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On K.L. Chung's problem of imbedding  
a time-discrete Markov chain in a  
time-continuous one for finitely many states I

by

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Call a stochastic matrix (i.e. a square matrix of order  $n$  with elements  $p_{jk} \geq 0$  for all  $j, k \in \{1, 2, \dots, n\}$  and  $\sum_{k=1}^n p_{jk} = 1$  for all  $j \in \{1, 2, \dots, n\}$ ) a P-matrix. Then the following problem (further called Chung's problem) is a specialization of a problem posed in CHUNG [1958] and CHUNG [1960].

Chung's problem: Find conditions in order that to a given P-matrix  $P_1$  there exists a matrix function  $P(t)$  such that  $P(t)$  is a P-matrix for each  $t \geq 0$ , for which <sup>1)</sup>

$$(1) \quad \left\{ \begin{array}{l} \lim_{t \downarrow 0} P(t) = I, \\ P(s+t) = P(s)P(t) \quad \text{for all real } s, t > 0, \\ P(1) = P_1. \end{array} \right.$$

For  $n=2$  Chung's problem has been solved by FRÉCHET [1952] and BELLMAN [1960]. In my thesis (RUNNENBURG [1960]) the case  $n=3$  was considered.

It is well-known (cf. DOOB [1953], BIRKHOFF and VARGA [1958]), that if  $P(t)$  exists, then

$$(2) \quad P(t) = e^{Q_1 t} \quad \text{for all real } t \geq 0,$$

where  $Q_1$  is a Q-matrix (i.e. a square matrix of order  $n$  with elements  $q_{jk}$  satisfying  $q_{jk} \geq 0$  if  $j \neq k$  for all  $j, k \in \{1, 2, \dots, n\}$  and  $\sum_{k=1}^n q_{jk} = 0$  for all  $j \in \{1, 2, \dots, n\}$ ).

Furthermore, if  $\mu_1, \mu_2, \dots, \mu_n$  are the eigenvalues of  $Q_1$ , then the eigenvalues  $\lambda_1(t), \lambda_2(t), \dots, \lambda_n(t)$  of  $P(t)$  are given by

$$(3) \quad \lambda_j(t) = e^{\mu_j t} \quad \text{for all real } t \geq 0 \text{ and all } j.$$

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1)  $I$  is the  $n \times n$  identity matrix.

According to GANTMACHER [1959] , in 1938 Kolmogorov posed the problem: Characterize the complex numbers  $z$  with  $|z| \leq 1$  which can occur as eigenvalues of an  $n^{\text{th}}$  order stochastic matrix. This problem was partly solved in DMITRIEV and DYNKIN [1946] and definitely in KARPELEWITSCH [1951] . Making use of their results, the next theorem can be proved.

Theorem: The eigenvalues  $\mu_1, \mu_2, \dots, \mu_n$  of a  $Q$ -matrix (of order  $n \geq 3$ ) satisfy

$$(4) \quad \left(\frac{1}{2} + \frac{1}{n}\right)\pi \leq \arg \mu_j \leq \left(\frac{3}{2} - \frac{1}{n}\right)\pi \quad \text{for } 1 \leq j \leq n;$$

the only  $Q$ -matrices  $Q_1^*$  (with elements  $q_{jk}^*$ ) with at least one  $\mu_j \neq 0$  on the boundary of this region are given (after a suitable renumbering of states, i.e. rows and columns at the same time) by

$$(5) \quad q_{jk}^* = \begin{cases} -\alpha & \text{for } j \equiv k \pmod{n}, \\ \alpha & \text{for } j \equiv k-1 \pmod{n}, \\ 0 & \text{otherwise,} \end{cases}$$

where  $\alpha$  is an arbitrary positive number.

