STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

AFDELING MATHEMATISCHE STATISTIEK

Report S 306

AN ORDER POLICY BASED ON APPROPRIATIONS

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(Revision of an essay written at the College of Aeronautics, Cranfield, England)

September 1962

CONTENTS

0.0 Introduction.

In Stock-Control, the decision whether or not to place an order for a particular item is usually based on the size of the "virtual" stock, that is: the size of the stock plus the size of the outstanding order(s) for that particular item. In this essay we will consider a policy where this decision is also based on the appropriations made by the customers. One advantage of a policy of this type is that it may be more profitable. Another advantage is that it lends itself more easily to the case of non-stationary demand than a policy based on the virtual stock-level only. However, in this essay we will limit our considerations largely to the case of stationary demand.

0.1 Object.

The object of this essay is to formulate a policy that uses the information available from the appropriations made, and to determine in what circumstances such a policy is better than one based on the virtual stock-level only.

0.2 Results.

The distribution of the appropriations _for_ a certain time has been determined, as well as the mean and the variance of this distribution, and of the distribution of the total appropriations.

A policy that uses the total appropriations as a basis for order-decisions, has been constructed.

Examination has been made of the conditions under which such a policy implies a lower average stock than the F.O.Q.S. does.

0.3 Recommendations.

It is recommended (i) that an investigation be made as to when it will be better to use the appropriations in other ways, as indicated in the text,

(ii) that an effort be made to extend the results of this essay to more general cases

(iii) that the constructed policy be used if the necessary information can easily be obtained, and if the conditions that make it more profitable are fulfilled.

1.0 Assumptions.

In the theory of stock-control, it is usually assumed that the customer, when he places an order, is supplied with the required amount immediately. (provided, of course, the stock is sufficiently large to do this)

It may, however, be more realistic to assume that the customer will have the goods delivered to him after a certain lapse of time. The length of this time interval will be called the notice of the order concerned. In practice, the notice will be fixed, e.g. in mutual agreement. In this model, we will assume that the notice has a stationary probability distribution. Further on in this section, this assumption will be stated more precisely.

At any given time the appropriations can be determined. By the appropriations at T, we understand the amount of goods ordered by the customers before T, to be delivered to the customers at time T or later. The appropriations at T can be divided into groups according to the time at which the goods are to be delivered.

Throughout this essay a discrete time parameter will be used, or more precisely: it will be assumed that any order by, and any delivery to customers takes place at integer values of the time. Since the unit of time is arbitrary, this is hardly a restriction.

It will also be assumed that the demand has a discrete distribution function, and that the notice for each unit is statistically independent of the notice of any other unit. The latter assumption, in spite of appearances, is in fact rather restrictive, as it virtually implies that the order size is constant and equal to one. It also makes it impossible to consider the case of a continuous distribution function of the demand as a limiting case. This point will be dealt with further in section 3.0.

As for the distribution of the notice, the above mentioned stationariness is used in the following sense. The (conditional) probability that the notice of a certain unit is less than or equal to, say, j, <u>given that the delivery time is T</u>, is defined, and is independent of T. In the case of stationary demand, it would make no difference if the underlined condition were replaced by "given that the time at which the unit has been ordered is T", but when demand is not stationary, it does make a difference.

It may finally be remarked that the special case where the notice is equal to zero with probability one, is equivalent to the practical situation where either no appropriations are made, or where the information concerned is neglected. The assumption about the statistical independency of the notices (that has been labelled "rather restrictive" in the above), is in <u>this</u> case (and even in the case of constant notice) automatically fulfilled, and has no influence whatsoever on the distribution of the order size.

1.1 Notation

Stochastic variables will be denoted by underlined capital letters, e.g. $\underline{V}$, $\underline{M}_{T,k}$.

An exception to this rule is made whenever a stochastic variable is immediately preceded by a symbol like $\mathcal{E}$ or σ^2. Hence we will write e.g. $\mathcal{E}(J)$, or even $\mathcal{E}J$, rather than $\mathcal{E}(\underline{J})$.

As far as possible, the values that are assumed by stochastic variables, will be denoted by the corresponding small letters, e.g. $\underline{V}=v$, $\underline{J}=j_1$.

The probability that an event A occurs will be abbreviated to $P[A]$, e.g. $P[\underline{M}_T=m]$.

The conditional probability that an event A occurs, given that the event B has occurred, will be abbreviated to $P[A|B]$, e.g. $P[\underline{V}\leqslant v \mid \underline{V}\geqslant m]$.

Although the meaning of all other symbols will be given as they occur, it is probably useful to remark that the letter V is reserved for the sales, J for the notice, and M for the appropriations.

