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A sequential distribution-free two-sample grouped test with three decisions

by

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1. Introduction

Before discussing the three-decision problem which is the subject of this paper, we start by making some general remarks about sequential analysis with two decisions.

A sequential test of a hypothesis H_o consists of a set of rules for making one of the following three decisions at each successive stage of experimentation:

1. Accept H_o
2. Reject H_o
3. Continue the experiment by making an additional observation (or observations).

Generally one only considers sequential tests for which the probability that the procedure will eventually terminate is 1 and hence that one ultimately takes decision 1 or 2.

Normally one uses the sequential probability ratio test for testing a hypothesis H_o against an alternative hypothesis H_1.

For instance when we have a random variable $\underline{x}$[1] with probability density function $f(x;\Theta)$ (Θ unknown) and we want to test

$$H_o:\ \Theta \leq \Theta_o \quad \text{against} \quad H_1:\Theta \geq \Theta_1$$

on the basis of m successive independent observations $x_1, x_2, \ldots, x_m$, then we compute:

$$p_{1m} = f(x_1;\Theta_1)\ f(x_2;\Theta_1)\ldots f(x_m;\Theta_1)$$
$$p_{om} = f(x_1;\Theta_o)\ f(x_2;\Theta_o)\ldots f(x_m;\Theta_o).$$

Now if:

$B < p_{1m}/p_{om} < A$ then we take additional observations
$p_{1m}/p_{om} \geq A$ " " accept H_1
$p_{1m}/p_{om} \leq B$ " " accept H_o,

with:

$$A = \frac{1-\beta}{\alpha}$$
$$B = \frac{\beta}{1-\alpha}.$$

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1) Stochastic variables will be distinguished from numbers (e.g. from values they take in an experiment) by underlining their symbols.

The test has the strength (α,β) approximately (A. WALD (1947) p. 46) where α is the type I error and β is the type II error. The interval (Θ_o,Θ_1) is called the indifference zone.

If one wants to test a composite hypothesis of the form H_o: $\Theta \leqslant \Theta_o$ against $H_1: \Theta > \Theta_o$ the problem can be reduced to the former one by choosing an indifference zone (Θ_1,Θ_2) containing Θ_o and testing $\Theta \leqslant \Theta_1$ against $\Theta \geqslant \Theta_2$.

2. <u>Sequential tests with three decisions</u>

When one wishes to test

$$H_o: \Theta = \Theta_o \quad \text{against} \quad H_1: \Theta \neq \Theta_o$$

one gets a three-decision problem. The three ultimate decisions are now:

1. $\Theta = \Theta_o$
2. $\Theta < \Theta_o$
3. $\Theta > \Theta_o$.

In this case one needs two-sided tests.

In chapter 10 of WALD's book "Sequential Analysis" some general remarks about multi-valued decision problems are made; M. SOBEL and A. WALD (1949) have developed a procedure to test the hypothesis: the expectation of a normal population with known standard deviation is equal to a given number. For the case of testing an unknown parameter p of a binomial distribution, an application with slight modifications of SOBEL and WALD's method has been given by J. DE BOER (1953).

The procedure is the following. For a two-sided test whether a certain parameter Θ of a distribution function is equal to Θ_o, one divides the real axis into three intervals by two points Θ_o' and Θ_o'' and sets up the following hypothesis ($\Theta_o' \leqslant \Theta_o \leqslant \Theta_o''$):

$$H_1: \Theta < \Theta_o \quad H_2: \Theta_o' \leqslant \Theta \leqslant \Theta_o'' \quad H_3: \Theta > \Theta_o''$$

For the execution of the test two points Θ_1 and Θ_2 must be chosen on both sides of Θ_o' and two points Θ_3 and Θ_4 on both sides of Θ_o'', satisfying the relations:

$$\Theta_1 < \Theta_o' < \Theta_2 < \Theta_3 < \Theta_o'' < \Theta_4.$$

The intervals (Θ_1, Θ_2) and (Θ_3, Θ_4) are called the zones of indifference. Of course it is possible to schoose Θ_2 equal to Θ_o' resp. Θ_3 equal to Θ_o''.

Then one performs two two-decision sequential tests of WALD, namely:

T_{12}: testing the hypothesis : $\Theta \leq \Theta_1$ against $\Theta \geq \Theta_2$
T_{34}: " " " : $\Theta \leq \Theta_3$ " $\Theta \geq \Theta_4$.