Remark 1: From this theorem we conclude, that for  $n \geq 3$  all eigenvalues  $\lambda_j(t)$  of  $P(t) = e^{Q_1^* t}$  satisfy

$$(6) \quad \lambda_j(t) \in H_n,$$

where  $H_n$  is a heart-shaped region in the complex plane, contained in the unit circle and symmetric with respect to the real axis, with the curve

$$(7) \quad \exp\left(-1 + \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}\right)t \quad \left(\text{where } 0 \leq t \leq \frac{\pi}{\sin \frac{2\pi}{n}}\right)$$

as boundary in the region  $\text{Im } z \geq 0$ . Hence Chung's problem can only be solved for those  $P_1$  for which all eigenvalues belong to the region  $H_n$ . The only matrices  $e^{Q_1}$  with a  $Q$ -matrix  $Q_1$  for which at least one eigenvalue  $\lambda \neq 1$  lies on the boundary curve of the region  $H_n$  are given by

$$(8) \quad P_1^* = e \cdot Q_1^*$$

Proof of the theorem: If  $P_1$  is a P-matrix, then  $Q_1 = P_1 - I$  is a Q-matrix; if  $Q_1$  is a Q-matrix, then  $P_{1,\beta} = I + \beta Q_1$  (where  $\beta > 0$  is chosen in such a way that  $0 \leq 1 + \beta q_{jj} \leq 1$  for  $1 \leq j \leq n$ , e.g.  $\beta = -\left(\min_{1 \leq j \leq n} q_{jj}\right)^{-1}$  if some  $q_{jj} < 0$ ) is a P-matrix. Hence if for an arbitrary Q-matrix  $Q_1$  we have

$$(9) \quad \det(Q_1 - \mu I) = 0,$$

then

$$(10) \quad \det(P_{1,\beta} - (\beta\mu + 1)I) = 0.$$

Now Karpelewitsch has shown that the roots of the characteristic equation

$$(11) \quad \det(P_1 - \lambda I) = 0$$

for an arbitrary P-matrix  $P_1$ , satisfy

$$(12) \quad \left(\frac{1}{2} + \frac{1}{n}\right)\pi \leq \arg(\lambda - 1) \leq \left(\frac{3}{2} - \frac{1}{n}\right)\pi$$

for  $n \geq 3$ , and so with  $\lambda = \beta\mu + 1$ , because of  $\beta > 0$  we have

$$(13) \quad \left(\frac{1}{2} + \frac{1}{n}\right)\pi \leq \arg \mu \leq \left(\frac{3}{2} - \frac{1}{n}\right)\pi.$$

If  $Q_1^*$  has an eigenvalue  $\mu_j$  for which  $\arg \mu_j = \left(\frac{1}{2} + \frac{1}{n}\right)\pi$  with  $\mu_j \neq 0$ , i.e.  $\mu_j = \gamma \left(e^{\frac{2\pi i}{n}} - 1\right)$  with a  $\gamma > 0$ . then  $P_{1,\beta}^*$  has an eigenvalue  $\lambda_j$  with  $\arg(\lambda_j - 1) = \left(\frac{1}{2} + \frac{1}{n}\right)\pi$  and  $\lambda_j - 1 \neq 0$ . As Dmitriev and Dynkin have shown, in that case the elements  $p_{jk}^*$  of  $P_{1,\beta}^*$  satisfy (after a suitable renumbering of the rows and columns at the same time)

$$(14) \quad p_{jk}^* = \begin{cases} 1 - \alpha_j^i & \text{for } j \equiv k \pmod{n}, \\ \alpha_j^i & \text{for } j \equiv k-1 \pmod{n}, \\ 0 & \text{otherwise,} \end{cases}$$

where  $0 \leq \alpha_j \leq 1$  for each  $j \in \{1, 2, \dots, n\}$ . Therefore the elements  $q_{jk}^*$  of  $Q_1^*$  are given by

$$(15) \quad q_{jk}^* = \begin{cases} -\alpha_j & \text{for } j \equiv k \pmod{n} \\ \alpha_j & \text{for } j \equiv k-1 \pmod{n} \\ 0 & \text{otherwise} \end{cases}$$

where we have written  $\alpha_j$  instead of  $\frac{\alpha_j}{\beta}$ , and so  $\alpha_j \geq 0$  for each  $j \in \{1, 2, \dots, n\}$ .

For the characteristic equation of  $Q_1^*$  we obtain

$$(16) \quad (\alpha_1 + \mu)(\alpha_2 + \mu) \dots (\alpha_n + \mu) = \alpha_1 \alpha_2 \dots \alpha_n.$$

In order that  $\mu = \gamma(e^{\frac{2\pi i}{n}} - 1)$ , with a constant  $\gamma > 0$ , is a root of this equation, all  $\alpha_j$  must be positive. We shall prove that

$$(17) \quad \alpha_j = \gamma \quad \text{for all } j \in \{1, 2, \dots, n\},$$

for which it is sufficient to consider  $\gamma = 1$ .

