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Two weak-order relations for distribution functions

Preliminary report

by

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1. SUMMARY

For the class $\mathcal{F}$ of distribution functions satisfying

(i) For $0 < F(x) < 1$, $F(x)$ is strictly increasing,

(ii) For $0 < F(x) < 1$, $F(x)$ is twice differentiable with continuous second derivative,

(iii) $F(x)$ possesses a finite absolute first moment,

the following order relation is defined: $F(x) \prec F^*(x)$ if $G^*F(x)$ is convex on the interval where $0 < F(x) < 1$. Here $G^*(y)$ denotes the inverse of $F^*(x)$. The ordering is independent of location and scale parameters.
Let $E\,\underline{x}_{i:n}$ and $E\,\underline{x}^*_{i:n}$ denote the expectation of the i-th order statistic of a sample of size n from $F(x)$ and $F^*(x)$ respectively. It is shown that if $F(x) \prec F^*(x)$, then $F(E\,\underline{x}_{i:n}) \leq F^*(E\,\underline{x}^*_{i:n})$ for all i and n. The converse is proved if the inequalities hold for sufficiently large n.
For the subclass $\mathcal{S} \subset \mathcal{F}$ of symmetric distributions a different order relation is defined: $F(x) \underset{s}{\prec} F^*(x)$ if $G^*F(x)$ is convex on the interval where $\frac{1}{2} < F(x) < 1$. Here again the ordering is independent of location and scale parameters. The same inequalities hold in this case for $i \geq \frac{n+1}{2}$, whereas the theorem may be reversed in the manner mentioned above.
Examples of both order relations and inequalities for expected values of order statistics are given.

2. A WEAK ORDERING AND AN EQUIVALENCE FOR A CLASS OF DISTRIBUTIONS.

We shall consider the class $\mathcal{F}$ of all probability distribution functions $F(x)$ on R_1 satisfying:

(2.1) For $0 < F(x) < 1$, $F(x)$ is strictly increasing,

(2.2) For $0 < F(x) < 1$, $F(x)$ is twice differentiable with continuous second derivative $F''(x)$,

(2.3) $F(x)$ possesses a finite absolute first moment

$$E|\underline{x}| = \int_{-\infty}^{+\infty} |x| dF(x).$$

Conditions (2.1) and (2.2) imply that $F(x)$ possesses a twice differentiable, strictly increasing, inverse function $G(y)$ with continuous second derivative, uniquely defined for $0 < y < 1$ by $G\,F(x)=x$. We shall denote distribution functions belonging to $\mathcal{F}$ by $F(x)$, $F^*(x)$,....., the corresponding (finite or infinite) open intervals where they increase strictly by I, I^*,....., their inverse functions by $G(y)$, $G^*(y)$,....., and random variables possessing these distributions by $\underline{x}$, $\underline{x}^*$,..... .[1)]
If $F(x) \in \mathcal{F}$ and $F^*(x) \in \mathcal{F}$ then $G^*F(x)$ is also uniquely defined on I where the function is strictly increasing and twice differentiable with continuous second derivative. We shall say that $G^*(y)$ is convex in $G(y)$ for $0 < y_1 < y < y_2 < 1$ if $G^*F(x)$ is convex for $G(y_1) < x < G(y_2)$, or equivalently, if $G\,F^*(x)$ is concave for $G^*(y_1) < x < G^*(y_2)$. Throughout this report we shall use the concepts of convexity and concavity in the weak sense thus referring to non-negative and non-positive second derivative respectively.
In the first part of this report we shall be concerned with the following order relation on $\mathcal{F}$.

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1) We distinguish random variables from algebraic variables and numbers by underlining their symbols.

DEFINITION 1. If $F(x) \in \mathcal{F}$ and $F^*(x) \in \mathcal{F}$, then $F(x) \prec F^*(x)$ (or equivalently $F^*(x) \succ F(x)$) if and only if $G^*(y)$ is convex in $G(y)$ for $0 < y < 1$.

We shall say in this case that $F(x)$ precedes $F^*(x)$ or that $F^*(x)$ follows $F(x)$ and that the two are comparable. Clearly $F(x) \prec F(x)$ for all $F(x) \in \mathcal{F}$; since an increasing convex function of a convex function is again convex, $F(x) \prec F^*(x) \prec F^{**}(x)$ yields $F(x) \prec F^{**}(x)$ for $F(x)$, $F^*(x)$ and $F^{**}(x) \in \mathcal{F}$. The relation $\prec$ is thus a weak ordering on $\mathcal{F}$. Hence by defining an equivalence relation $\sim$ by

DEFINITION 2. If $F(x) \in \mathcal{F}$ and $F^*(x) \in \mathcal{F}$, then $F(x) \sim F^*(x)$ if and only if $F(x) \prec F^*(x)$ and $F^*(x) \prec F(x)$,

and passing to the collection $\mathcal{F}'$ of equivalence classes we may define a partial ordering on $\mathcal{F}'$ by ordering equivalence classes according to the ordering of their representatives. The structure of the equivalence classes is given by

THEOREM 1. $F(x) \sim F^*(x)$ if and only if $F^*(x)=F(ax+b)$ for some constants $a > 0$ and b.

PROOF. $F(x) \sim F^*(x)$ if and only if $G^*F(x)$ is convex as well as concave on I and hence linear. Since it is also strictly increasing on I the result of the theorem follows.

In statistical parlance theorem 1 asserts that the ordering is independent of location and scale parameters and that we may consequently restrict our attention to a comparison of standardized distribution functions.

3. A NECESSARY AND SUFFICIENT CONDITION FOR ORDERING

To establish the significance of this weak ordering in statistical terms we shall need a well known result on convex functions and some equally well known properties of order statistics. Moreover we shall have to prove the latter properties for a slightly more general class of random variables than order statistics only.
The result on convex functions is the celebrated JENSEN inequality [1].

LEMMA 1. Let $\underline{x}$ be a real valued random variable assuming values in a (finite or infinite) interval I with probability 1, and let $\varphi(x)$ be a real valued continuous convex function on I. Then,

$$\varphi(E\,\underline{x}) \leq E\,\varphi(\underline{x}) \;,$$

provided both expectations exist.

PROOF. Let L(x) be a line of support of $\varphi(x)$ through the point $(E\,\underline{x},\ \varphi(E\,\underline{x}))$. Since $L(x) \leq \varphi(x)$ on I and L(x) is linear

$$E\,\varphi(\underline{x}) \geq E\,L(\underline{x}) = L(E\,\underline{x}) = \varphi(E\,\underline{x}).$$

We define the following extension of the concept of an order statistic.

DEFINITION 3. A random variable $\underline{x}_{\lambda:n}$ with distribution function $H_{\lambda:n}(x)$ satisfying

$$d\,H_{\lambda:n}(x) = \frac{\Gamma(n+1)}{\Gamma(\lambda).\Gamma(n+1-\lambda)}\;F(x)^{\lambda-1}\;[1-F(x)]^{n-\lambda}dF(x),$$

where n=1,2,... and λ is any real number $1 \leq \lambda \leq n$ will be called a generalized order statistic from the distribution F(x).