To avoid the possibility of conflicting decisions (T_{12} decides: $\Theta < \Theta_o$ and T_{34} decides $\Theta > \Theta_o'$), one has to set up some restrictions on the α's and β's for the two tests T_{12} resp. T_{34} and on the values chosen for Θ_1, Θ_2, Θ_3 and Θ_4.

The probability of an incorrect decision is a function of Θ; its maximum over the whole interval of possible values of Θ (excepting the indifference zones) is called the true level of significance α_T of the test.

3. A sequential distribution-free two-sample test with three decisions.

F. WILCOXON, L.J. RHODES and R.A. BRADLEY (1963) have constructed two sequential distribution-free two-sample tests with two decisions. This is done by using LEHMAN's alternative hypothesis $H_1 : G(x) \equiv F^k(x)$ for evaluating the probabilities of configurations under the non-null hypothesis. This alternative has been chosen in order to simplify the computation of the test; although it does not on first sight seem realistic, the authors point out that it may be interpreted as an approximation of a translation (cf their table 5, p. 73).

Using their method one may construct the following two-sided test with three decisions.

Let $\underline{x}$ and $\underline{y}$ be random variables with continuous cumulative distribution functions $F(x)$ resp. $G(y)$. Then the general null and alternative hypothesis would be:

$$H_o:\ F(x) \equiv G(x) \qquad H_1:\ G(x) \equiv F^k(x) \quad (k \geq 0, k \neq 1). \tag{1}$$

In view of:

$$p \overset{def}{=} P[\,\underline{x} \leq \underline{y}\,] = \int F(y)\, d\, F^k(y) = \frac{k}{k+1}$$

these hypotheses may be rewritten as:

$$H_o:\ p = \tfrac{1}{2} \qquad H_1:\ p \neq \tfrac{1}{2}$$

which is equivalent to:

$$H_o:\ k=1 \qquad H_1:\ k \neq 1 \tag{2}$$

or, in a form that is more convenient for our purposes:

$$H_o:\ k_o \leq k \leq k_o' \qquad H_1:\ k < k_o \qquad H_2:\ k > k_o' \tag{3}$$

with: $k_o \leq 1 \leq k_o'$.

One determines numbers k_1, k_2, k_3 and k_4, satisfying the following inequalities:

$$k_1 < k_o < k_2 < 1 < k_3 < k_o' < k_4.$$

The choice is of course dependent on the practical problem and the applications. In practice one will usually try to choose a value of k that corresponds roughly to a given translation. The intervals (k_1,k_2) and (k_3,k_4) are the zones of indifference. That is to say: if $k \in (k_1,k_2)$ it is for practical purposes irrelevant whether one decides $k \leq k_1$ or $k \geq k_2$. Similarly for $k \in (k_3,k_4)$. It is possible to choose some of the k's equal to k_o resp. k_o'.

Then one performs (usual) sequential tests:

T_{12}: testing the hypothesis $k \leq k_1$ against $k \geq k_2$
T_{34}: " " " $k \leq k_3$ " $k \geq k_4$.

If the g^{th} group of observations consists of the (sample) values: $x_{1,g}, x_{2,g}, \ldots, x_{m,g}$ and $y_{1,g}, y_{2,g}, \ldots, y_{n,g}$, being ranked with the sets of ranks $r_{1,g}, r_{2,g}, \ldots, r_{m,g}$ and $s_{1,g}, s_{2,g}, \ldots, s_{n,g}$ respectively, one evaluates:

$$r_g(k_a,k_b) = \frac{P[s_{1,g},s_{2,g},\ldots,s_{n,g} /m,n,k_a]}{P[s_{1,g},s_{2,g},\ldots,s_{n,g} /m,n,k_b]} \quad , \tag{4}$$

where $P[s_{1,g},s_{2,g},\ldots,s_{n,g} /m,n,k]$ is the probability, given m,n and k, that the y-sample has the given ranks $s_{1,g},s_{2,g},\ldots,s_{n,g}$.

