STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

AFDELING MATHEMATISCHE STATISTIEK

Report S 321a

# Some remarks on ballot problems

by

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(Revision of S 321)

Paper presented at the I.M.S.-conference
at Berne (14-18 September 1964)

August 1964

Printed at the Mathematical Centre at Amsterdam,49,2nd Boerhaavestraat.
The Netherlands.

The Mathematical Centre, founded the 11th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications, and is sponsored by the Netherlands Government through the Netherlands Organization for Pure Scientific Research (Z.W.O.) and the Central National Council for Applied Scientific Research in the Netherlands (T.N.O.), by the Municipality of Amsterdam and by several industries.

Summary

This report consists of two parts. In the first part a combinatorial proof by HODGES of the symmetric version of the CHUNG-FELLER theorem will be used to extend this symmetric case in such a way that the homogeneity of the distribution concerned is preserved.

In the second part some relations are given for the number of paths below an increasing but otherwise arbitrary function. If this function is linear, a result is obtained that is closely related to a formula by PÓLYA for the sum $\sum_{j=0}^{\infty} \binom{\alpha j+\beta}{j} x^j$ .

1. In a ballot with two candidates A and B, let the total number of votes for A be denoted by a, and the total for B by b. It will always be assumed that each of the $\binom{a+b}{b}$ possible arrangements of the votes has probability $\binom{a+b}{b}^{-1}$.

Let $a_m$ be the number of votes for A after m votes have been counted, and let $b_m = m - a_m$.

The situation $(a_m, b_m)$ may be represented by a point in a two-dimensional lattice of points with integer-valued non-negative co-ordinates. A lattice point that corresponds to a situation occurring during the counting will be called a <u>vertex</u>. Consecutive vertices may be joined by line segments (<u>sides</u>) to form a <u>path</u> from the origin to (a,b) representing the counting. The line y=cx will be denoted by L(c). It will be assumed that $a \geq b$.

CHUNG and FELLER [1] have determined the distribution of $\underline{k}$, the number of sides above L(1) of a path chosen at random. In our notation their result is: if $a > b$, then[1)]

(1) $$P[\underline{k}=2j] = \sum_{i=j}^{b} \frac{1}{i+1} \binom{2i}{i} \frac{a-b}{a+b-2i} \binom{a+b-2i}{b-i} \qquad (j=0,\dots,b)$$

and if a=b, then

(2) $$P[\underline{k}=2j] = \frac{1}{b+1} \qquad (j=0,\dots,b)$$

The original proof by CHUNG and FELLER, which is rather complicated, has been simplified considerably by HODGES [2] who gave a combinatorial proof of (2). Once (2) is known, it is not difficult to find (1), hence an elementary proof of (1) can be given.

In the next section it will be shown that (2) can be extended in such a way that the homogeneous distribution is preserved.

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1) The restriction "a+b even", made by CHUNG and FELLER presumably to keep their formulae simpler, has no influence on the form of (1) in the present notation, and can be dropped.

1.1 Instead of L(1), we will now consider the line L(b/a)=D. There are several ways of generalizing the variable "number of sides above L(1)" to a quantity that can be used in the present case. We choose the length $\underline{k}$ above D of the path.

Let us first look into the values that can be taken on by $\underline{k}$ for arbitrary integers a and b. The part above D of any path may be divided in pieces that lie entirely above D, except for their end-points. The length of such a piece (cf. fig. 1) equals

$$QR + QS = y - \frac{bx}{a} + \frac{ay}{b} - x = (a+b)\left(\frac{y}{b} - \frac{x}{a}\right)$$

where x and y are the coordinates of Q. Multiplication of the

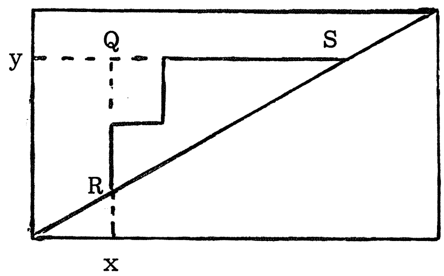



Figure 1.

second factor by $\left[a,b\right]$ (the least common multiple of a and b) yields an integer, hence QR + QS can be written in the form $\frac{j(a+b)}{[a,b]}$ where j is an integer, and the same is true for the sum of a number of these lengths. By considering the extreme values of the length above D of a path one then sees that

$$P\left[\underline{k} = k\right] > 0 \quad \text{implies} \quad k = \frac{j(a+b)}{[a,b]}\ , \quad 0 \leqslant j \leqslant \left[a,b\right].$$

From now on, we will confine ourselves to the case where a is a multiple of b. The possible values for $\underline{k}$ are now restricted to 0, $\frac{a+b}{a}$, $2\,\frac{a+b}{a}, \ldots, a.\ \frac{a+b}{a}$ .