If we introduce the finite positive numbers  $\theta_j = \frac{1}{\alpha_j}$  in (16) and substitute  $e^{\frac{2\pi i}{n}} - 1$  for  $\mu$ , we obtain

$$(18) \quad \left\{ \theta_1 e^{\frac{2\pi i}{n}} + (1 - \theta_1) \right\} \left\{ \theta_2 e^{\frac{2\pi i}{n}} + (1 - \theta_2) \right\} \dots \left\{ \theta_n e^{\frac{2\pi i}{n}} + (1 - \theta_n) \right\} = 1.$$

Consider a line in the complex plane which does not pass through the origin. Let  $r(\varphi)$  be the absolute value of the complex number on this line with argument  $\varphi$ . Then  $\log r(\varphi)$  is a strictly convex analytic function of  $\varphi$  for finite  $r(\varphi)$ . Hence if  $\varphi_1 \neq \varphi_2$ , we have for any  $p_1, p_2 > 0$  with  $p_1 + p_2 = 1$

$$(19) \quad p_1 \log r(\varphi_1) + p_2 \log r(\varphi_2) > \log r(p_1 \varphi_1 + p_2 \varphi_2).$$

Therefore

$$(20) \quad r(\varphi_1) r(\varphi_2) \dots r(\varphi_n) > r\left(\frac{\varphi_1 + \varphi_2 + \dots + \varphi_n}{n}\right)^n,$$

unless  $\varphi_1 = \varphi_2 = \dots = \varphi_n$ .

Now consider for  $n \geq 3$  the points  $\theta_j \cdot e^{\frac{2\pi i}{n}} + (1-\theta_j) \cdot 1$  for finite  $\theta_j > 0$ , which lie on the line passing through 1 and  $e^{\frac{2\pi i}{n}}$ . Then  $0 < \varphi_j < (\frac{1}{2} + \frac{1}{n})\pi$ . Under the restriction  $\varphi_1 + \varphi_2 + \dots + \varphi_n \equiv 0 \pmod{2\pi}$ , the product  $r(\varphi_1)r(\varphi_2)\dots r(\varphi_n)$  assumes its smallest value for  $\varphi_1 = \varphi_2 = \dots = \varphi_n = \frac{2\pi}{n}$ . Any other choice of values for the  $\varphi_j$  leads to a larger value for the product. Hence if we introduce polar coordinates in (18) by writing

$$(21) \quad \theta_j e^{\frac{2\pi i}{n}} + 1 - \theta_j = r(\varphi_j) e^{i\varphi_j},$$

we find that under the condition "all  $\theta_j$  are finite positive numbers" the left-hand side of (18) has smallest positive value 1, where the value 1 is only obtained if  $\varphi_j = \frac{2\pi}{n}$  for all  $j$ . Therefore, if equation (18) holds, we must have  $\theta_j = 1$  for all  $j$  or  $\alpha_1 = \alpha_2 = \dots = \alpha_n = 1$ .

Remark 2: For  $n=3$  one may easily verify that there exist  $P$ -matrices  $P_1$  for which Chung's problem does not have a solution, although  $P_1$  has all eigenvalues inside  $H_3$ . If  $P_1$  is a circulant (i.e. a square matrix of order  $n$  with elements  $p_{jk} = p_{k-j}$ , where  $k-j$  is taken modulo  $n$ ), then for  $n=3$  a circulant  $Q_1$  can be found which is a  $Q$ -matrix and for which  $P_1 = e^{Q_1}$  holds, on condition that all eigenvalues of  $P_1$  are inside  $H_3$ .

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Report S 299

On K.L. Chung's problem of imbedding  
a time-discrete Markov chain in a  
time-continuous one for finitely many states II<sup>\*\*)</sup>

by

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\* ) Part I of this report was written before we knew of KINGMAN [5]. The problem is really due to G. ELFVING [6].



1

The purpose of this note is to show that one of the results obtained in the previous one [1], can be deduced without using the work of DMITRIEV and DYNKIN [2] and KARPELEWITSCH [3]. For a statement of the problem we refer to [1].