One can prove that:

$$P[s_{1,g},s_{2,g},\ldots,s_{n,g} /m,n,k] = \frac{k^n}{\binom{m+n}{n}} \prod_{j=1}^{n} \frac{\Gamma(s_{j,g}+jk-j)\Gamma(s_{j+1,g})}{\Gamma(s_{j+1,g}+jk-j)\Gamma(s_{j,g})} \tag{5}$$

(cf (3) p. 62).

To carry out T_{12} one determines after G steps:

$$p_{12} = \prod_{g=1}^{G} r_g \,(k_2,k_1), \tag{6}$$

and one concludes either:

$k < k_o$ if $p_{12} \leqslant B$ or

$k \geqslant k_o$ if $p_{12} \geqslant A$,

or one takes a next group of observations if $B < p_{12} < A$.

To carry out T_{34} one determines:

$$p_{34} = \prod_{g=1}^{G} r_g \,(k_4,k_3) \tag{7}$$

and concludes either:

$k < k'_o$ if $p_{34} \leqslant B'$ or

$k \geqslant k'_o$ if $p_{34} \geqslant A'$

or one takes a next group of observations if $B' < p_{34} < A'$.

The test is then as follows. One continuous to take observations as long as either test T_{12} or test T_{34} has not lead to a conclusion. As soon as both tests have lead to conclusions, the test is ended and one gets the following decisions:

1. $k < k_o$ if T_{12} gives the decision $k < k_o$ and T_{34} gives $k < k'_o$
2. $k > k'_o$ if T_{12} gives the decision $k \geqslant k_o$ and T_{34} gives $k > k'_o$
3. $k_o \leqslant k \leqslant k'_o$ if T_{12} gives the decision $k \geqslant k_o$ and T_{34} gives $k \leqslant k'_o$.

Naturally one must choose $\alpha,\beta,\alpha',\beta'$ and k_1,k_2,k_3,k_4 in such a way that it is impossible to decide $k < k_o$ according to T_{12}, and $k > k'_o$ according to T_{34}.

Hence we assume that:

$$A \leqslant A' \quad \text{and} \quad B \leqslant B' \tag{8}$$

or

$$\frac{1-\beta}{\alpha} \leqslant \frac{1-\beta'}{\alpha'} \quad \text{and} \quad \frac{\beta}{1-\alpha} \leqslant \frac{\beta'}{1-\alpha'}$$

Then the following relation is sufficient. For every G and n and every set of ranks $s_{1,1},\ldots,s_{n,G}$:

$$\prod_{g=1}^{G} \left(\frac{k_2}{k_1}\right)^n \frac{\prod_{j=1}^{n} \frac{\Gamma(s_{j,g}+jk_2-j)}{\Gamma(s_{j+1,g}+jk_2-j)}}{\prod_{j=1}^{n} \frac{\Gamma(s_{j,g}+jk_1-j)}{\Gamma(s_{j+1,g}+jk_1-j)}} \geqslant$$

$$\prod_{g=1}^{G} \left(\frac{k_4}{k_3}\right)^n \frac{\prod_{j=1}^{n} \frac{\Gamma(s_{j,g}+jk_4-j)}{\Gamma(s_{j+1,g}+jk_4-j)}}{\prod_{j=1}^{n} \frac{\Gamma(s_{j,g}+jk_3-j)}{\Gamma(s_{j+1,g}+jk_3-j)}} \, . \tag{9}$$

We shall find sufficient conditions for this inequality. These conditions are in general also necessary. A sufficient condition for the above inequality to hold is:

$$\frac{k_2}{k_1} \frac{\frac{\Gamma(s_{j,g}+jk_2-j)}{\Gamma(s_{j+1,g}+jk_2-j)}}{\frac{\Gamma(s_{j,g}+jk_1-j)}{\Gamma(s_{j+1,g}+jk_1-j)}} \geqslant \frac{k_4}{k_3} \frac{\frac{\Gamma(s_{j,g}+jk_4-j)}{\Gamma(s_{j+1,g}+jk_4-j)}}{\frac{\Gamma(s_{j,g}+jk_3-j)}{\Gamma(s_{j+1,g}+jk_3-j)}} \quad (j=1,2,\ldots,n \text{ and } g=1,2,\ldots,G). \tag{10}$$

