STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

AFDELING MATHEMATISCHE STATISTIEK

S 321

Ballot Problems

by

F. Göbel

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Summary

In each of the sections 1 and 2 an extension is presented of a well-known result on Ballots, the CHUNG-FELLER theorem.

In section 3 two recurrence relations are obtained for the number of paths that do not overshoot a monotone but otherwise arbitrary boundary function.

0. Preliminaries to sections 1 and 2.

In a ballot with two candidates A and B, let the total number of votes for A be denoted by a, and the total for B by b. It will always be assumed that each of the $\binom{a+b}{b}$ possible arrangements of the votes has probability $\binom{a+b}{b}^{-1}$.

Let a_m be the number of votes for A after m votes have been counted, and let $b_m=m-a_m$.

The counting process can be completely described by a path in a two-dimensional lattice of points (x,y) where x and y are non-negative integers. To be explicit: the situation $a_m=x$, $b_m=y$ is represented by the point (x,y). Consecutive points are joined by line segments, which will be termed sides of the path.

In section 1 and 2 we will consider, roughly speaking, the distribution of the number of steps above a given line, which is either the line x=y, or the line bx=ay. These lines will be denoted by L and D, respectively. When a=b, L coincides with D, and for this case FELLER and CHUNG [1] proved that the number of paths with exactly 2j sides above L is equal to

$$\frac{1}{a+1}\binom{2a}{a} \qquad (j=0,\ldots,a) \qquad (1)$$

independent of j. In section 1, the distribution of the number of sides above L is derived for the case $a>b$.

When, instead of L, the line D is considered, we have the difficulty that a side of the path may be only partly above D. It will be shown in section 2 that, when $b|a$ (and, of course, also when $a|b$), the length of the path above D has a homogeneous distribution.

1. First extension

The problem of determining the distribution of the number of sides above L is mentioned, in a slightly different formulation, in [2]. It can be solved by a method, that has also been used by WHITWORTH |3| for similar questions.

It can be assumed without restriction that $a > b$. Let $N_j(a,b)$ be the number of paths that have exactly $2j$ sides above L.

Consider all paths for which (i,i) is the last point in common with L. Then, from $(0,0)$ to (i,i) there are $\frac{1}{i+1}\binom{2i}{i}$ paths with exactly $2j$ sides above L (provided $j \leq i$). From (i,i) to (a,b) there are $\frac{a-b}{a+b-2i}\binom{a+b-2i}{b-i}$ paths that have no point in common with L, according to a classical ballot formula. Finally, the total number of paths that have exactly $2j$ sides above L is obtained by summing the product of the two above-mentioned expressions over all admissible values of i, that is: from j to b, inclusive. Hence

$$\boxed{N_j(a,b) = \sum_{i=j}^{b} \frac{1}{i+1}\binom{2i}{i}\frac{a-b}{a+b-2i}\binom{a+b-2i}{b-i}} \qquad (2)$$

Putting $j=0$ in (2), an alternative expression is obtained for the number $\frac{a+1-b}{a+1}\binom{a+b}{b}$ of paths with all sides below L, and we have the identity

$$\sum_{i=0}^{b} \frac{1}{i+1}\binom{2i}{i}\frac{a-b}{a+b-2i}\binom{a+b-2i}{b-i} = \frac{a+1-b}{a+1}\binom{a+b}{b}, \qquad (3)$$

which will be of use in the next sub-section.

1.1 Asymptotic properties of $N_j(a,b)$

The probability $p_j(a,b)$ that a randomly chosen path has $2j$ sides above L is is given by

$$p_j(a,b) = N_j(a,b)\cdot\binom{a+b}{b}^{-1} \qquad (4)$$

We will let a and b tend to infinity in such a way that $\frac{a}{b}$ tends to a constant value $\lambda > 1$. The corresponding limiting values of $p_j(a,b)$ will be denoted by $p_j(\lambda)$.