We use the fact that a Markov-chain on a finite set consisting of  $n$  states  $A_1, A_2, \dots, A_n$ , induces an abelian semi-group  $G$  of real linear transformations on an  $R_n$ : with the Markov matrix  $(p_{jk})$  (where  $p_{jk} = P\{x_{n+1} = A_k | x_n = A_j\}$ ) corresponds the linear transformation  $\pi$  defined by

$$\pi e_j = (p_{j1}, p_{j2}, \dots, p_{jn}),$$

where  $e_j$  is the  $j$ 'th unit vector.

A necessary and sufficient condition that the matrix of a given linear transformation  $\varphi$  is a Markov matrix is

$$(1) \quad \varphi W \subset W,$$

where  $W$  is the set of all probability distributions on  $\{A_1, A_2, \dots, A_n\}$

$$(2) \quad W = \left\{ x | x = (\xi_1, \xi_2, \dots, \xi_n); \forall_j \xi_j \geq 0; \sum_{j=1}^n \xi_j = 1 \right\}.$$

2

Now let  $G$  be a continuous one-parameter abelian semi-group of linear transformations satisfying (1) ( $\iota$  is the identical transformation on  $R_n$ )

$$G = \left\{ \varphi_t | t \in (0, \infty); \forall_t^{(0, \infty)} \varphi_t W \subset W; \forall_t^{(0, \infty)} \forall_s^{(0, \infty)} \varphi_t \varphi_s = \varphi_{t+s}; \lim_{t \downarrow 0} \varphi_t = \iota \right\}.$$

It is well-known (cf e.g. FRÉCHET [4]) that under these conditions the eigenvalues of  $\varphi_t$  are  $1, \lambda_1^t, \lambda_2^t, \dots, \lambda_m^t$  ( $m \leq n-1$ ), where  $1, \lambda_1, \lambda_2, \dots, \lambda_m$  are eigenvalues of  $\varphi_1$ , for some choice of the arguments of  $\lambda_1, \dots, \lambda_m$ . Moreover there is at least one vector  $p \in W$ , which is invariant under  $G$ .

Let  $\lambda = \xi e^{i\psi}$  ( $\xi$  real and  $\geq 0$ ) be an eigenvalue of  $\varphi_1$ . We suppose that  $\psi$  is chosen in such a way that  $\xi^t e^{it\psi}$  is, for all  $t > 0$ , an eigenvalue of  $\varphi_t$ . Moreover we suppose for the moment  $\psi \neq 0$ ;  $\xi \neq 0$  and  $n \geq 3$ . (The inequality (20) which

we want to establish is trivially satisfied for  $\psi=0$  or  $\xi=0$ ).

Let

$$z_0 = x_0 + iy_0,$$

where  $x_0$  and  $y_0$  are real vectors, be an eigenvector of  $\varphi_1$  belonging to the eigenvalue  $\lambda$ . As  $\psi \neq 0$ , it follows that  $x_0$  and  $y_0$  are independent. Then the two-dimensional subspace  $V$  spanned by  $x_0$  and  $y_0$

$$V = \{ u | u = \alpha x_0 + \beta y_0; \alpha, \beta \text{ real} \}$$

is invariant under  $G$ , and the movement induced by  $G$  in  $V$  is easily seen to be described by the equations

$$(3) \quad \begin{cases} \varphi_t x_0 = \xi^t (x_0 \cos t\psi - y_0 \sin t\psi), \\ \varphi_t y_0 = \xi^t (x_0 \sin t\psi + y_0 \cos t\psi). \end{cases}$$

Introducing

$$W_0 = \{ x | x \in W; \exists_j \xi_j = 0 \},$$

$$U = W \cap (p+V),$$

$$U_0 = W_0 \cap (p+V),$$

the condition (1) together with the invariance of  $p$  and  $V$  yields

$$(4) \quad \forall_t^{(0, \infty)} \varphi_t U \subset U.$$

As  $U$  is a closed convex polygon<sup>1)</sup> in the plane  $p+V$  and

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 1) It might seem that this statement is false for  $n=3$ . However in that case our hypothesis  $\psi \neq 0$  entails that  $1, \xi e^{it\psi}$  and  $\xi e^{-it\psi}$  are the only eigenvalues of  $\varphi_t$  (and hence of the adjoint transformation).

Therefore  $\{ x | \xi_1 + \xi_2 + \xi_3 = 0 \}$  is the only possible real invariant two-dimensional subspace and thus  $V = \{ x | \xi_1 + \xi_2 + \xi_3 = 0 \}$ , implying  $W \subset p+V$ , whence  $U=W$  and  $U_0=W_0$ .