Let us assume that: $s_{j+1,g} = s_{j,g} + l$ (possible values for l are $1,2,\ldots,m+1$), then we get after cancellations:

$$\frac{k_2}{k_1} \frac{(s_{j,g}+l-1+jk_1-j)(s_{j,g}+l-2+jk_1-j)\ldots\ldots(s_{j,g}+jk_1-j)}{(s_{j,g}+l-1+jk_2-j)(s_{j,g}+l-2+jk_2-j)\ldots\ldots(s_{j,g}+jk_2-j)} \geqslant$$

$$\frac{k_4}{k_3} \frac{(s_{j,g}+l-1+jk_3-j)(s_{j,g}+l-2+jk_3-j)\ldots\ldots(s_{j,g}+jk_3-j)}{(s_{j,g}+l-1+jk_4-j)(s_{j,g}+l-2+jk_4-j)\ldots\ldots(s_{j,g}+jk_4-j)} \, . \tag{11}$$

We can also write this as:

$$\frac{k_2}{k_1}\;\frac{(k_1+\varepsilon_1)(k_1+\varepsilon_2)\ldots.(k_1+\varepsilon_l)}{(k_2+\varepsilon_1)(k_2+\varepsilon_2)\ldots.(k_2+\varepsilon_l)} \gg \frac{k_4}{k_3}\;\frac{(k_3+\varepsilon_1)(k_3+\varepsilon_2)\ldots.(k_3+\varepsilon_l)}{(k_4+\varepsilon_1)(k_4+\varepsilon_2)\ldots.(k_4+\varepsilon_l)} \qquad (12)$$

with:

$$\varepsilon_i \overset{\text{def}}{=} \frac{s_{j,g} + i-1-j}{j} \geqslant 0 \quad (i=1,2,\ldots,l). \qquad (13)$$

It is easy to prove that $\frac{k_1}{k_2} \geqslant \frac{k_3}{k_4}$ implies:

$$\frac{k_1 + \varepsilon_i}{k_2 + \varepsilon_i} \geqslant \frac{k_3 + \varepsilon_i}{k_4 + \varepsilon_i} \qquad (i=1,2,\ldots,l). \qquad (14)$$

because we may write:

$$\frac{(k_1/k_2) + \varepsilon_i/k_2}{1 + \varepsilon_i/k_2} \text{ and } \frac{(k_3/k_4) + \varepsilon_i/k_4}{1 + \varepsilon_i/k_4} \;(i=1,2,\ldots,l). \qquad (15)$$

However, the factors $\frac{k_2}{k_1}$ and $\frac{k_4}{k_3}$ in (11) make it in general impossible to choose $\frac{k_1}{k_2} > \frac{k_3}{k_4}$.

But if one chooses:

1. $\frac{k_2}{k_1} = \frac{k_4}{k_3}$

2. $\frac{1-\beta}{\alpha} \leqslant \frac{1-\beta'}{\alpha'}$

3. $\frac{\beta}{1-\alpha} \leqslant \frac{\beta'}{1-\alpha'}$,

it is certain that one cannot decide according to T_{12} $k < k_o$ before one has decided according to T_{34} that $k \leqslant k'_o$, and similarly one cannot decide according to T_{34} $k > k'_o$ before one has decided according to T_{12} that $k \geqslant k_o$.

4. Properties of the sequential test with three decisions.

This test terminates with probability one, because both tests T_{12} and T_{34} terminate with probability one (cf A. WALD (1947) p. 157).

3. $\frac{\ln B}{mn(p_2-p_1)} < \sum_{g=1}^{G} \frac{U_g - mn\frac{p_1+p_2}{2}}{\sigma^2} < \frac{\ln A}{mn(p_2-p_1)}$ then we take

a next group of observations.

Since we do not know the variance σ^2 of U_g we may at first take two samples of sizes m_o resp. n_o to estimate the variance. Then $\frac{U(m_o,n_o)}{m_o n_o}$ is an unbiased estimate $\hat{p}$ of the true p and

$$m\ n\ \{(m-1)\ \hat{\phi}^2 + (n-1)\ \hat{\gamma}^2 + \hat{p}_2\ (1-\hat{p})\} \qquad (19)$$

is an estimate of σ^2, with:

$$\hat{\phi}^2 \overset{def}{=} \int_0^1 \{F(t) - 1 + \hat{p}\}^2\ dF^{k_o}(t) = \frac{k_o}{k_o+2} - \frac{2k_o(1-\hat{p})}{k_o+1} + (1-\hat{p})^2 \qquad (20)$$

$$\hat{\gamma}^2 \overset{def}{=} \int_0^1 \{F^{k_o}(t) - \hat{p}\}^2\ dF(t) = \frac{1}{2k_o+1} - \frac{2\hat{p}}{k_o+1} + \hat{p}^2\ . \qquad (21)$$