The formulae have been numbered consecutively 1,2,3, etc., and for the sections the decimal numbering has been used.

1.2 Use of Appropriations.

The information available from the appropriations may be used in several ways.

In the first place, appropriations may be considered as quantities that change the distribution of the sales into a conditional distribution. However, the expression for this distribution is almost unmanageable if, in addition to the distribution of the notice, and the unconditional distribution of the sales, only the total appropriations at an earlier time are known. One way of surmounting this difficulty is to consider a breakdown of the appropriations according to the date of delivery. For each time-unit the sales can be divided into two parts: a known quantity, viz. the appropriations, and a stochastic part with known (conditional) distribution. In the case of known lead time, this approach would make it possible to split the problem into two separate ones. However, when the lead time is stochastic, this split is not possible, as the known demand now also requires a buffer stock, which can partly coincide with the one required for the stochastic demand.

In the approach that we have chosen, the appropriations are considered purely as quantities on which a sales-forecast is based. It is taken into account that this forecast is subject to an error.

1.3 Some formulae for the non-stationary case.

Let q_j be the probability that the notice $\underline{J}$ for a certain unit has the value j, given that the unit has to be delivered at a certain time, and let Q_j be defined by

$$Q_j = \sum_{i=0}^{j} q_i \qquad (1)$$

Further, let $\underline{M}_T$ be the total appropriations made at times T-1, T-2,... for times T, T+1,... .

Then, if the sales at time i are v_i, on the average, of v_T a fraction $1-Q_0$ will have been ordered by the customers at T-1 or earlier, etc.
Hence

$$\mathcal{E} M_T = (1-Q_0)\ v_T+(1-Q_1)v_{T+1}+\dots \quad . \tag{2}$$

As is easily proved, this formula may also be written in the following manner:

$$\mathcal{E} M_T = q_1\ v_T + q_2\ (v_T+v_{T+1})+ \dots \quad . \tag{3}$$

If the sales are considered to be variates, (3) changes into

$$\mathcal{E} M_T = q_1 \mathcal{E} V_T + q_2\ (\mathcal{E} V_T + \mathcal{E} V_{T+1}) + \dots \quad . \tag{4}$$

1.31 Special choices for V_T and J.

If demand is stationary, it follows from (4) that

$$\mathcal{E} M_T = \mathcal{E} J \,.\, \mathcal{E} V \quad . \tag{5}$$

If, during a certain time interval, $\mathcal{E} V_T$ increases (or decreases) linearly with time: $\mathcal{E} V_T=aT+b$, formula (4) changes into

$$\mathcal{E} M_T = \mathcal{E} J \,.\, \mathcal{E} V_T + \frac{a}{2}\,(\mathcal{E} J^2 - \mathcal{E} J) \,. \tag{6}$$

If $\mathcal{E} V_T$ changes exponentially with time: $v_T=ae^{bT}$, formula (4) changes into

$$\mathcal{E} M_T = \mathcal{E} V_T\ \frac{1-\mathcal{E}(e^{b\underline{J}})}{1-e^b} \quad .\ (b \neq 0) \tag{7}$$

The proofs of these formula are straightforward, and will be omitted.
The calculation of the quantities $\mathcal{E} J$, $\mathcal{E} J^2$, and $\mathcal{E}\, e^{b\underline{J}}$ offers no difficulties in simple cases like Poissonian, binomial or geometrical distribution of the notice.

2.0 Some lemmata.

Lemma 1. If $|x| < 1$, and m is an integer $\geqslant 0$, then

$$\sum_{i=m}^{\infty} \binom{i}{m}\ x^i = \frac{x^m}{(1-x)^{m+1}} \quad . \tag{8}$$

Proof: denote the left hand side of (8) by $f(x,m)$; then it follows from the relation $\binom{i}{m}=\binom{i-1}{m-1}+\binom{i-1}{m}$ that

$$f(x,m)=x\ f(x,m-1) + x\ f(x,m) \quad . \tag{9}$$

Solving (9) for $f(x,m)$, we get

$$f(x,m) = \frac{x}{1-x}\ f(x,m-1) \tag{10}$$

and, since $f(x,0)=\sum_0^\infty x^i = \frac{1}{1-x}$, we obtain the desired result from (10) by induction.