Clearly the i-th order statistic $\underline{x}_{i:n}$ of a random sample of

size n ($\underline{x}_{1:n} \leqslant \underline{x}_{2:n} \leqslant \ldots \leqslant \underline{x}_{n:n}$) from the distribution F(x) satisfies the above definition for integer values of λ . We do not claim that for non-integer values of λ these generalized order statistics have any statistical meaning whatsoever and they are merely introduced to facilitate the proof of theorem 2.
If $F(x) \in \mathcal{F}$ condition (2.3) asserts that $E|x_{\lambda:n}| < \infty$ for all n=1,2,... and $1 \leqslant \lambda \leqslant n$. If G(y) and I are defined as in section 2

$$E\, \underline{x}_{\lambda:n} = \int_I x\, d\, H_{\lambda:n}(x) = \int_0^1 G(y) b(y;\lambda,n+1-\lambda) dy,$$

where $b(y;\lambda,n+1-\lambda)$ denotes the density function of the beta-distribution with parameters λ and $n+1-\lambda$. Concerning $E\, \underline{x}_{\lambda:n}$ the following properties will be needed in the sequel.

LEMMA 2. For fixed n and $F(x) \in \mathcal{F}$, $E\, \underline{x}_{\lambda:n}$ is a continuous and strictly increasing function of λ for $1 \leqslant \lambda \leqslant n$.

PROOF. For $1 < \lambda < n$, $b(y;\lambda,n+1-\lambda)$ is a continuous function of λ , uniformly for $0 \leqslant y \leqslant 1$. Since, by (2.3),

$$\int_0^1 |G(y)|\, dy = \int_I |x|\, dF(x) < \infty ,$$

$E\, \underline{x}_{\lambda:n}$ is a continuous function of λ for $1 < \lambda < n$. For $\lambda = 1$ (or n), $b(y;\lambda,n+1-\lambda)$ is continuous to the right (c.q. left) in λ, uniformly in y as long as y is bounded away from 0 (c.q. 1). Since the function remains bounded if y tends to 0 (c.q. 1) this suffices to prove continuity of $E\, \underline{x}_{\lambda:n}$ to the right (c.q.left) also for $\lambda=1$ (c.q. n).
Strict monotonicity is proved by noting that for fixed λ ($1 \leqslant \lambda \leqslant n$)

$$\frac{d}{d\lambda} b(y;\lambda,n+1-\lambda) = \left[-\frac{\Gamma'(\lambda)}{\Gamma(\lambda)} + \frac{\Gamma'(n+1-\lambda)}{\Gamma(n+1-\lambda)} + \log y - \log(1-y) \right] b(y;\lambda,n+1-\lambda)$$

$$> 0 \quad \text{for } y > y_\lambda$$

$$< 0 \quad \text{for } y < y_\lambda$$

for some $0 < y_\lambda < 1$, since the expression within brackets increases strictly from $-\infty$ to $+\infty$ for $0 < y < 1$. As $G(y)$ is strictly increasing $E\,\underline{x}_{\lambda:n}$ increases strictly for $1 \leq \lambda \leq n$.

LEMMA 3. If $F(x) \in \mathcal{F}$ and $x_0 \in I$ then there exists an integer $N \geq 1$ and a unique sequence of real numbers λ_j satisfying $1 \leq \lambda_j \leq N+j$, $E\,\underline{x}_{\lambda_j:N+j} = x_0$ for $j=0,1,2,\ldots$, and $\lim_{j\to\infty} \frac{\lambda_j}{N+j} = y_0 = F(x_0)$.

PROOF. To prove the lemma we shall make use of a result due to HOEFFDING [2] stating that if $F(x) \in \mathcal{F}$ and i_n is a sequence of integers satisfying $1 \leq i_n \leq n$ and $\lim_{n\to\infty} i_n/n = y_0 = F(x_0)$, then $\lim_{n\to\infty} E\,\underline{x}_{i_n:n} = x_0$. In fact Hoeffding proved this theorem for a far more general class of distribution functions but we shall restrict ourselves to the class $\mathcal{F}$.
In the first place this theorem asserts that $E\,\underline{x}_{1:n}$ and $E\,\underline{x}_{n:n}$ converge towards the end-points of the open interval I. As $E\,\underline{x}_{\lambda:n}$ is a continuous and strictly increasing function of λ, this establishes the existence of an integer N and a uniquely defined sequence $1 \leq \lambda_j \leq n$ with $E\,\underline{x}_{\lambda_j:N+j} = x_0$, $j=0,1,2,\ldots$, for any point $x_0 \in I$.
Now if $\lambda_j/N+j$ would not converge towards $y_0 = F(x_0)$, a subsequence $\lambda_{j_k}/N+j_k$ would exist converging towards some value $y_0' = F(x_0') \neq F(x_0)$, since the sequence is bounded. As a result the sequences $[\lambda_{j_k}]/N+j_k$ and $[\lambda_{j_k}]+1/N+j_k$ would also converge towards y_0'. (Here $[\lambda]$ denotes the largest integer $\leq \lambda$). Applying Hoeffding's result for the second time and using the monotonicity of $E\,\underline{x}_{\lambda:n}$ we should find

$$x_0' = \lim_{k\to\infty} E\,\underline{x}_{[\lambda_{j_k}]:N+j_k} \leq \lim_{k\to\infty} E\,\underline{x}_{\lambda_{j_k}:N+j_k} \leq \lim_{k\to\infty} E\,\underline{x}_{[\lambda_{j_k}]+1:N+j_k} = x_0'$$

This contradicts $E\,\underline{x}_{\lambda_j:N+j} = x_0$, which proves the lemma.
We are now in a position to prove

THEOREM 2. If $\underline{x}_{\lambda:n}$ and $\underline{x}^*_{\lambda:n}$ denote generalized order statistics from the distributions $F(x)\in\mathcal{F}$ and $F^*(x)\in\mathcal{F}$ respectively, then the following statements are equivalent:

(i) $F(x) \prec F^*(x)$

(ii) $F(E\,\underline{x}_{\lambda:n}) \leq F^*(E\,\underline{x}^*_{\lambda:n})$ for all $n=1,2,\dots$ and $1\leq\lambda\leq n$

(iii) $F(E\,\underline{x}_{i:n}) \leq F^*(E\,\underline{x}^*_{i:n})$ for all $n=1,2,\dots$ and $i=1,2,\dots,n$

(iiii) $F(E\,\underline{x}_{i:n}) \leq F^*(E\,\underline{x}_{i:n})$ for all $n=M,M+1,\dots$ and $i=1,2,\dots,n$,

where M denotes an arbitrary integer.

We note that (ii), (iii) and (iiii) are independent of location and scale parameters as indeed they should be (cf section 2).