From (2) and (3) it follows that

$$p_j(\lambda) = \lim \left\{ \frac{a+1-b}{a+1} - \sum_{i=0}^{j-1} \frac{1}{i+1} \binom{2i}{i} \frac{a-b}{a+b-2i} \binom{a+b-2i}{b-i} \binom{a+b}{b}^{-1} \right\} =$$

$$= \frac{\lambda-1}{\lambda} - \frac{\lambda-1}{\lambda+1} \sum_{i=0}^{j-1} \frac{1}{i+1} \binom{2i}{i} \lim \frac{a!}{(a-i)!} \frac{b!}{(b-i)!} \frac{(a+b-2i)!}{(a+b)!} \quad ,$$

whence

$$p_j(\lambda) = \frac{\lambda-1}{\lambda} - \frac{\lambda-1}{\lambda+1} \sum_{i=0}^{j-1} \frac{1}{i+1} \binom{2i}{i} \left\{ \frac{\lambda}{(\lambda+1)^2} \right\}^i \qquad (5)$$

A slightly different formula for $p_j(\lambda)$ may be obtained as follows. The series

$$\sum_0^\infty \binom{2i}{i} x^i = (1-4x)^{-\frac{1}{2}}$$

is uniformly convergent when $|x| \leqslant |\xi| < \frac{1}{4}$, so it may be integrated term by term:

$$\sum_0^\infty \frac{1}{i+1} \binom{2i}{i} x^i = \frac{1-(1-4x)^{\frac{1}{2}}}{2x} \qquad (6)$$

If we substitute $x = \frac{\lambda}{(\lambda+1)^2}$ into (6), we see that [1]

$$\sum_0^\infty \frac{1}{i+1} \binom{2i}{i} \left\{ \frac{\lambda}{(\lambda+1)^2} \right\}^i = \frac{\lambda+1}{\lambda} \quad , \qquad (7)$$

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1) Actually, the result is $\frac{(\lambda+1)^2-(\lambda+1)\sqrt{(\lambda-1)^2}}{2\lambda}$, which is $\frac{\lambda+1}{\lambda}$ when $\lambda > 1$, and $\lambda+1$ when $0 < \lambda < 1$. This explains the "contradiction" that, for $\lambda=0$, the left side of (7) is convergent, whereas the right hand side is infinite.

whence [2)]

$$p_j(\lambda)=\frac{\lambda-1}{\lambda+1}\sum_{i=j}^{\infty}\frac{1}{i+1}\binom{2i}{i}\left\{\frac{\lambda}{(\lambda+1)^2}\right\}^i \tag{8}$$

In order to determine the generating function of $p_j(\lambda)$,

$$P(x,\lambda)=\sum_{j=0}^{\infty} p_j(\lambda)x^j \tag{9}$$

first consider

$$q_i(\lambda)=\frac{1}{i+1}\binom{2i}{i}\left\{\frac{\lambda}{(\lambda+1)^2}\right\}^i \tag{10}$$

The quickest way to find $Q(x,\lambda)=\Sigma q_i(\lambda)x^i$ is to replace x in (6) by $\frac{\lambda x}{(\lambda+1)^2}$:

$$Q(x,\lambda)=\frac{(\lambda+1)^2-(\lambda+1)\sqrt{(\lambda+1)^2-4\lambda x}}{2\lambda x} \tag{11}$$

Using (8) and a derivation similar to that in [4], p.249, it follows from (11) and the relation between $p_j(\lambda)$ and $q_j(\lambda)$ that

$$P(x,\lambda)=\frac{(\lambda-1)\sqrt{(\lambda+1)^2-4\lambda x}-(\lambda-1)^2}{2\lambda(1-x)} \tag{12}$$

$P(x,\lambda)$ may be written as a power series in $1-x$:

$$P(x,\lambda)=1-\frac{\lambda(1-x)}{(\lambda-1)^2}+\frac{2\lambda^2(1-x)^2}{(\lambda-1)^4}-\dots+(-1)^i\frac{1}{i+1}\binom{2i}{i}\left\{\frac{\lambda(1-x)}{(\lambda-1)^2}\right\}^i+\dots \tag{13}$$

From (13) it follows that $P'(1,\lambda)=\frac{\lambda}{(\lambda-1)^2}$, i.e. in the limiting case, the average number of sides above L is

$$\boxed{M(\lambda)=\frac{2\lambda}{(\lambda-1)^2}} \tag{14}$$

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2) I am indebted to Mr. W. van ZWET who pointed out an error in an earlier version of the proof of (8), and suggested some ways of improving it.

1.2 Some numerical results

For finite a and b, $a > b$, the average number of sides above L is

$$M(a,b)= \sum_{1}^{b} i\binom{2i}{i} \frac{a-b}{a+b-2i} \binom{a+b-2i}{b-i} \binom{a+b}{b}^{-1} \qquad (15)$$

as can be easily verified from (2).