In this case, for a path chosen at random, each of these values has the probability $\frac{1}{a+1}$ . The proof, an outline of which will be given below, is based on that by HODGES for the symmetric case of the CHUNG-FELLER theorem. Instead of $\underline{k}$ the variable $\underline{j}$, as defined above, will be used.

The basic idea is to prove the existence of a one-to-one mapping[1] of paths with $\underline{j} = j > 0$ onto paths with $\underline{j} = j-1$. Consider an arbitrary path with $\underline{j} = j > 0$.

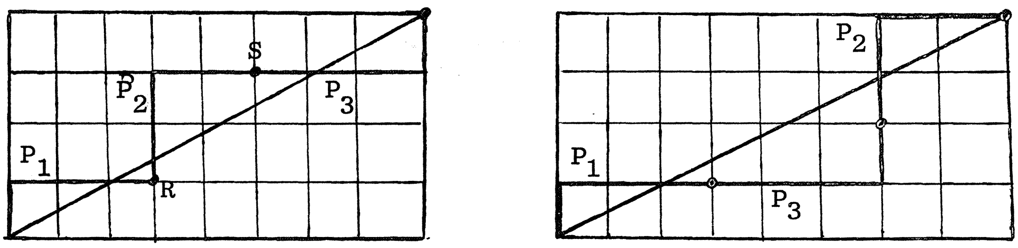
Let S and R be vertices of this path such that

(a) S is the last vertex above D,

(b) R is the last vertex preceding S that lies on or below D.

The existence of S is ensured by $j > 0$, the existence of R is trivial.

Let $P_1$ be the path from the origin to R, $P_2$ the path from R to S, and $P_3$ the path from S to (a,b). The mapping then changes $P=P_1P_2P_3$ into $P'=P_1P_3P_2$ , as exemplified by fig. 2.




Original, $\underline{j}=5$ Image, $\underline{j}=4$

Fig. 2.

It remains to be shown that the length of a path is always diminished by 1 (in terms of j), and that the inverse image exists and is unique.

The first assertion can easily be verified by considering the first sides $F_2$ and $F_3$ of $P_2$ and $P_3$, respectively. $F_2$ is a vertical side, and the length of its part above D is diminished by an amount $\frac{b}{a}$ by the mapping. $F_3$ is a horizontal side above D, and its image lies below D. After verifying that the points of $P_2$ not belonging to $F_2$ remain

1) Thanks are due to Dr. W.R. van ZWET who observed that in my adaptation of HODGES' proof, the inductive definition of the mapping could be simplified to the present one.

above D, one concludes that k is diminished by $1 + \frac{b}{a}$ , and hence j by 1.

The proof of the existence and uniqueness of the inverse mapping is easy , but tedious, and will be omitted.

When neither of a and b divides the other, the distribution of $\underline{j}$ is no longer homogeneous (in general), and very little more can be said. It is to be expected from GROSSMAN's formula [1)] (cf [3] ) for the number of paths that have no point above D, that the answer will be complicated. In the cases: a is odd, b=2, the distribution is a mixture of "triangular" distributions, as exemplified by fig. 3.

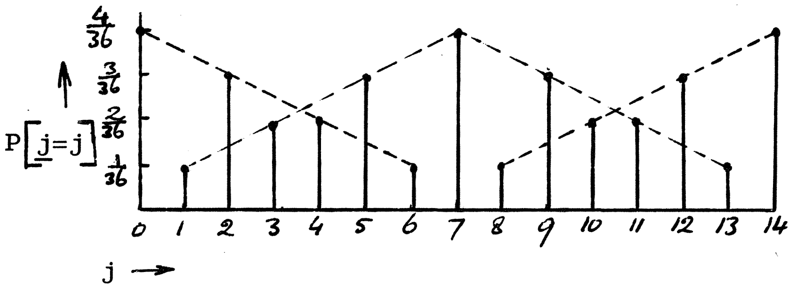



Fig. 3

$P[\underline{j} = j]$ for a = 7, b = 2.

1.2 It may be noted here that one of the distributions that TÁKACS has determined in his article [5] is also a homogeneous distribution. His result is as follows.

Suppose $a = \mu b + 1$, where $\mu$ is a positive integer, and let $P_j$ be the probability that the inequality $a_r > \mu b_r$ holds for exactly j values among r = 1, ... ,a+b. Then

$$P_j = \frac{1}{a+b}, \quad j = 1, \ldots, a+b$$

1) Proved by BIZLEY [4].