PROOF. By substitution we find

$$E\,\underline{x}^*_{\lambda:n} = \int_{I^*} x\, d\,H^*_{\lambda:n}(x) = \int_0^1 G^*(y)\, b(y;\lambda,n+1-\lambda)dy =$$

$$= \int_I G^*F(x)\, d\,H_{\lambda:n}(x) = E\,G^*F(\underline{x}_{\lambda:n}) ,$$

where $H_{\lambda:n}(x)$ and $H^*_{\lambda:n}(x)$ denote the distribution functions of $\underline{x}_{\lambda:n}$ and $\underline{x}^*_{\lambda:n}$ (cf definition 3).
If $F(x) \prec F^*(x)$, or $G^*F(x)$ is convex on I, application of lemma 1 to the random variable $\underline{x}_{\lambda:n}$ and the function $G^*F(x)$ gives

$$G^*F(E\,\underline{x}_{\lambda:n}) \leq E\,G^*F(\underline{x}_{\lambda:n}) = E\,\underline{x}^*_{\lambda:n} \quad , \text{ or}$$

$$F(E\,\underline{x}_{\lambda:n}) \leq F^*(E\,\underline{x}^*_{\lambda:n}),$$

which proves (i)$\Longrightarrow$(ii). As (ii)$\longrightarrow$(iii)$\longrightarrow$(iiii) is trivially true it remains to be proved that (iiii) yields (i).
Suppose that (i) is false and hence that $\varphi(x)=G^*F(x)$ is not convex on I. Since $\varphi(x)$ is twice differentiable on I and $\varphi''(x)$ is continuous (cf.section 2) there exist an open interval J_1 and a closed interval K, such that $J_1\subset K\subset I$ and $\varphi''(x) < 0$ on J_1. We consider an arbitrary point $\xi\in J_1$ and

denote the tangent to $\varphi(x)$ at $x=\xi$ by $L(x,\xi)$. Then $\psi(x,\xi)=L(x,\xi)-\varphi(x)>0$ for $x\in J_1$, $x\neq\xi$, and $\psi(\xi,\xi)=0$. According to lemma 3 we may choose an integer N and a sequence $1<\lambda_j(\xi)<N+j$ in such a way that $E\,\underline{x}_{\lambda_j(\xi):N+j}=\xi$ for $j=0,1,2,\ldots$, and $\lim_{j\to\infty}\frac{\lambda_j(\xi)}{N+j}=\eta=F(\xi)$.
We note that N may be chosen independent of $\xi\in J_1$ as J_1 is bounded away from the endpoints of the open interval I by K. Now

$$E\ \psi(\underline{x}_{\lambda_j(\xi):N+j},\xi)=$$

$$=\frac{\Gamma(N+j+1)}{\Gamma(\lambda_j(\xi))\Gamma(N+j+1-\lambda_j(\xi))}\int_I \psi(x,\xi)\,F^{\lambda_j(\xi)-1}(x)(1-F(x))^{N+j-\lambda_j(\xi)}dF(x)=$$

$$=C_j(\xi)\int_I \psi(x,\xi)\,f_j^{N+j-1}(x,\xi)\,d\,F(x)\ ,$$

where $C_j(\xi)>0$ and $f_j(x,\xi)=F^{\mu_j(\xi)}(x)\ (1-F(x))^{1-\mu_j(\xi)}$

with $\mu_j(\xi)=\frac{\lambda_j(\xi)-1}{N+j-1}$, and hence $\lim_{j\to\infty}\mu_j(\xi)=\eta=F(\xi)$.

Furthermore, since $E\,\underline{x}_{\lambda:n}$ is a continuous and strictly increasing function of λ, $\lambda_j(\xi)$ and $\mu_j(\xi)$ are continuous and strictly increasing functions of ξ. Hence $f_j(x,\xi)$ is continuous in x and ξ and so of course is

$$f(x,\xi)=\lim_{j\to\infty} f_j(x,\xi)=F^{\eta}(x)\ (1-F(x))^{1-\eta}\ .$$

For fixed $\xi\in J_1$, ξ_0 say, $f(x,\xi_0)$ possesses a single maximum at $x=\xi_0$. Therefore we can find a non-degenerate open interval $J_2\subset J_1$, with $\xi_0\in J_2$, and constants $A>0$ and $\delta>0$ satisfying

$$\max_{x\notin J_1} f(x,\xi_0)<A<A+\delta<\inf_{x\in J_2} f(x,\xi_0).$$

Since $\max_{x\notin J_1} f(x,\xi)$ and $\inf_{x\in J_2} f(x,\xi)$ are continuous functions of $\xi\in J_2$ these inequalities will continue to hold for some interval $\xi_1<\xi<\xi_2$, $\xi_1,\xi_2\in J_2$, or, which is exactly the same,

$$\max_{x \notin J_1} f_j(x,\xi) < A < A+\delta < \inf_{x \in J_2} f_j(x,\xi)$$

will hold for $\eta_1 = F(\xi_1) < \mu_j(\xi) < F(\xi_2) = \eta_2$, $j=0,1,2,\ldots$.
Now for $j \longrightarrow \infty$ this interval tends to $\xi_1 < \xi < \xi_2$, so we can find an open interval $J_3 \subset (\xi_1, \xi_2) \subset J_2$ such that the inequalities hold for $\xi \in J_3$ and sufficiently large $j (\geq M_1)$.
Thus for $j \geq M_1$ and $\xi \in J_3$

$$E \ \psi(\underline{x}_{\lambda_j(\xi):N+j}, \xi) >$$

$$> C_j(\xi) \left\{ \int_{J_2} \psi(x,\xi) \ f_j^{N+j-1}(x,\xi) dF(x) + \int_{I-J_1} \chi(x,\xi) \ f_j^{N+j-1}(x,\xi) dF(x) \right\}$$

$$> C_j(\xi) \left\{ (A+\delta)^{N+j-1} \int_{J_2} \psi(x,\xi) dF(x) + A^{N+j-1} \int_{I-J_1} \chi(x,\xi) dF(x) \right\}$$

$$= C_j(\xi) . \ A^{N+j-1} \left\{ (1+\frac{\delta}{A})^{N+j-1} \ B(\xi) - D(\xi) \right\} ,$$

where $\chi(x,\xi) = \min \left\{ \psi(x,\xi), \ 0 \right\} \leq 0$,

$$B(\xi) = \int_{J_2} \psi(x,\xi) d \ F(x) > 0, \text{ and}$$

$$D(\xi) = -\int_{I-J_1} \chi(x,\xi) d \ F(x) \geq 0.$$

Since $\psi(x,\xi)$ and $\chi(x,\xi)$ an continuous in x and ξ and the latter integral converges uniformly for all $\xi \in J_3$, $B(\xi)$ and $D(\xi)$ are continuous on J_3. Hence for some non-degenerate open interval $J_4 \subset J_3$

$$\inf_{\xi \in J_4} B(\xi) = B > 0, \quad \text{and}$$

$$0 \leq \sup_{\xi \in J_4} D(\xi) \leq D \quad .$$

For $\zeta \in J_4$ and $j \geq M_1$

$$E\ \psi(\underline{x}_{\lambda_j(\zeta):N+j}, \zeta) > C_j(\zeta).\ A^{N+j-1}\left\{ B.\left(1+\frac{\delta}{A}\right)^{N+j-1} - D\right\},$$

