	9,17	9,02	9,02	8,99	9,01	9,10	9,12

Table N

Exact percentage points of the non-central t-distribution for

α = 0,99;0,975;0,95;0,90;0,70;0,50;0,30;0,10;0,05;0,025;0,01

f = 4(1)9

found from the tables of JOHNSON and WELCH

α	f	n						
		0,45	0,40	0,35	0,30	0,25	0,15	0,125
0,99	4	-3,13	-2,56	-2,02	-1,50	-0,99	-0,01	0,25
	5	-2,80	-2,26	-1,76	-1,25	-0,77	0,22	0,49
	6	-2,59	-2,08	-1,57	-1,09	-0,60	0,42	0,70
	7	-2,46	-1,94	-1,44	-0,95	-0,45	0,60	0,89
	8	-2,35	-1,83	-1,32	-0,83	-0,32	0,77	1,08
	9	-2,27	-1,75	-1,23	-0,72	-0,20	0,92	1,25
0,975	4	-2,279	-1,812	-1,366	-0,935	-0,511	0,364	0,605
	5	-2,092	-1,635	-1,195	-0,763	-0,334	0,576	0,832
	6	-1,969	-1,512	-1,067	-0,627	-0,186	0,767	1,039
	7	-1,882	-1,417	-0,963	-0,512	-0,054	0,943	1,232
	8	-1,815	-1,341	-0,874	-0,409	0,065	1,108	1,414
	9	-1,760	-1,275	-0,795	-0,315	0,176	1,266	1,587
0,95	4	-1,707	-1,303	-0,913	-0,529	-0,147	0,669	0,902
	5	-1,589	-1,182	-0,785	-0,390	0,008	0,876	1,127
	6	-1,510	-1,092	-0,682	-0,272	0,145	1,065	1,335
	7	-1,451	-1,020	-0,595	-0,168	0,268	1,242	1,530
	8	-1,403	-0,958	-0,518	-0,074	0,383	1,409	1,714
	9	-1,363	-0,903	-0,447	0,014	0,491	1,567	1,888
0,90	4	-1,166	-0,810	-0,463	-0,117	0,237	1,020	1,251
	5	-1,096	-0,727	-0,364	0,003	0,380	1,227	1,478
	6	-1,045	-0,660	-0,277	0,110	0,510	1,417	1,689
	7	-1,004	-0,601	-0,200	0,207	0,631	1,596	1,887
	8	-0,970	-0,550	-0,131	0,297	0,744	1,764	2,073
	9	-0,940	-0,503	-0,065	0,382	0,850	1,924	2,250
0,70	4	-0,261	0,045	0,355	0,676	1,016	1,819	2,066
	5	-0,229	0,101	0,436	0,784	1,154	2,027	2,297
	6	-0,200	0,152	0,510	0,883	1,281	2,221	2,512
	7	-0,175	0,198	0,580	0,977	1,400	2,402	2,713
	8	-0,152	0,242	0,645	1,064	1,512	2,574	2,904
	9	-0,131	0,283	0,706	1,148	1,619	2,737	3,084
0,50	4	0,299	0,603	0,919	1,252	1,614	2,491	2,769
	5	0,323	0,653	0,993	1,354	1,744	2,688	2,987
	6	0,347	0,699	1,064	1,450	1,867	2,875	3,194
	7	0,368	0,743	1,131	1,540	1,982	3,053	3,391
	8	0,389	0,784	1,194	1,626	2,093	3,222	3,578
	9	0,409	0,824	1,254	1,707	2,198	3,382	3,756

Table N (continued)

α	f	p: 0,45	0,40	0,35	0,30	0,25	0,15	0,125
0,30	4	0,883	1,209	1,554	1,926	2,334	3,348	3,674
	5	0,896	1,244	1,611	2,005	2,437	3,505	3,847
	6	0,912	1,282	1,670	2,086	2,543	3,667	4,026
	7	0,928	1,319	1,728	2,166	2,647	3,827	4,203
	8	0,945	1,355	1,786	2,245	2,748	3,982	4,375
	9	0,961	1,391	1,841	2,321	2,846	4,131	4,541
0,10	4	1,9[illegible]3	2,328	2,771	3,255	3,796	5,167	5,613
	5	1,872	2,290	2,737	3,228	3,772	5,144	5,590
	6	1,850	2,281	2,741	3,240	3,797	5,192	5,644
	7	1,839	2,285	2,761	3,275	3,847	5,275	5,737
	8	1,838	2,298	2,789	3,320	3,906	5,374	5,847
	9	1,840	2,316	2,822	3,369	3,973	5,481	5,967
0,05	4	2,579	3,062	3,586	4,163	4,809	6,458	7,001
	5	2,461	2,936	3,450	4,012	4,642	6,238	6,758
	6	2,396	2,874	3,390	3,954	4,584	6,173	6,689
	7	2,356	2,844	3,368	3,940	4,576	6,176	6,696
	8	2,333	2,833	3,368	3,950	4,596	6,220	6,746
	9	2,320	2,832	3,379	3,973	4,633	6,286	6,821
0,025	4	3,303	3,873	4,494	5,181	5,954	7,934	8,579
	5	3,077	3,620	4,208	4,856	5,584	7,436	8,042
	6	2,947	3,483	4,062	4,698	5,408	7,210	7,797
	7	2,870	3,406	3,985	4,618	5,325	7,112	7,694
	8	2,819	3,361	3,944	4,581	5,293	7,085	7,669
	9	2,783	3,336	3,928	4,573	5,290	7,100	7,687
0,01	4	4,41	5,12	5,89	6,74	7,73	10,25	11,08
	5	3,96	4,61	5,31	6,09	6,96	9,22	9,95
	6	3,73	4,34	5,01	5,76	6,59	8,70	9,41
	7	3,57	4,18	4,84	5,56	6,38	8,43	9,10
	8	3,46	4,07	4,73	5,46	6,26	8,29	8,94
	9	3,39	4,00	4,66	5,38	6,18	8,21	8,87

Fig. 1

α \ p		0,50				0,45				0,40				0,35				0,30				0,25				0,15				0,125			
		a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d
0,99	I																																
	II																																
0,975	I																																
	II																																
0,95	I																																
	II																																
0,90	I																																
	II																																
0,70	I																																
	II																																
0,50	I																																
	II																																
0,30	I																																
	II																																
0,10	I																																
	II																																
0,05	I																																
	II																																
0,025	I																																
	II																																
0,01	I																																
	II																																

In this figure we indicated, for all values of α and p used in the tables E-L, the approximations which may be considered as good approximations to $t(f,\delta,\alpha)$.
The approximations IIb and IId cannot be used as an approximation for cases with p=0,50; they lead in this case to the equality $t(f,0,\alpha) = t(f,0,\alpha)$. However they are important for values of p near 0,50; therefore we indicated these approximations in the

column $p = 0,50$.
This figure may also be used for values of α and p, not used in the tables, with $0,01 < \alpha < 0,99$ and $0,125 < p < 0,50$. E.g. for $\alpha = 0,95$ and any p with $0,35 < p < 0,30$. IIa may be used as an approximation to $t(f,\delta,\alpha)$.