Lemma 2. If the variate $\underline{X}$ assumes non-negative integer values only, then

$$\sum_{j=0}^{\infty} P[\underline{X} > j] = \mathcal{E}X \quad . \tag{11}$$

Proof: $\sum_{j=0}^{\infty} P[\underline{X} > j] = \sum_{j=0}^{\infty} \sum_{i=j+1}^{\infty} P[\underline{X} = i] =$

$$= \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} P[\underline{X} = i] = \sum_{i=1}^{\infty} i\ P[\underline{X} = i] = \mathcal{E}X .$$

Lemma 3. If the variate $\underline{X}$ assumes non-negative integer values only, then

$$\sum_{j=0}^{\infty} (P[\underline{X} > j])^2 = \mathcal{E}\min(X_1,X_2) \tag{12}$$

where $\underline{X}_1$ and $\underline{X}_2$ are independent variates with the same distribution as X.

Proof: according to lemma 2

$$\sum_{j=0}^{\infty} P[\min(\underline{X}_1,\underline{X}_2) > j] = \mathcal{E}\min(X_1,X_2) \tag{13}$$

and since

$$P[\min(\underline{X}_1,\underline{X}_2) > j] = P[\underline{X}_1 > j;\underline{X}_2 > j] = (P[\underline{X} > j])^2$$

it follows that (13) is equivalent to (12).

2.1 The distribution of $M_{T,k}$.

Let p_v be the probability that the sales $\underline{V}_k$ at time k equal v, and let $\underline{M}_{T,k}$ be the appropriations made at times T-1, T-2,... for time k. Then:

$$P\left[\underline{M}_{T,k}=m\right] = \sum_{v=m}^{\infty} P\left[\underline{M}_{T,k}=m;\ \underline{V}_k=v\right] = \qquad (14)$$
$$= \sum_{v=m}^{\infty}{}^{*}\, p_v \cdot P\left[\underline{M}_{T,k} = m \mid \underline{V}_k=v\right].$$

The asterisk at the second summation sign means that the sum runs only through the values of v for which $p_v \neq 0$. This is necessary, since the conditional probability $P\left[A|B\right]$ has a meaning only when $P\left[B\right] \neq 0$.

As the probability that a unit is ordered before T is equal to the probability that the notice is greater than k-T, we can rewrite (20) as follows:

$$P\left[\underline{M}_{T,k}=m\right] = \sum_{v=m}^{\infty} p_v \binom{v}{m} (1-Q_{k-T})^m\, Q_{k-T}^{v-m} \qquad (15)$$

where Q_j has the same meaning as in section 1.3.
Taking the factors not depending on v outside the summation sign, we finally get:

$$P\left[\underline{M}_{T,k}=m\right] = \left(\frac{1-Q_{k-T}}{Q_{k-T}}\right)^m \sum_{v=m}^{\infty} p_v \binom{v}{m} Q_{k-T}^{v}. \qquad (16)$$

2.11 Examples.

In the case $p_v=(1-\alpha)\,\alpha^v$ for v=0,1,... formula (16) can, with the aid of lemma 1, be reduced to

$$P\left[\underline{M}_{T,k}=m\right] = \frac{1-\alpha}{1-\alpha Q_{k-T}} \left(\frac{\alpha-\alpha Q_{k-T}}{1-\alpha Q_{k-T}}\right)^m \qquad (17)$$

which is also a geometrical distribution.

In the more important case of binomially distributed sales:

$$\left.\begin{array}{l} p_v = \binom{n}{v} \alpha^v (1-\alpha)^{n-v} \quad \text{if } 0 \leqslant v \leqslant n \\ p_v = 0 \quad \text{if } v > n \end{array}\right\} \tag{18}$$

(16) can be reduced to:

$$P\left[\underline{M}_{T,k} = m\right] = \binom{n}{m} \beta^m (1-\beta)^{n-m} \tag{19}$$

where $\beta = \alpha(1-Q_{k-T})$.

It is not surprising that $\underline{M}_{T,k}$ has a binomial distribution when $\underline{V}_k$ does. Each of the n units that may be sold at k has a probability α of actually being sold, and a probability $1-Q_{k-T}$ of being ordered before T, given that it will be sold at k. Hence, $\underline{M}_{T,k}$ is just the number of "successes" in a binomial experiment consisting of n trials, with probability of success equal to $\alpha(1-Q_{k-T}) = \beta$.