so for sufficiently large $j \geq M_2 \geq M_1$ and for all $\zeta \in J_4$

$$E\ \psi(\underline{x}_{\lambda_j(\zeta):N+j}, \zeta) > 0.$$

Now $\lambda_j(\zeta)$ being continuous in ζ maps J_4 on an interval L_j. As $\lim_{j\to\infty} \frac{\lambda_j(\zeta)}{N+j} = \eta = F(\zeta)$ the length of L_j tends to infinity for $j \to \infty$, and as a consequence L_j contains an integer $i_j = \lambda_j(\zeta)$, $\zeta \in J_4$, for sufficiently large $j \geq M_3 \geq M_2$. Hence

$$E\ \psi(\underline{x}_{i_j:N+j}, \zeta) = \varphi(E\ \underline{x}_{i_j:N+j}) - E\ \varphi(\underline{x}_{i_j:N+j}) > 0, \text{ or}$$

$$F(E\ \underline{x}_{i_j:N+j}) > F^*(E\ \underline{x}^*_{i_j:N+j})$$

for all $j \geq M_3$ and at least one integer i_j, $1 \leq i_j \leq N+j$. This contradicts (iiii) which completes the proof.

Theorem 2 presents two equivalent approaches to the problem of finding inequalities for expected values or order statistics by comparison with distributions for which these quantities are either analytically tractable or numerically known. The equivalence of (i) and (iii) permits an approach by means of a convexity proof whereas the equivalence of (iiii) and (iii) enables us to start from known asymptotic results.

We conclude this section with two remarks. The first one is simply that as $F(x) < F^*(x)$ implies $F(E\ \underline{x}) \leq F^*(E\ \underline{x}^*)$ we may, roughly speaking, expect distributions following on one another to show a tendency for increasing skewness to the left c.q. decreasing skewness to the right. The second remark concerns conditions (2.2) and (2.3). We note that condition (2.2) has only been fully exploited to prove (iiii) $\Longrightarrow$ (i) in theorem 2. For the remainder of the text continuity of $F(x)$ would have been sufficient. Condition (2.3) might have been relaxed by replacing it by the condition $E|\underline{x}_{i_o:n_o}| < \infty$ for some integers $1 \leq i_o \leq n_o$, and adding "if both expectations exist" to statements (ii), (iii) and (iiii) of theorem 2. The proof requires only minor changes. We shall make use of this in example 4.2.

4. SOME EXAMPLES OF ORDERING.

In this section we shall give three examples of the order relation considered in the preceding sections, ranging from the almost trivial case of comparison with the rectangular distribution to the more intricate problem of mutual comparison of gamma distributions. Especially the first two examples are meant to provide simple illustrations of the theory rather than sharp inequalities for use in specific cases.

4.1. Comparison with the rectangular distribution.

We take $F^*(x)=x$, $0 \leq x \leq 1$, or $G^*(y)=y$. Since $E\,\underline{x}^*_{i:n} = \frac{i}{n+1}$ application of theorem 2 gives a result mentioned by BLOM ([3],p.68):

If the density function $F'(x)$ is non-decreasing ($F(x)$ convex), then $F(E\,\underline{x}_{i:n}) \leq \frac{i}{n+1}$ for all $n=1,2,...$ and $i=1,2,..,n$; if the density function $F'(x)$ is non-increasing ($F(x)$ concave) the inequalities are reversed.

4.2. Comparison with $F^*(x) = -\frac{1}{x}$ and $F^*(x) = \frac{x-1}{x}$.

For $F^*(x) = -\frac{1}{x}$, $-\infty < x \leq -1$, or $G^*(y) = -\frac{1}{y}$, we find for $i \geq 2$ $E\,\underline{x}^*_{i:n} = -\frac{n}{i-1}$ and $F^*(E\,\underline{x}^*_{i:n}) = \frac{i-1}{n}$. Although $E|\underline{x}^*|$ is not finite we may apply theorem 2 (cf. the remark at the end of section 3) to obtain for all $n=2,3,...$ and $i=2,3,...,n$:

If $\frac{1}{F(x)}$ is concave on I ($-\frac{1}{F(x)}$ convex) then $F(E\,\underline{x}_{i:n}) \leq \frac{i-1}{n}$; if $\frac{1}{F(x)}$ is convex on I the inequalities are reversed.

For $F^*(x) = \frac{x-1}{x}$, $1 \leq x < \infty$, or $G^*(y) = \frac{1}{1-y}$, we find for $i \leq n-1$ $E\,\underline{x}^*_{i:n} = \frac{n}{n-i}$ and $F^*(E\,x^*_{i:n}) = \frac{i}{n}$. Applying theorem 2 we obtain for n=2,3,... and i=1,2,...,n-1 :
If $-\frac{1}{1-F(x)}$ is convex on I then $F(E\,\underline{x}_{i:n}) \leq \frac{i}{n}$; if $\frac{1}{1-F(x)}$ is concave on I the inequalities are reversed.

Combining the results of 4.1 and 4.2 we may set up crude bounds for the expected values of order statistics in terms of the distribution quantiles for many distribution functions, for instance

A. Gamma distributions: $F'(x) = \frac{1}{\Gamma(\sigma)} e^{-x} x^{\sigma-1}$, $\sigma > 0$, $0 \leq x < \infty$.
For $\sigma \leq 1$, $F'(x)$ is non-increasing and 4.1 is applicable. Furthermore one easily shows by repeated differentiation that $\frac{1}{F(x)}$ is convex for all values of σ, and $\frac{1}{1-F(x)}$ is convex for $\sigma \geq 1$. Summarizing we obtain

$\sigma > 1$ $\qquad \frac{i-1}{n} \leq F(E\,\underline{x}_{i:n}) \leq \frac{i}{n}$

$\sigma = 1$ $\qquad (\frac{i-1}{n} <)\frac{i}{n+1} \leq F(E\,\underline{x}_{i:n}) \leq \frac{i}{n}$

$\sigma < 1$ $\qquad (\frac{i-1}{n} <)\frac{i}{n+1} \leq F(E\,\underline{x}_{i:n})$.

B. Beta distributions: $F'(x) = \frac{1}{B(\sigma,\tau)} x^{\sigma-1} (1-x)^{\tau-1}$, $\sigma > 0$, $\tau > 0$, $0 \leq x \leq 1$.

$F'(x)$ is non-decreasing for $\sigma \geq 1$, $\tau \leq 1$, and non-increasing for $\sigma \leq 1$, $\tau \geq 1$. Repeated differentiation shows that $\frac{1}{F(x)}$ is convex for $\tau \geq 1$ and $\frac{1}{1-F(x)}$ is convex for $\sigma \geq 1$. Hence we obtain .