2. Let $f_0$, $f_1$,... be a non-decreasing sequence of positive integers, and let $N_m$ be the number of paths from (0,0) to $(m,f_m)$ with the property that $(m,f_m)$ is the first point $(i,f_i)$ on the path. Such paths will be called <u>permitted paths</u> to $(m,f_m)$. For $m + f_m$ the abbreviation $h_m$ is used.

<u>Theorem</u> $N_m$ satisfies the recurrence

$$\boxed{\sum_{i=0}^{m} (-1)^i \; N_i \binom{f_i}{m-i} = 0,} \qquad (m > 0) \tag{3}$$

<u>Proof</u>. Consider the right hand end points of the horizontal steps of a path. These points constitute a non-decreasing sequence of non-negative integers $y_i$ $(i=1,\ldots,m)$.

For a sequence corresponding to a permitted path we have

$$y_i < f_{i-1}, \quad i = 1,\ldots,m \tag{4}$$

We now define a more general type of sequences, viz. sequences of non-negative integers that satisfy (4), $y_0 = 0$, and

$$y_0 \leq y_1 \leq \ldots \leq y_j \tag{5}$$

$$y_j > y_{j+1} > \cdots > y_m \tag{6}$$

for some value of j. These sequences will be termed j-sequences.

Let $A_j$ be the number of j-sequences (m is fixed). As $y_o = 0$ cannot exceed $y_1$, we have $A_0 = 0$. Also, $A_m = N_m$.

Now both the j-sequences and the (j+1)-sequences have properties (4), (5), and

$$y_{j+1} > y_{j+2} > \cdots > y_m \tag{7}$$

Conversely, a sequence of non-negative integers with $y_o = 0$, satisfying (4), (5) and (7) must be either a j-sequence or a (j+1)-sequence. The number of sequences of non-negative integers with $y_o = 0$ having properties (4), (5) and (7) is equal to $N_j\binom{f_j}{m-j}$, the second factor being the number of ways in which the m-j values $y_{j+1},\ldots,y_m$ can be chosen to satisfy (7) and $y_{j+1} < f_j$. Hence

(8) $$N_j \binom{f_j}{m-j} = A_j + A_{j+1} .$$

From (8) it follows that

$$\sum_{j=0}^{m-1} (-1)^j N_j \binom{f_j}{m-j} = \sum_{j=0}^{m-1} (-1)^j (A_j+A_{j+1}) = (-1)^{m-1} A_m = (-1)^{m-1} N_m,$$

and the theorem is proved.

2.1 As an application consider the classical ballot problem of "weak sense lead throughout ". This is equivalent to the case $f_i = \min(i+1, b+1)$ where b is the number of votes obtained by the loser. If both candidates obtain the same number of votes, $N_i$ is equal to

$\frac{1}{i+1}\binom{2i}{i}$, and we have the identity

$$\sum_{i=0}^{r} \frac{(-1)^i}{i+1} \binom{2i}{i}\binom{i+1}{r-i} = 0 \qquad (r > 0)$$

2.2 $N_m$ also satisfies the relation

(9) $$N_m = \binom{h_m}{m} - \sum_{i=0}^{m-1} \binom{h_m - h_i}{m - i} N_i$$

The proof is immediate: the number of all paths to $(m, f_m)$ is $\binom{h_m}{m}$. The non-permitted paths may be distinguished by the first point $(i, f_i)$ on it. The number of such paths is $N_i$ times $\binom{h_m - h_i}{m - i}$, and (9) follows.

2.3 By induction it can be proved that

(10) $$N_m = \sum_{\mathcal{L}(m)} (-1)^{m+k(\mathcal{L})} \binom{f_o}{c_1} \binom{f_{c_1}}{c_2} \binom{f_{c_1} + c_2}{c_3} \cdots$$

where the sum is over all compositions (ordered partitions) $\mathcal{L}(m)$ of m; $c_1$, $c_2$, ... are the parts of the composition, $k(\mathcal{L})$ is the number of parts.

Defining

(11) $$\mathcal{N}(t) = \sum_{m=0}^{\infty} N_m t^m$$

[5] L. TAKÁCS, The Distribution of Majority Times in a Ballot; Z. Wahrscheinlichkeitstheorie 2 (1963), p.118-121.

[6] G. PÓLYA, Sur les séries entières, dont la somme est une fonction algébrique; l'Enseignement Mathématique 22 (1922), p.38-47.

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