2.2 Laplace Transforms and Moments.

Let the Laplace Transform of the distribution of $\underline{M}_{T,k}$ be denoted by $\mathcal{L}_\tau(M_{T,k})$. It then follows from (16) that

$$\mathcal{L}_\tau(M_{T,k}) = \sum_{m=0}^{\infty} e^{\tau m} \; P\left[\underline{M}_{T,k} = m\right] = \tag{20}$$
$$= \sum_{m=0}^{\infty} e^{\tau m} \left(\frac{1-Q_{k-T}}{Q_{k-T}}\right)^m \sum_{v=m}^{\infty} p_v \binom{v}{m} Q_{k-T}^v \, .$$

After some calculations, this reduces to:

$$\mathcal{L}_\tau(M_{T,k}) = \sum_{v=0}^{\infty} p_v \left\{Q_{k-T} + e^{\tau}(1-Q_{k-T})\right\}^v . \tag{21}$$

Since the total appropriations at T are given by

$$\underline{M}_T = \sum_{k=T}^{\infty} \underline{M}_{T,k} \tag{22}$$

the Laplace Transform of the distribution of $\underline{M}_T$ equals

$$\mathcal{L}_\tau(M_T) = \sum_{k=T}^{\infty} \mathcal{L}_\tau(M_{T,k}) \ . \qquad (23)$$

The quantities p_v, which occur in $\mathcal{L}_\tau(M_{T,k})$, and hence in the right hand member of (23) will, in the case of non-stationary demand, obviously depend on k.
From now on, however, we will assume that the demand is stationary.

Theoretically, all moments (in particular the first and second moments) of $\underline{M}_{T,k}$ and $\underline{M}_T$, can be found from the expressions (21) and (23). However, in this case there is a simpler way.

Since, on the average, of the expected sales at k a fraction $1-Q_{k-T}$ will have been ordered before T, $\mathcal{E}(M_{T,k})$ is given by

$$\mathcal{E}(M_{T,k}) = (1-Q_{k-T})\, \mathcal{E}\, V \ . \qquad (24)$$

The expected value of M_T has already been determined in sub-section 1.31:

$$\mathcal{E}\, M_T = \mathcal{E}\, J \ . \ \mathcal{E}\, V \ . \qquad (25)$$

The variance of $\underline{M}_{T,k}$ can be found as follows:

$$\sigma^2(M_{T,k}) = \sum_{m=0}^{\infty} P\left[\underline{M}_{T,k}=m\right] \quad (m-\mathcal{E}\, M_{T,k})^2 =$$

$$= \sum_{m=0}^{\infty} \left(\frac{1-Q_{k-T}}{Q_{k-T}}\right)^m \left\{\sum_{v=m}^{\infty} p_v \binom{v}{m} Q_{k-T}^v\right\} (m-\mathcal{E}\, M_{T,k})^2 =$$

$$= \sum_{v=0}^{\infty} p_v \cdot \left[\sum_{m=0}^{v} \binom{v}{m} (1-Q_{k-T})^m\, Q_{k-T}^{v-m}\, (m-\mathcal{E}\, M_{T,k})^2\right] .$$

The expression in the square brackets is just the second

moment of a binomial distribution with respect to $\mathcal{E}\, M_{T,k}$, whereas the mean of this distribution is $v(1-Q_{k-T})$.
Hence:

$$\sigma^2(M_{T,k})=\sum_{v=0}^{\infty} p_v \cdot \left[v(1-Q_{k-T})Q_{k-T} + \left\{v(1-Q_{k-T})-\mathcal{E}\, M_{T,k}\right\}^2\right]$$

with the aid of (24) this can be reduced to

$$\sigma^2(M_{T,k})=(1-Q_{k-T})Q_{k-T}\ \mathcal{E}\, V+(1-Q_{k-T})^2\sigma^2\ V\ . \qquad (26)$$

The variance of $\underline{M}_T$ can be found by summing (26) over k, and applying lemmata 2 and 3. The result is

$$\sigma^2(M_T)=\mathcal{E}\, V\left\{\mathcal{E}\, J-\varphi J\right\} + \sigma^2 V.\ \varphi J \qquad (27)$$

where $\varphi J=\mathcal{E}\left\{\min(J_1,J_2)\right\}$.

As was to be expected, $\mathcal{E}\, M_T$ and $\sigma^2(M_T)$ do not depend on T in the case of stationary demand; hence the index T will be omitted in sections 4.0 and 4.1.

It is interesting to note the occurence of $\mathcal{E}\left\{\min(J_1,J_2)\right\}$ in this connection.

3.0 Continuous distribution function of demand.

If we want to extend the results of the previous sections to the case of demand with a continuous distribution function, it would be impossible to do this without breaking down the demand into the orders by the customers.
For, if the sales v were considered as N orders of size $\frac{v}{N}$, then the conditional probability

$$P\left[\underline{M}_{T,k} = m \,\middle|\, \underline{V}_k = v\right]$$

in formula (20) tends to 1 for $m=v(1-Q_{k-T})$, and to 0 for all other m, when N tends to infinity.