$\sigma > 1,\ \tau > 1$ $\qquad \frac{i-1}{n} \leq F(E\ \underline{x}_{i:n}) \leq \frac{i}{n}$

$\sigma > 1,\ \tau = 1$ $\qquad \frac{i-1}{n} \leq F(E\ \underline{x}_{i:n}) \leq \frac{i}{n+1} (< \frac{i}{n})$

$\sigma = 1,\ \tau > 1$ $\qquad (\frac{i-1}{n} <) \frac{i}{n+1} \leq F(E\ \underline{x}_{i:n}) \leq \frac{i}{n}$

$\sigma \geq 1,\ \tau < 1$ $\qquad F(E\ \underline{x}_{i:n}) \leq \frac{i}{n+1} (< \frac{i}{n})$

$\sigma < 1,\ \tau \geq 1$ $\qquad (\frac{i-1}{n} <) \frac{i}{n+1} \leq F(E\ \underline{x}_{i:n})$.

The case $\sigma = \tau = 1$ is trivial and the case $\sigma < 1,\ \tau < 1$ is not covered by 4.1 or 4.2.

C. Normal distribution: $F'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, $-\infty < x < \infty$.

Here $\frac{1}{F(x)}$ and $\frac{1}{1-F(x)}$ are both convex, so we find

$$\frac{i-1}{n} \leq F(E\ \underline{x}_{i:n}) \leq \frac{i}{n} \quad ,$$

corresponding to $\sigma \longrightarrow \infty$ and $\sigma, \tau \longrightarrow \infty$ in cases A and B.
We note that in all three cases the bounds $\frac{i-1}{n}$ and $\frac{i}{n}$ derived from 4.2 hold trivially for i=1 and i=n respectively.

4.3. The maximal chain of gamma distributions.

In a partially ordered set we define a chain to be a subset in which any two elements are comparable. A maximal chain is a chain which is not included, in the strict sense, in any other chain. We recall KURATOWSKI's lemma stating that any partially ordered set contains at least one maximal chain.
If we start looking for a chain in the partially ordered class $\mathcal{F}'$ of standardized distribution functions (cf. section 2) and keep in mind that the ordering is related, in a sense, to the skewness of the distributions (cf. section 3), a plausible candidate seems to be the class of gamma distributions $F_\sigma(x) = \frac{1}{\Gamma(\sigma)} \int_0^x e^{-t}\, t^{\sigma-1}\, dt$.

We shall first sketch a proof that $F_\tau(x) < F_\sigma(x)$ for $0 < \sigma < \tau$,

i.e. the gamma distributions follow one another with decreasing values of the parameter.
This means we have to prove that $\varphi(x)=G_\tau F_\sigma(x)$, $0<\sigma<\tau$, is concave for $0<x<\infty$, where $G_\tau(y)$ denotes the inverse of $F_\tau(x)$. The rather forbidding appearance of $\varphi(x)$ leads to the following indirect approach.
Consider the function $\psi(x)=F_\sigma(x)-F_\tau(b(x+a))$, $b>0$, $a\geqslant 0$, and $0\leqslant x<\infty$. As $F_\sigma(x)-F_\tau(\varphi(x))\equiv 0$, and $F_\tau(x)$ is strictly increasing, $\psi(x)$ has the same sign as $\varphi(x)-b(x+a)$ for all $x\geqslant 0$. Also $\psi'(x)=F'_\sigma(x)-b\,F'_\tau(b(x+a))$ has the same sign as $\chi(x)=\log F'_\sigma(x)-\log F'_\tau(b(x+a))-\log b$, and
$\chi'(x)=(b-1)+\frac{\sigma-1}{x}-\frac{\tau-1}{x+a}$.
A detailed study of the sign of $\chi'(x)$ for $x\geqslant 0$ and different values of a,b,σ and τ, and of the signs of $\chi(x)$ and $\psi(x)$ for $x=0$ and $x\longrightarrow\infty$ reveals that $\psi(x)$, and hence $\varphi(x)-b(x+a)$, can have at most two distinct zeros and is positive between these zeros. For $b>0$, $a<0$ a comparison with the case $b>0$, $a=0$ shows that $\varphi(x)-b(x+a)$ can have at most one zero, whereas for $b\leqslant 0$ the same holds since $\varphi(x)$ is strictly increasing. Thus the graph of $\varphi(x)$ lies above any chord which proves concavity of $\varphi(x)$.
To construct a maximal chain we add the normal distribution $F_\infty(x)$ and the class of distribution functions $F_{-\sigma}(x)=1-F_\sigma(-x)$, $\sigma>0$, $-\infty<x\leqslant 0$, to the family of gamma distributions $F_\sigma(x)$, $\sigma>0$.
Now $G_{-\tau}\,F_{-\sigma}(x)=G_{-\tau}(1-F_\sigma(-x))=-G_\tau F_\sigma(-x)$ is convex for $0<\sigma<\tau$, $x\leqslant 0$, so $F_{-\sigma}(x)\prec F_{-\tau}(x)$ for $0<\sigma<\tau$. Also $F_\infty(x)\prec F_\sigma(x)$ for all $\sigma>0$ since $G_\infty\,F_\sigma(x)$ is the limit of the (standardized) concave functions $G_\tau\,F_\sigma(x)$, $0<\sigma<\tau$, $\tau\longrightarrow\infty$; $F_{-\sigma}(x)\prec F_\infty(x)$ for all $\sigma>0$ follows by the same argument. Hence the class $F_\sigma(x)$, $-\infty<\sigma\leqslant+\infty$, is indeed a chain in $\mathcal{F}'$, where

$$F_\sigma(x)\prec F_\tau(x) \qquad \text{for } \frac{1}{\sigma}\leqslant\frac{1}{\tau} \;.$$

To show that this chain is maximal we remark that

$F_\sigma(x) \prec F(x) \prec F_\tau(x)$ for fixed σ and all $\frac{1}{\tau} > \frac{1}{\sigma}$, implies that $G\,F_\sigma(x) = \lim_{\tau \to \sigma} G\,F_\tau(x)$ is convex as well as concave, and hence that $F(x)$ and $F_\sigma(x)$ are equivalent and may be identified; for fixed τ and all $\frac{1}{\sigma} < \frac{1}{\tau}$, $F_\sigma(x) \prec F(x) \prec F_\tau(x)$ implies that $F(x)$ and $F_\tau(x)$ are equivalent. Finally we note that $\lim_{\sigma \to 0} F_\sigma(E\,\underline{x}(\sigma)) = 1$, where $\underline{x}(\sigma)$ denotes a random variable with distribution $F_\sigma(x)$, so $F_\sigma(x) \prec F(x)$ for all σ implies $F(E\,\underline{x}) = 1$. But this again implies that either $E\,\underline{x}$ is not finite or $\underline{x} = E\,\underline{x}$ with probability 1, and hence that $F(x) \notin \mathcal{F}$ by (2.2) and (2.3) A similar argument shows that $F(x) \prec F_\sigma(x)$ for all σ also implies $F(x) \notin \mathcal{F}$. This concludes the proof that the chain is maximal.

To illustrate the results obtained in this section and in section 7 table 1 shows the values of $F(E\,\underline{x}_{i:10})$, $i=1,2,..,10$, for the gamma distributions $F_\sigma(x)$, $\sigma=1,2,...,5$, and the normal distribution $F_\infty(x)$. For gamma distributions up to $\sigma=5$ values of $E\,\underline{x}_{i:n}$ are given by GUPTA [4], whereas the expected values of normal order statistics were taken from TEICHROEW [5].