$U_0$  its boundary the condition (4) is satisfied if and only if

$$(5) \quad \forall_t^{(0, \infty)} \varphi_t U_0 \subset U$$

is satisfied.

To derive a geometric condition equivalent with (5), consider a vector  $w \in U_0$ . The curve  $\{\varphi_t w | t > 0\}$  must, on account of (5), be in  $U$ . Hence a vector  $v_1$  tangent to  $\{\varphi_t w | t \geq 0\}$  in  $w$  must point from  $w$  to the interior of  $U$ . To make this condition more precise consider a vector  $v_2$  pointing from  $w$  along  $U_0$  in such a way that  $v_1$  lies inside the angle<sup>1)</sup> formed by  $p-w$  and  $v_2$ .

If now we define

$$(6) \quad \theta_w = \angle(p-w, v_1); \theta_{0,w} = \angle(p-w, v_2),$$

then it is easily seen that (5) is satisfied if and only if

$$(7) \quad \forall w \in U_0 \quad \theta_w \leq \theta_{0,w}$$

is true.

3 It does not seem possible to deduce useful results directly from (7), as  $\theta_w$  is in general a complicated expression depending on  $w$ . The only case in which  $\theta_w$  is independent of  $w$  obtains when  $|x_0| = |y_0|$  and the inner product  $(x_0, y_0) = 0$ , which will be shown below. The general case can be reduced to this case by means of the following considerations:

Let  $\sigma$  be a real non-singular linear transformation on  $R_n$ . Then the semi-group

$$G^* = \sigma G \sigma^{-1}$$

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 1) There is a slight difficulty here on account of the fact that  $p$  may be in  $U_0$ . This case does not occur if  $\psi \neq 0$ ,  $\xi \neq 0$ : On account of (3) any vector  $w-p$  performs a complete rotation around  $p$  in the time interval  $(0, 2\pi/\psi]$ . Hence (5) and  $p \in U_0$  are inconsistent.

consists of those linear transformations  $\varphi_t^*$  for which

$$\varphi_t^* = \sigma \varphi_t \sigma^{-1}.$$

These  $\varphi_t^*$  have the same eigenvalues as  $\varphi_t$  and eigenvectors

$$z^* = \sigma z, \text{ with } z^* = x^* + iy^* ; x^* = \sigma x; y^* = \sigma y,$$

where  $z=x+iy$  is an eigenvector of  $\varphi_t$ .

Application of the transformation  $\sigma$  to  $W, p, x_0, y_0, V, U, U_0, v_1, v_2$  yields  $W^*, p^*, x_0^*, y_0^*, V^*, U^*, U_0^*, v_1^*, v_2^*$ , i.e.  $W^* = \sigma W$ , etc.

We can now repeat the argument of 2 and arrive at equivalent conclusions. In particular, if we define

$$(6^*) \quad \theta_w^* = \rightarrow (p^* - w, v_1^*); \theta_{0,w}^* = \rightarrow (p^* - w, v_2^*),$$

then

$$(7^*) \quad \forall w \in U_0^* \quad \theta_w^* = \theta_{0,w}^*$$

is equivalent with (7).

In view of the remark at the beginning of this section, we choose  $\sigma$  in such a way that

$$(8) \quad |x_0^*| = |y_0^*| \quad \text{and} \quad (x_0^*, y_0^*) = 0.$$

Using (8) it is easy to calculate  $\theta_w^*$ . For an arbitrary  $w \in p^* + V^*$  we have

$$(9) \quad w = p^* + \alpha(x_0^* \sin \gamma + y_0^* \cos \gamma) \quad (\alpha, \gamma \text{ real}).$$

Consider  $\{\varphi_t w \mid t \geq 0\}$ . A vector tangent to this curve in  $w$  is given by

$$(10) \quad v_1^* = \lim_{h \downarrow 0} \frac{\varphi_h^* w - w}{h}.$$

Substitution of (9) and

$$(3^*) \quad \begin{cases} \varphi_t^* x_0^* = g^t(x_0^* \cos t \psi - y_0^* \sin t \psi) \\ \varphi_t^* y_0^* = g^t(x_0^* \sin t \psi + y_0^* \cos t \psi) \end{cases}$$

in (10) gives after some calculation

$$(11) \quad v_1^* = -\alpha A [x_0^* \sin(\gamma-\chi) + y_0^* \cos(\gamma-\chi)] ,$$

where

$$(12) \quad A^2 = \psi^2 + \log^2 \xi ; \quad 0 < \chi < \pi \quad \text{and} \quad \text{ctn } \chi = -\psi^{-1} \log \xi .$$