To make a test with three decisions, one can use SOBEL and WALD's method.

$$\sum \left[\frac{k_2}{k_1} \frac{\prod_{j=1}^{n} \frac{\Gamma(s_j+jk_2-j)}{\Gamma(s_{j+1}+jk_2-j)}}{\prod_{j=1}^{n} \frac{\Gamma(s_j+jk_1-j)}{\Gamma(s_{j+1}+jk_1-j)}} \right]^{h(k)} \frac{k^n}{\binom{m+n}{n}} \frac{\Gamma(s_j+jk-j)\ (s_{j+1})}{\Gamma(s_{j+1}+jk-j)\ (s_j)} = 1, \quad (18)$$

where the summation is extended over all possible configurations.

A second important characteristic is the avarage sample number function ASN(k). In this case it gives the expected number of groups required by the test.

To get a rough sketch of the ASN-function, one can use the following approximations:

$$\begin{array}{lll} k \leq k_2 & \text{then} & ASN(k) \approx ASN_{12}(k) \\ k \geq k_3 & \text{"} & \text{"} \approx ASN_{34}(k), \end{array}$$

where $ASN_{12}(k)$ and $ASN_{34}(k)$ are the ASN-function of the tests T_{12} and T_{34} respectively. In (k_2,k_3) ASN(k) possesses a local minimum.

If one wants to use the appropriate formulae in A. WALD (1947) p. 53 and p. 176, one also needs $L_{12}(k)$ and $L_{34}(k)$.
If one should be able to evaluate $L_{12}(k)$ and $L_{34}(k)$, it would also be possible to use some results of M. SOBEL and A. WALD (1949) p. 513 ff.

5. Normal approximation.

For simplification, when we use small zones of indifference and large samples, it may be convenient to use the normal approximation.

Then we get for the usual sequential test (H_0: $p \leq p_0$, H_1: $p > p_0$) the following decision rules (U_g is the Wilcoxon statistic of the g^{th} group; p_1 and p_2 are chosen on both sides of p_0):

1. $$\sum_{g=1}^{G} \frac{U_g - mn\,\frac{p_1+p_2}{2}}{\sigma^2} \geq \frac{\ln A}{mn(p_2-p_1)} \quad \text{then: } p > p_0$$

2. $$\sum_{g=1}^{G} \frac{U_g - mn\,\frac{p_1+p_2}{2}}{\sigma^2} \leq \frac{\ln B}{mn(p_2-p_1)} \quad \text{then: } p \leq p_0$$

3. $\frac{\ln B}{mn(p_2-p_1)} < \sum_{g=1}^{G} \frac{U_g - mn\frac{p_1+p_2}{2}}{\sigma^2} < \frac{\ln A}{mn(p_2-p_1)}$ then we take a next group of observations.

Since we do not know the variance σ^2 of U_g we may at first take two samples of sizes m_o resp. n_o to estimate the variance. Then $\frac{U(m_o,n_o)}{m_o n_o}$ is an unbiased estimate $\hat{p}$ of the true p and

$$m\ n\ \left\{(m-1)\ \hat{\phi}^2 + (n-1)\ \hat{\gamma}^2 + \hat{p}_2\ (1-\hat{p})\right\} \qquad (19)$$

is an estimate of σ^2, with:

$$\hat{\phi}^2 \overset{def}{=} \int_0^1 \{F(t) - 1+\hat{p}\}^2\ dF^{k_o}(t) = \frac{k_o}{k_o+2} - \frac{2k_o(1-\hat{p})}{k_o+1} + (1-\hat{p})^2 \qquad (20)$$

$$\hat{\gamma}^2 \overset{def}{=} \int_0^1 \{F^{k_o}(t) - \hat{p}\}^2\ dF(t) = \frac{1}{2k_o+1} - \frac{2\hat{p}}{k_o+1} + \hat{p}^2\ . \qquad (21)$$

To make a test with three decisions, one can use SOBEL and WALD's method.

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