TABLE 1. Values of $F(E\,\underline{x}_{i:10})$ for gamma distributions $F_\sigma(x)$.

	$\sigma=1$	$\sigma=2$	$\sigma=3$	$\sigma=4$	$\sigma=5$	$\sigma=\infty$
i=1	0,095	0,080	0,075	0,072	0,071	0,062
2	0,190	0,177	0,172	0,170	0,168	0,158
3	0,285	0,274	0,269	0,267	0,266	0,256
4	0,381	0,370	0,367	0,365	0,363	0,354
5	0,476	0,467	0,464	0,462	0,461	0,451
6	0,571	0,563	0,560	0,559	0,558	0,549
7	0,666	0,660	0,657	0,656	0,655	0,646
8	0,760	0,756	0,754	0,753	0,752	0,744
9	0,855	0,851	0,850	0,849	0,848	0,842
10	0,947	0,945	0,944	0,943	0,943	0,938

We note that the inequalities derived from 4.1 and 4.2 are indeed rather crude. On the other hand the smooth appearance of curves of the tabled values for fixed i suggests that computation of E $\underline{x}_{i:n}$ for different values of σ may perhaps largely proceed by interpolation for $F(E\,\underline{x}_{i:n})$ with respect to σ , for which the monotonicity proved in 4.3 provides a firm basis.

5. A WEAK ORDERING FOR A CLASS OF SYMMETRIC DISTRIBUTIONS

In the remaining part of this report we consider the subclass $\mathcal{S} \subset \mathcal{F}$ of symmetric distributions F(x) defined by:

(5.1) $F(x) \in \mathcal{F}$

(5.2) $F(\mu - x) + F(\mu + x) = 1$ for some real μ and all values of x.

By (2.3) $E\,\underline{x} = \int_I x \, d\,F(x)$ exists and is therefore equal to μ.
We adopt the same notation and conventions as in section 2.
Condition (5.2) may also be written

$$G(y) + G(1-y) = 2\mu \qquad \text{for } 0 < y < 1,$$

so for $F(x) \in \mathcal{S}$ and $F^*(x) \in \mathcal{S}$

$$G^*F(\mu - x) + G^*F(\mu + x) = 2\mu^* \qquad \text{for all } \mu - x \in I,$$

where μ^* denotes the point of symmetry of $F^*(x)$. Consequently convexity (c.q. concavity) of $G^*F(x)$ for $x > \mu$ implies concavity (c.q. convexity) of $G^*F(x)$ for $x < \mu$, and conversely. This may also be expressed as follows: if $F(x) \in \mathcal{S}$ and $F^*(x) \in \mathcal{S}$ and $G^*(y)$ is convex (c.q. concave) in $G(y)$ for $\frac{1}{2} < y < 1$ then $G^*(y)$ is concave (c.q. convex) in $G(y)$ for $0 < y < \frac{1}{2}$, and conversely. It follows immediately that $F(x) \in \mathcal{S}$, $F^*(x) \in \mathcal{S}$ and $F(x) \prec F^*(x)$ implies $F(x) \sim F^*(x)$, i.e. symmetric distributions are not comparable unless they are equivalent.
We may however define a different order-relation on $\mathcal{S}$ which is better adapted to this situation:

DEFINITION 4. If $F(x) \in \mathcal{S}$ and $F^*(x) \in \mathcal{S}$, then $F(x) \underset{s}{\prec} F^*(x)$ if and only if $G^*(y)$ is convex in $G(y)$ for $\frac{1}{2} < y < 1$.

We shall say in this case that $F(x)$ s-precedes $F^*(x)$ or that $F^*(x)$ s-follows $F(x)$ and that the two are s-comparable. We shall also speak of s-ordering, s-comparison, etc. . Clearly $F(x) \underset{s}{<} F(x)$ for all $F(x) \in \mathcal{S}$; since $G^*F(x)$ maps μ on μ^*, and an increasing convex function of a convex function is again convex, $F(x) \underset{s}{<} F^*(x) \underset{s}{<} F^{**}(x)$ yields $F(x) \underset{s}{<} F^{**}(x)$ for $F(x)$, $F^*(x)$, $F^{**}(x) \in \mathcal{S}$. The relation $\underset{s}{<}$ is thus a weak ordering on $\mathcal{S}$. Defining an equivalence relation $\underset{s}{\sim}$ by

DEFINITION 5. If $F(x) \in \mathcal{S}$ and $F^*(x) \in \mathcal{S}$, then $F(x) \underset{s}{\sim} F^*(x)$ if and only if $F(x) \underset{s}{<} F^*(x)$ and $F^*(x) \underset{s}{<} F(x)$,

and passing to the collection $\mathcal{S}'$ of equivalence classes, the relation $\underset{s}{<}$ defines a partial ordering on $\mathcal{S}'$. Again we have

THEOREM 3. $F(x) \underset{s}{\sim} F^*(x)$ if and only if $F^*(x)=F(ax+b)$ for some constants $a > 0$ and b.

PROOF. $F(x) \underset{s}{\sim} F^*(x)$ if and only if $G^*F(x)$ is concave-convex as well as convex-concave about μ, and thus linear on I and strictly increasing.

Hence this order relation is also independent of location and scale parameters. The symbol $\underset{s}{\sim}$ is superfluous and may be replaced by $\sim$.

6. A NECESSARY AND SUFFICIENT CONDITION FOR S-ORDERING.

To obtain a theorem analogous to theorem 2 for the weak-order relation $\underset{s}{<}$, we only have to prove an analogue of lemma 1 for the symmetric case.

LEMMA 4. Let $\underline{x}$ be a real valued random variable assuming values in a (finite or infinite) interval I with probability 1, with distribution function $H(x)$ satisfying $d\,H(x_0+x) \geq d\,H(x_0-x)$ for some $x_0 \in I$ and all $x \geq 0$. Let $\varphi(x)$ be a real valued continuous function on I, convex for $x \geq x_0$, $x \in I$, and satisfying $\varphi(x_0+x)+\varphi(x_0-x)=2\varphi(x_0)$ for all $x_0-x \in I$. Then

$$\varphi(E\,\underline{x}) \leq E\,\varphi(\underline{x}) \;,$$

provided both expectations exist.

We remark that the condition $d\,H(x_0+x) \geq d\,H(x_0-x)$ for all $x \geq 0$ ensures that, if $x_0-x \in I$, then also $x_0+x \in I$. The condition $\varphi(x_0+x)+\varphi(x_0-x)=2\varphi(x_0)$ for all $x_0-x \in I$ is therefore compatible with the fact that $\varphi(x)$ is only defined on I.