Then, using (8), (11) and (9), it follows that

$$(13) \quad \cos \theta_w^* = \frac{(p^* - w, v_1^*)}{|p^* - w| \cdot |v_1^*|} = \cos \chi ,$$

and hence (as  $\theta_w^*$  and  $\chi$  are both in  $[0, \pi]$  )

$$(14) \quad \theta_w^* = \chi .$$

We thus see that  $\theta_w^*$  is indeed independent of  $w$ . Our condition (7\*) yields

$$(15) \quad \chi \cong \inf_{w \in U_0^*} \theta_{0,w}^* .$$

4

As  $\sigma$  is non-singular,  $W^*$  is like  $W$  an  $(n-1)$ -simplex. Hence  $U_0^*$ , being the boundary of the intersection of  $W^*$  with a plane that has at least one point (namely  $p^*$ ) in common with  $W^*$ , is a convex polygon with at most  $n$  vertices, say  $a_1, a_2, \dots, a_n$ . It is then easy to see, that the infimum in the right-hand side of (15) is reached if  $w$  is in one of the vertices of  $U_0^*$ . Therefore, if we introduce <sup>1)</sup>

$$\left. \begin{aligned} \alpha_j &= \angle p^* a_j a_{j+1} \\ \alpha'_j &= \angle p^* a_j a_{j-1} \end{aligned} \right\} (j=1, 2, \dots, n; \text{ indices taken mod } n),$$

then

$$(16) \quad \inf_{w \in U_0^*} \theta_{0,w}^* = \min_j \alpha_j \quad \text{or} \quad \min_j \alpha'_j ,$$

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 1) For any three vectors  $a, b$  and  $c$  we use the following notation:  
 $\angle abc \stackrel{\text{df}}{=} \angle (a-b, c-b)$ .

depending on whether the sense of rotation in  $p^*+V^*$  given by  $p^*, w, v_1^*$  in that order is the same as the rotational sense given by  $a_1, a_2, \dots, a_n$  or that given by  $a_n, a_{n-1}, \dots, a_1$  respectively.

In 5 we shall give a proof of the fact that

$$(17) \quad \min_j \alpha_j \leq \frac{n-2}{2n} \pi \quad ; \quad \min_j \alpha'_j \leq \frac{n-2}{2n} \pi,$$

with equality in at least one of these two relations if and only if the polygon under consideration is regular and  $p^*$  its centre.

Combination of (15), (16) and (17) yields

$$\chi \leq \frac{n-2}{2n} \pi,$$

from which it follows that

$$(18) \quad \varrho \leq e^{-\psi} \operatorname{tg} \pi/n.$$

Now  $\psi$  is of the form  $\psi'_0 + 2k\pi$  with  $0 \leq \psi'_0 < 2\pi$  and  $k$  an integer. As  $\bar{\lambda}$  is also an eigenvalue of  $\varphi_1$ , we may conclude

$$(19) \quad \exists_k \quad \varrho \leq e^{-(\psi'_0 + 2k\pi) \operatorname{tg} \pi/n} \quad \text{and} \quad \varrho = e^{-(-\psi'_0 - 2k\pi) \operatorname{tg} \pi/n}.$$

If we put

$$\psi_0 = \min \{ \psi'_0, 2\pi - \psi'_0 \},$$

than either  $\psi'_0 + 2k\pi \geq \psi_0$  or  $-\psi'_0 - 2k\pi \geq \psi_0$ , hence

$$(20) \quad \varrho \leq e^{-\psi_0 \operatorname{tg} \pi/n}.$$

If  $\psi=0$ , then  $\psi_0=0$  also and as  $\varrho \leq 1$  always, (20) is trivially satisfied in this case. The restriction  $\varrho \neq 0$  can now be removed also, as again (20) is satisfied in this case. Our final result is therefore: if  $\lambda$  is an eigenvalue of  $\varphi_1$ ,  $\varrho$  its absolute value,  $\psi_0$  the minimum value of the non-negative arguments of  $\lambda$  and  $\bar{\lambda}$ , then (20) must be satisfied. This result is equivalent to (6) in [1].