With the aid of (15), $M(2b,b)$ has been computed for some values of b. The results are presented below. For $b \geqslant 8$, the X1 computer has been used. It is seen that the convergence to the limiting value 4 is very slow.

b	M(2b,b)	b	M(2b,b)
1	0.6667	8	2.1680
2	1.0667	9	2.2681
3	1.3571	10	2.3568
4	1.5838		
5	1.7682	17	2.7805
6	1.9224	25	3.0512
7	2.0534	40	3.3259

Table 1

M(2b,b) for some values of b

2. Second extension.

As stated before, is this section we will consider the distribution of $\underline{l}$, the length of the part above D of a randomly chosen path. Let us first look into the values that can be taken on by $\underline{l}$ for arbitrary integers a and b. It is easily shown that

$$P[\underline{l}=l] > 0 \text{ implies } l= \frac{k(a+b)}{[a,b]}$$

where $[a,b]$ is the l.c.m. of a and b, and k is integer with $0 \leqslant k \leqslant [a,b]$. Consider fig. 1.

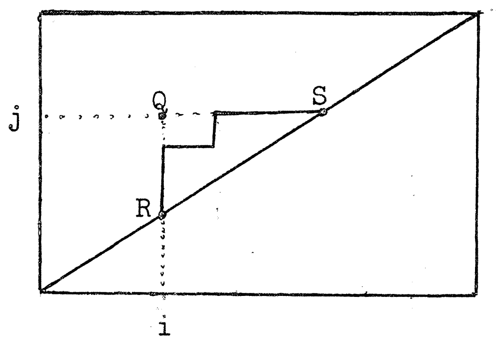

Figure 1

The part above D of any path may be divided in pieces that lie entirely above D, except for their end-points. The length of such a piece equals

$$QR+QS=j-\frac{bi}{a}+\frac{aj}{b}-i=(a+b)\left(\frac{j}{b}-\frac{i}{a}\right),$$

where i and j are the coordinates of Q. Multiplication of the second factor by $[a,b]$ yields an integer, hence QR+QS can be written in the form $\frac{k(a+b)}{[a,b]}$, and the same is true for the sum of a number of these lengths.

It follows that in the special case $b|a$ the possible values for $\underline{l}$ are $0, \frac{a+b}{a}, 2.\frac{a+b}{a}, \ldots, a\frac{a+b}{a}$. In this case, for a randomly chosen path, each of these values has probability $\frac{1}{a+1}$. The proof, an outline of which will be given below, is along the same lines as that by HODGES [5] of the CHUNG-FELLER theorem. Instead of $\underline{l}$ the variable $\underline{k}$, as defined above, will be used.

The basic idea is to prove the existence of a one-to-one mapping of paths with $\underline{k}=k>0$ onto paths with $\underline{k}=k-1$. Consider an arbitrary path with $\underline{k}=k>0$. Let P be the first point (excluding the origin) on the path which either lies on D, or has the property that the side starting at P has an interior point in common with D [1)]. There are two cases to be considered:

I The path has points above D between P and (a,b)

II Between P and (a,b) the path is entirely below D.

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1) From $b|a$ it follows that P is necessarily below D in the latter case.

In case I the mapping is as follows: between the origin and P the path is left as it is; on the part between P and (a,b) the mapping is again applied . This inductive definition may cause difficulties only when P is strictly below D, as in fig 2. In a situation like this the mapping be applied as if D had a pit near P, as indicated in fig 2.

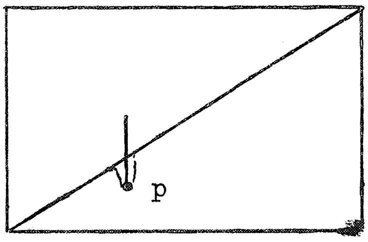

Figure 2

In case II the first side of the path can only be vertical, and the side that has P as its end-point must be horizontal. Also, P lies on D in this case. Hence, using a self-explanatory notation, the path is of the form vW_1hW_2 where P separates h and W_2. The mapping changes this into hW_2vW_1. It remains to be shown that every path with $k > 0$ has a uniquely determined image, that the length is always diminished by 1 (in terms of k), and that the inverse image exists and is unique. We will not enter into the details, except for noting that an additional complication, as compared to the case a=b, is to prove that when II applies, W_1 has no points below D after being interchanged with W_2.