PROOF. Since $d\,H(x_0+x) \geq d\,H(x_0-x)$ for $x \geq 0$, it follows that $E\,\underline{x} \geq x_0$. Let $L(x)$ be a line of support of $\varphi(x)$ for $x \geq x_0$ through the point $(E\,\underline{x},\ \varphi(E\,\underline{x}))$. Then

$\varphi(x_0+x)-L(x_0+x) \geq 0$ for $x \geq 0$, $x+x_0 \in I$, and

$L(x_0+x)+L(x_0-x)=2L(x_0) \leq 2\varphi(x_0)=\varphi(x_0+x)+\varphi(x_0-x)$ for $x_0-x \in I$,

or $L(x_0-x)-\varphi(x_0-x) \leq \varphi(x_0+x)-L(x_0+x)$ for $x_0-x \in I$.

Therefore

$$\int_I \left\{\varphi(x)-L(x)\right\} d\,H(x) =$$

$$= \int_0^\infty \left\{\varphi(x_0+x)-L(x_0+x)\right\} d\,H(x_0+x) - \int_0^\infty \left\{ L(x_0-x)-\varphi(x_0-x)\right\} dH(x_0-x) \geq 0,$$

or
$$E\,\varphi(\underline{x}) \geq E\,L(\underline{x}) = L(E\,\underline{x}) = \varphi(E\,\underline{x}) \;.$$

We may now pass immediately to

THEOREM 4. If $\underline{x}_{\lambda:n}$ and $\underline{x}^*_{\lambda:n}$ denote generalized order statistics from the distributions $F(x) \in \mathcal{F}$ and $F^*(x) \in \mathcal{F}$ respectively, then the following statements are equivalent:

(i) $F(x) \underset{s}{\prec} F^*(x)$

(ii) $F(E\,\underline{x}_{\lambda:n}) \leq F^*(E\,\underline{x}^*_{\lambda:n})$ for all $n=1,2,\ldots$ and $\frac{n+1}{2} \leq \lambda \leq n$

(iii) $F(E\,\underline{x}_{i:n}) \leq F^*(E\,\underline{x}^*_{i:n})$ for all $n=1,2,\ldots$ and integer values of i, $\frac{n+1}{2} \leq i \leq n$

(iiii) $F(E\,\underline{x}_{i:n}) \leq F^*(E\,\underline{x}^*_{i:n})$ for all $n=M,M+1,\ldots$ and integer values of i, $\frac{n+1}{2} \leq i \leq n$, where M denotes an arbitrary integer.

PROOF. From the proof of theorem 2 we recall

$$E\,\underline{x}^*_{\lambda:n} = E\,G^*F(\underline{x}_{\lambda:n}) \; .$$

For $\frac{n+1}{2} \leq \lambda \leq n$ the distribution function $H_{\lambda:n}(x)$ of $\underline{x}_{\lambda:n}$ (cf. definition 3) satisfies $d\,H_{\lambda:n}(\mu+x) \geq d\,H_{\lambda:n}(\mu-x)$ for $x \geq 0$, since μ is the point of symmetry of the distribution $F(x)$. Also (cf. section 5)
$G^*F(\mu-x) + G^*F(\mu+x) = 2\mu^* = 2\,G^*F(\mu)$ for $\mu-x \in I$.
If $F(x) \underset{s}{\prec} F^*(x)$, or $G^*F(x)$ is convex for $x \geq \mu$, $x \in I$, application of lemma 4 to the random variable $\underline{x}_{\lambda:n}$, the point $x_0=\mu$, and the function $G^*F(x)$ gives for $\frac{n+1}{2} \leq \lambda \leq n$

$$G^*F(E\,\underline{x}_{\lambda:n}) \leq E\,G^*F(\underline{x}_{\lambda:n}) = E\,\underline{x}^*_{\lambda:n} \;, \text{ or}$$
$$F(E\,\underline{x}_{\lambda:n}) \leq F^*(E\,\underline{x}^*_{\lambda:n}) \;,$$

which proves (i)$\Longrightarrow$(ii). The rest of the proof is an obvious modification of the proof of theorem 2.

We note that for $F(x) \in \mathcal{S}$, $F(E\,\underline{x}_{\lambda:n}) = 1 - F(E\,\underline{x}_{n+1-\lambda:n})$. Consequently theorem 4 may of course also be formulated for $1 \leq \lambda \leq \frac{n+1}{2}$ and $1 \leq i \leq \frac{n+1}{2}$ by reversing the inequalities in (ii), (iii) and (iiii).

At the end of section 3 we remarked that the order relation $\prec$ is related to the skewness of comparable distributions.

Theorem 5 shows that the relation $\underset{s}{\prec}$ has implications for the even central moments of standardized s-comparable distributions. Let μ_k and μ_k^* denote the k-th central moments of F(x) and $F^*(x)$, if they exist. We find

THEOREM 5. If $F(x)\in\mathcal{S}$, $F^*(x)\in\mathcal{S}$ and $F(x)\underset{s}{\prec} F^*(x)$, and if μ_{2k} and μ_{2k}^* exist, then

$$\frac{\mu_{2k}}{(\mu_2)^k} \leq \frac{\mu_{2k}^*}{(\mu_2^*)^k} \qquad (k=1,2,\dots).$$

PROOF. Without loss of generality we set

$$E\,\underline{x}=\int_I x\ d\ F(x)=0, \quad E\,\underline{x}^*=\int_{I^*} x\ d\ F^*(x)=\int_I \varphi(x)d\ F(x)=0,$$

$$\mu_2=E\,\underline{x}^2=\int_I x^2\ d\ F(x)=1 \text{ and } \mu_2^*=E\,\underline{x}^{*2}=\int_{I^*} x^2 d\ F^*(x)=\int_I \varphi^2(x)dF(x)=1,$$

where $\varphi(x)=G^*F(x)$ is concave-convex about x=0 on I and strictly increasing, $\varphi(0)=0$ and hence $\varphi(-x)=-\varphi(x)$. We drop the trivial case $F^*(x)\equiv F(x)$, or $\varphi(x)\equiv x$ on I.

Now $\varphi(x)-x$ cannot be non-negative (or non-positive) for all $x\geq 0$, $x\in I$, for in that case $\varphi^2(x)-x^2=(\varphi(x)-x)(\varphi(x)+x)$ would be non-negative (or non-positive) for all $x\in I$; since $\varphi(x)$ is continuous and we have supposed $\varphi(x)\not\equiv x$, this would mean that $\mu_2^*-\mu_2$ would not be equal to zero.

As $\varphi(x)-x$ is convex for $x\geq 0$, $x\in I$, and $\varphi(0)=0$, it follows that $\varphi(x)-x\leq 0$ for $0\leq x\leq x_0$ and $\varphi(x)-x\geq 0$ for $x\geq x_0, x\in I$, for some $x_0>0$, $x_0\in I$. Hence $\varphi^2(x)-x^2\leq 0$ for $|x|\leq x_0$ and $\varphi^2(x)-x^2\geq 0$ for $|x|\geq x_0, x\in I$. Now

$$\mu_{2k}^*-\mu_{2k}=\int_I (\varphi^{2k}(x)-x^{2k})d\ F(x)=\int_I (\varphi^2(x)-x^2)\ \psi(x)d\ F(x),$$

where $\psi(x)=\sum_{j=0}^{k-1} x^{2j}\,\varphi^{2k-2j-2}(x)\geq 0$ on I, and $\psi(x)$ is even and increasing for $x\geq 0$, $x\in I$. So

$$\mu^*_{2k}-\mu_{2k}\geq\psi(x_o)\int_{|x|>x_o}(\varphi^2(x)-x^2)dF(x)-\psi(x_o)\int_{|x|\leq x_o}(x^2-\varphi^2(x))dF(x)=$$

$$=\psi(x_o)\int_I(\varphi^2(x)-x^2)dF(x)=0,$$

which completes the proof.