5 It remains to prove (17). To this purpose suppose, with the notation of section 4

$$(21) \quad \forall_j \alpha_j \geq \frac{n-2}{2n} \pi .$$

Then it is possible to find vectors <sup>1)</sup>  $p_1 \in \langle p^*, a_1 \rangle, p_2 \in \langle p^*, a_2 \rangle, \dots, p_n \in \langle p^*, a_n \rangle$  in such a way that

$$\sphericalangle p^* a_n p_1 = \sphericalangle p^* p_1 p_2 = \dots = \sphericalangle p^* p_{n-1} p_n = \frac{n-2}{2n} \pi .$$

Then, as  $p_n \in \langle p^* a_n \rangle$

$$\frac{|p_n - p^*|}{|a_n - p^*|} \leq 1,$$

with equality if and only if  $\forall_j p_j = a_j$ . Introducing  $\alpha_j'' = \sphericalangle p^* p_j p_{j-1}$  ( $j=2, \dots, n$ ) and  $\alpha_1'' = \sphericalangle p^* p_1 a_n$ , we find

$$\begin{aligned} 1 &\geq \frac{|p_n - p^*|}{|a_n - p^*|} = \frac{|p_n - p^*|}{|p_{n-1} - p^*|} \cdot \frac{|p_{n-1} - p^*|}{|p_{n-2} - p^*|} \cdots \frac{|p_1 - p^*|}{|a_n - p^*|} = \\ &= \prod_{j=1}^n \frac{\sin \frac{n-2}{2n} \pi}{\sin \alpha_j''} . \end{aligned}$$

Hence

$$(22) \quad \prod_{j=1}^n \sin \alpha_j'' \geq \sin^n \frac{n-2}{2n} \pi ,$$

with equality if and only if  $\forall_j \alpha_j'' = \alpha_j'$ .

On the other hand, as the sum of the angles ( $n\pi$ ) of the  $n$  triangles  $\langle a_n, p_1, p^* \rangle, \langle p_1, p_2, p^* \rangle, \dots, \langle p_{n-1}, p_n, p^* \rangle$  is equal to  $\sum_{j=1}^n \alpha_j'' + 2\pi + n \cdot \frac{n-2}{2n} \pi$ , we have

$$\sum_{j=1}^n \alpha_j'' = n \cdot \pi - 2\pi - \sum_{j=1}^n \frac{n-2}{2n} \pi = \frac{n-2}{2} \pi .$$

As  $-\log \sin x$  is a convex function for  $0 < x < \pi$ , we have

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1) We denote by  $\langle b_1, b_2, \dots, b_k \rangle$  the closed convex hull of  $\{b_1, \dots, b_k\}$ :  $\langle b_1, b_2, \dots, b_k \rangle = \{x \mid x = \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_k b_k;$

$$\forall_j \alpha_j \geq 0; \sum_{j=1}^k \alpha_j = 1\} .$$

$$(23) \quad \prod_{j=1}^n \sin \alpha_j'' \leq \sin^n \frac{1}{n} \sum_{j=1}^n \alpha_j'' = \sin^n \frac{n-2}{2n} \pi ,$$

with equality if and only if  $\forall_j \alpha_j'' = \frac{n-2}{2n} \pi$ .

Combination of (22) and (23) yields:

$$(24) \quad \forall_j \alpha_j' = \alpha_j'' = \frac{n-2}{2n} \pi .$$

We can now repeat this argument, starting from

$$(25) \quad \forall_j \alpha_j' \geq \frac{n-2}{2n} \pi ,$$

which follows from (24). We then arrive at the conclusion

$$(26) \quad \forall_j \alpha_j = \frac{n-2}{2} \pi .$$

Hence, either of (21) or (25) entails (24) and (26).  
In other words: Either

$$\forall_j \alpha_j = \alpha_j' = \frac{n-2}{2n} \pi ,$$

in which case  $\langle a_1, a_2, \dots, a_n \rangle$  is regular with centre  $p^*$ ,  
or

$$\min_j \alpha_j < \frac{n-2}{2n} \pi \quad \text{and} \quad \min_j \alpha_j' < \frac{n-2}{2n} \pi ,$$

which proves (17).



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