When neither of a and b divides the other, the distribution of k is no longer homogeneous (in general), and very little more can be said. It is to be expected from GROSSMAN's formula [1)] (cf [6]) for the number of paths that have no point above D, that the answer will be complicated. In the case: a is odd, b=2, the distribution is a mixture of "triangular" distributions, as exemplified by fig 3. This can be

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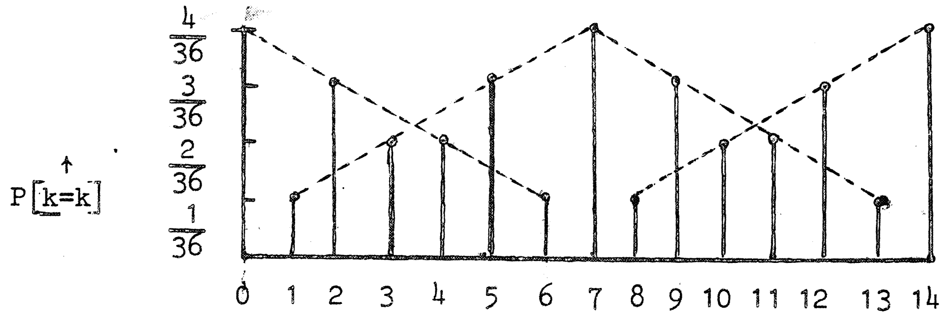
1) Proved by BIZLEY [7].



k→ Fig. 3 P[k=k] for a=7, b=2

shown by considering k as a function of the places of the (two) vertical sides of the path.

3. Two recurrence relations

Let $f_0, f_1, \ldots$ be a non-decreasing sequence of positive integers, and let N_r be the number of paths from (0,0) to (r, f_r) with the property that (r, f_r) is the first point (i, f_i) on the path. Such paths will be called permitted paths to (r, f_r). For $r+f_r$ the abbreviation h_r is used.

Theorem 1. N_r satisfies the recurrence

$$N_r = \binom{h_r}{r} - \sum_{i=0}^{r-1} \binom{h_r - h_i}{r-i} N_i \qquad (16)$$

Proof. The number of all paths to (r, f_r) is $\binom{h_r}{r}$. The non-permitted paths may be distinguished by the first point (i, f_i) on it. The number of such paths is N_i times $\binom{h_r - h_i}{r-i}$, and the theorem follows.

Perhaps less trivial is the following

Theorem 2. N_r satisfies the recurrence

$$\sum_{i=0}^{r} (-1)^i N_i \binom{f_i}{r-i} = 0 \quad (r > 0) \qquad (17)$$

Proof. Consider the right hand end points of the horizontal steps of a path. These points constitute a non-decreasing function y_i defined on $\{1,\ldots,r\}$. y_0 is defined as 0.

For a function corresponding to a permitted path we have

$$y_i < f_{i-1}, \quad i=1,\ldots,r \tag{18}$$

We now define a more general type of functions, viz. functions that satisfying (18) and

$$y_0 \leqslant y_1 \leqslant \ldots \leqslant y_j \tag{19}$$

$$y_j > y_{j+1} > \ldots > y_r \tag{20}$$

for some value of j. These functions will be termed j-sequences. Let A_j be the number of j-sequences (r is fixed). As $y_0=0$ cannot exceed y_1, we have $A_0=0$. Also, $A_r=N_r$.

Now both the j-sequences and the (j+1)-sequences have properties (18),(19), and

$$y_{j+1} > y_{j+2} > \ldots > y_r \tag{21}$$

Conversely, a function satisfying (18),(19) and (21) must be either a j-sequence or a (j+1)-sequence. The number of functions having properties (18),(19) and (21) is equal to $N_j\binom{f_j}{r-j}$, the second factor being the number of ways in which the r-j values $y_{j+1},\ldots,y_r$ can be chosen to satisfy (21) and $y_{j+1} < f_j$. Hence

$$N_j\binom{f_j}{r-j} = A_j + A_{j+1} \tag{22}$$

From (22) it follows that

$$\sum_{j=0}^{r-1} (-1)^j N_j\binom{f_j}{r-j} = \sum_{j=0}^{r-1} (-1)^j (A_j + A_{j+1}) = (-1)^{r-1}A_r = (-1)^{r-1}N_r \tag{23}$$

and theorem 2 is proved.

Application:

The classical ballot problem of "weak sense lead throughout" is equivalent to the case $f_i = \min(i+1, b+1)$ where b is the number of votes obtained by the loser. If both candidates obtain the same number of votes, $N_i = \frac{1}{i+1} \binom{2i}{i}$, whence the identity

$$\sum_{i=0}^{r} \frac{(-1)^i}{i+1} \binom{2i}{i} \binom{i+1}{r-i} = 0 \quad (r > 0) \tag{24}$$

References

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