The other remarks at the end of section 3 continue to apply.

7. SOME EXAMPLES OF S-ORDERING.

The first two examples given here are similar to those treated in section 4. They refer to s-comparison with the rectangular distribution and to mutual s-comparison of symmetric beta distributions. The third example treats the s-ordering of the normal and logistic distributions.

7.1 s-Comparison with the rectangular distribution.

We take $F^*(x)=x$, $0\leq x\leq 1$, or $G^*(y)=y$, and $E\,\underline{x}^*_{i:n}=\frac{i}{n+1}$.
For $F(x)$ we consider the class of symmetric distributions having a density function $F'(x)$ which possesses a single extreme, and is therefore either U-shaped (single minimum; $F(x)$ concave-convex) or unimodal (single maximum; $F(x)$ convex-concave). By theorem 4 we have:
If $F'(x)$ is symmetric and U-shaped, then $F(E\,\underline{x}_{i:n})\leq\frac{i}{n+1}$ for $i\geq\frac{n+1}{2}$; If $F'(x)$ is symmetric and unimodal, then $F(E\,\underline{x}_{i:n})\geq\frac{i}{n+1}$ for $i\geq\frac{n+1}{2}$.
BLOM ([3] p.66) proved the latter inequality asymptotically for $n\to\infty$, which by theorem 4 is equivalent to the result stated here.
VAN DANTZIG and HEMELRIJK [6] mention the result for all n in connection with a comparison of TERRY's and VAN DER WAERDEN's tests where respectively the quantities

$E\,\underline{x}_{i:n}$ and $G(\frac{i}{n+1})$ for the normal distribution are involved in the test statistic.

7.2. The s-chain of symmetric beta distributions.

Consider the class $F_\sigma(x)=\dfrac{\Gamma(2\sigma)}{2^{2\sigma-1}(\Gamma(\sigma))^2}\displaystyle\int_{-1}^{x}(1-t^2)^{\sigma-1}\,dt$, $\sigma>0$, $-1\leq x\leq 1$, representing an increasing linear transform ($x=2u-1$) of the symmetric beta distributions.
We shall sketch a proof that $F_\sigma(x)\underset{s}{<}F_\tau(x)$ for $0<\sigma<\tau$, i.e. the symmetric beta distributions s-follow one another with increasing values of the parameter.
Hence we have to show that $\varphi(x)=G_\tau F_\sigma(x)$ is convex for $0<x<1$, where $G_\tau(y)$ denotes the inverse of $F_\tau(x)$. As in 4.3 we consider the function $\psi(x)=F_\sigma(x)-F_\tau(b(x+a))$, $b>0$, $ba\geq -1$, $b(1+a)\leq 1$, which has the same sign as $\varphi(x)-b(x+a)$ for $0\leq x\leq 1$. Also $\psi'(x)$ has the same sign as

$$\chi(x)=\log F'_\sigma(x)-\log F'_\tau(b(x+a))-\log b\ .$$

$$\chi'(x)=-\frac{2(\sigma-1)x}{1-x^2}+\frac{2\,b^2(\tau-1)(x+a)}{1-b^2(x+a)^2}\ .$$

As in 4.3 we study the sign of $\chi'(x)$ for $0\leq x\leq 1$ and the signs of $\chi(x)$ and $\psi(x)$ for $x=0$ and $x=1$. In this way we find that $\psi(x)$ and hence $\varphi(x)-b(x+a)$ can have at most two zeros for $0\leq x\leq 1$, in which case the function is negative between these zeros. For $b>0$, $ba<-1$, $b(1+a)\leq 1$ the representation of $\psi(x)$ remains valid for $-a-1/b\leq x\leq 1$, and we may prove the same result in this interval. However, $\varphi(x)-b(x+a)<0$ for $0\leq x\leq -a$ and hence the result continues to apply for $0\leq x\leq 1$. For $b>0$, $b(1+a)>1$ a comparison with the case $b>0$, $b(1+a)=1$ shows that $\varphi(x)-b(x+a)$ can have most one zero for $0\leq x\leq 1$; for $b\leq 0$ the same holds since $\varphi(x)$ is strictly increasing. Hence the graph of $\varphi(x)$ for $0\leq x\leq 1$ lies below any chord, which proves convexity of $\varphi(x)$ for $0 \leqslant x \leqslant 1$.

7.3. s-Comparison of normal and logistic distributions.

Consider $F(x)=\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\, dt$, $-\infty < x < \infty$, and $F^*(x)=\frac{1}{1+e^{-x}}$, $-\infty < x < \infty$. Clearly $F(x) \in \mathcal{S}$ and $F^*(x) \in \mathcal{S}$; furthermore one easily shows by repeated differentiation that $G^* F(x) = \log F(x) - \log(1-F(x))$ is convex for $x \geqslant 0$, so $F(x) \underset{s}{\prec} F^*(x)$.

Now $E\, \underline{x}^*_{i:n}$ is simple to evaluate, giving

$$E\, \underline{x}^*_{i:n} = \sum_{k=n+1-i}^{i-1} \frac{1}{k} \qquad \text{for } i \geqslant \frac{n+1}{2}$$

$$= 0 \qquad \text{for } i = \frac{n+1}{2} .$$

Since $\sum_{k=n+1-i}^{i-1} \frac{1}{k} \leqslant \log \frac{i-\frac{1}{2}}{n-i+\frac{1}{2}}$, we find $F^*(E\, \underline{x}^*_{i:n}) \leqslant \frac{i-\frac{1}{2}}{n}$ for $i \geqslant \frac{n+1}{2}$ and as a consequence

$$F(E\, \underline{x}_{i:n}) \leqslant \frac{i-\frac{1}{2}}{n} \qquad \text{for } i \geqslant \frac{n+1}{2} .$$

We note that BLOM [3] proved the corresponding asymptotic result for $n \longrightarrow \infty$. The inequality can not be sharpened for all n and all $i \geqslant \frac{n+1}{2}$ since $F(E\, \underline{x}_{i:n}) \sim \frac{i-\frac{1}{2}}{n}$ for $i=n$, $n \longrightarrow \infty$.

The easy derivation of this inequality (and of course also of those proved in 4.1, 4.2 and 7.1) is a consequence of the fact that $G^*(y)$ is an incomplete beta function. The properties of distributions $F^*(x)$ of this type (cf. [3]) make them particularly well suited as standards for comparison ans s-comparison, and a further study of inequalities to be obtained in this way is in progress.

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