In other words, we would have to introduce a (discrete) distribution function for the number of orders at T, and a (continuous) distribution function for the order size.

However, this would make the problem of finding the distribution of $\underline{M}_{T,k}$ very complicated indeed.

3.1 Normal approximation.

In section 2.11 we have seen that, if $\underline{V}$ has a binomial distribution, $\underline{M}_{T,k}$ also has one. Suppose this binomial distribution of $\underline{V}$ can be approximated sufficiently well by a normal distribution (according to a relevant criterion). Since the probability of success in the distribution of $\underline{M}_{T,k}$ is smaller, viz. by a factor $1-Q_{k-T}$, it does not follow that this distribution is also approximately normal. In fact, when k-T is large, the distribution of $\underline{M}_{T,k}$ is bound to be very skew. However, as $\underline{M}_T=\sum_{k=T}^{\infty} \underline{M}_{T,k}$, it would not be surprising if the distribution of $\underline{M}_T$ still has a good normal approximation. Hence, once we have assumed that $\underline{V}$ has a binomial distribution with a good normal fit, the assumption that $\underline{M}_T$ also has an approximately normal distribution, is probably not very restrictive.

4.0 Order-policy.

Let us consider a policy which tells the stockholder to review the stocks and the appropriations at all times $T+\delta$, (where δ is a fixed number between 0 and 1, and where T assumes all integer values) and to order a fixed amount A whenever the virtual stock S drops below a certain level Z, which will depend on the appropriations made. (such a policy will be called an S.M.-policy in the sequel).

Suppose the lead time is constant and equal to t. As a matter of fact, if the lead time varies between $t-\delta$ and $t+1-\delta$, the argument would still be valid.

Let the probable maximum demand be defined as the value of the

demand that is exceeded in only a given, small percentage of all cases. Then Z should be equal to the probable maximum demand during the lead time + one unit of time.
So, if $t_1=t+1$, then Z equals the expected demand during t_1, plus a buffer stock.
The first term can be estimated by the "sales-forecast" $\frac{t_1 \underline{M}}{\mathcal{E} J}$ (provided, of course $\mathcal{E} J > 0$). According to formula (5) this is an unbiassed estimate.
To find the buffer-stock, we assume that the distribution of $\underline{M}$ is approximately normal. Then, if γ is the normal deviate corresponding to the given risk of running out of stock, the buffer stock should be $\gamma \frac{t_1}{\mathcal{E} J} \sigma M$. This term is, as it were, the maximum error in the sales-forecast.
So the S.M.-policy becomes: review at all times $T+\delta$, and order A when

$$\underline{S} - \frac{t_1}{\mathcal{E} J} \underline{M} < \gamma \frac{t_1}{\mathcal{E} J} \sigma M .$$

4.1 Comparison of policies.

In the S.M.policy, the average stock carried will be $\frac{A}{2} + \gamma \frac{t_1}{\mathcal{E} J} \sigma M$. In the corresponding F.O.Q.S. (Fixed Order Quantity System), adapted to a discrete time parameter, the average stock is $\frac{A}{2} + \gamma \sqrt{t_1} \sigma V$.
Since the number of orders per year, as well as the probability of running out are the same for both policies, it is the average stock that turns the scale: the S.M. policy is better than the corresponding F.O.Q.S. if and only if

$$\frac{A}{2} + \gamma \frac{t_1}{\mathcal{E} J} \sigma M < \frac{A}{2} + \gamma \sqrt{t_1} \sigma V \quad ; \qquad (28)$$

with the aid of the formula $\sigma^2 M = \mathcal{E} V(\mathcal{E} J - \varphi J) + \sigma^2 V . \varphi J$, the inequality (28) can be shown to be equivalent to

$$t_1 < \frac{\mathcal{E}^2 J}{\varphi J + \frac{\mathcal{E} V}{\sigma^2 V} (\mathcal{E} J - \varphi J)} \quad . \qquad (29)$$

4.11 Example.

In the special case where the notice is constant and equal to j_o, we have $\mathcal{E} J = \varphi J = j_o$, so that (29) reduces to

$$t_1 < j_o \ .$$

This is very much in accordance with what one would intuitively expect. One might, however, be surprised at the fact that the buffer stock is not zero in this case, but equal to

$$\frac{\gamma t_1 \sigma V}{\sqrt{j_o}} \ .$$

This is due to the fact that only the total appropriations are considered: if the demand at time $T+j_o-1$ happens to be low, then $\frac{t_1 \underline{M}_T}{j_o}$ underestimates the sales during t_1 .