First draft: On the use of the method of collective marks in queuing theory

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The method of collective marks was introduced in Van Dantzig (1947, 1948) and applied by him and his pupils to derive generating functions by means of probabilistic interpretations. In queuing problems the method has been used in Kesten and Runnenburg (1957), Bloemena (1958) and Runnenburg (1958). Because the method is only known to very few people, it seems very desirable to use the present opportunity to indicate what can be done with it. The main advantage of the method is, that it supplies a simple approach to simple problems and gives some insight as to why different problems have nearly the same answer. Moreover, one always knows what is happening from the probabilistic point of view.

In this paper a number of known results are again derived, but here we use each time a suitable interpretation to get quickly at the desired generating functions. In order to do that we have to generalize the original problems somewhat. Two methods are used to this end:

I. We put a mark on customers. Each customer has probability $1-X$ of being marked and probability X of remaining unmarked. X may have any value in the interval $[0,1]$. Customers are marked independently and the marking is independent of the (original) process studied. With this marking we derive generating functions with X as generating variable. By analytic continuation our results can be extended to hold for complex X with $|X| \leq 1$ as well. We say: a) "no C_X present at [2)] $\underline{t}_n$" to describe the event that none of the customers present at time $\underline{t}_n$ is marked, b) "no C_X in $\underline{w}_n$" to describe the event that during the interval of time the n^{th} customer is waiting no marked customers arrive (the n^{th} customer himself is excluded).

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1) Report S 327 of the Statistical Department of the Mathematical Centre.

2) Random variables are underlined.

II. <u>We consider an extra Poisson process producing catastrophes</u>. This process has parameter ξ (sometimes <u>two</u> independent Poisson processes with parameters ξ and η are used at the same time) and does not depend on the (original) process studied. Here of necessity we have $\xi \geq 0$, but our results can be extended to hold for complex ξ with Re $\xi \geq 0$ by analytic continuation. We say: "no C_ξ in $\underline{w}_n$" to describe the event, that during the interval of time the n^{th} customer is waiting the Poisson process does not produce a catastrophe.

We do not intend to give a theory, but show our point of view by treating examples, which can be confronted with the existing literature.

<u>Example 1: Infinitely many counters</u>

At time 0 there are k customers present at infinitely many counters. New customers arrive in a Poisson stream with parameter λ and all servicetimes are exponentially distributed with parameter μ. All arrivalintervals and servicetimes are independent. We write $p_n(t)$ for the probability, that exactly n customers are present at time t and ask for

$$(1.1) \qquad p(t,X) = \sum_{n=0}^{\infty} p_n(t) X^n .$$

Customers are considered marked in accordance with method I. Hence each has probability X of being unmarked. Clearly p(t,X) is the probability, that there are no marked customers present at time t. With probability $e^{-\lambda t} \frac{(\lambda t)^n}{n!}$ exactly n customers arrive in the timeinterval (0,t]. The moments of arrival of these customers may be regarded as independent drawings $\underline{h}_1, \underline{h}_2, \ldots, \underline{h}_n$ from a rectangular distribution over (0,t] under the condition that there are exactly n. Hence each of these customers has probability

$$(1.2) \qquad 1-(1-X)P\{\underline{h}+\underline{s}>t\} = 1-(1-X)\frac{1-e^{-\mu t}}{\mu t}$$

of not being a marked customer present at time t. Here $\underline{h}$ and $\underline{s}$ are independent with

$$(1.3) \qquad P\{\underline{h} \leq h\} = \frac{h}{t} \text{ for } 0 \leq h \leq t, \quad P\{\underline{s} \leq s\} = 1-e^{-\mu s} \text{ for } s \geq 0.$$

A customer present at time 0 is with probability

$$1-(1-X)P\{\underline{s} > t\} = 1-(1-X)e^{-\mu t} \tag{1.4}$$

not a marked customer present at time t. But then

$$p(t,X) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \left(1-(1-X) \frac{1-e^{-\mu t}}{\mu t}\right)^n (1-(1-X)e^{-\mu t})^k. \tag{1.5}$$

<u>Example 2: Discrete aspects of M/G/1 queue</u>

Customers arrive at a counter to be served in the order of arrival. The waitingtime $\underline{w}_1$ of the first customer has a given distribution

$$C_1(w) = P\{\underline{w}_1 \leq w\}, \tag{2.1}$$

the arrivalintervals $\underline{y}_1, \underline{y}_2, \ldots$ between successive customers satisfy

$$P\{\underline{y}_n \leq y\} = 1-e^{-\lambda y} \qquad (y \geq 0), \tag{2.2}$$

where λ is a positive constant and the servicetimes $\underline{s}_1, \underline{s}_2, \ldots$ all have distribution

$$B(s) = P\{\underline{s} \leq s\} \tag{2.3}$$

with $B(0-) = 0$. The random variables $\underline{w}_1, \underline{y}_1, \underline{y}_2, \ldots, \underline{s}_1, \underline{s}_2, \ldots$ are independent. The n^{th} customer arrives at time $\underline{t}_n$ at the counter, where

$$\begin{cases} \underline{t}_n = \underline{y}_1 + \ldots + \underline{y}_{n-1} & (n > 1), \\ \underline{t}_1 = 0. \end{cases} \tag{2.4}$$

We write ($\mathcal{E}$ = mathematical expectation)

$$\gamma_n(\xi) = \mathcal{E}\, e^{-\xi \underline{w}_n} \qquad (\text{Re}\, \xi \geq 0) \tag{2.5}$$

and

$$\beta(\xi) = \mathcal{E}\, e^{-\xi \underline{s}_n} \qquad (\text{Re}\, \xi \geq 0), \tag{2.6}$$

where $\underline{w}_n$ is the waitingtime of the n^{th} customer and $\underline{s}_n$ his servicetime.

Customers are considered marked with probability (1-X) (method I). Clearly

$$(2.7)\qquad P\{\text{no } C_X \text{ in } \underline{w}_n\} = \sum_{n=0}^{\infty} \int_0^{\infty} e^{-\lambda w} \frac{(\lambda w)^n}{n!} X^n \, dP\{\underline{w}_n \le w\} = = \gamma_n(\lambda(1-X))$$

and

$$(2.8)\qquad P\{\text{no } C_X \text{ in } \underline{s}_n\} = \beta(\lambda(1-X)).$$

Hence

$$(2.9)\qquad \gamma_n(\lambda(1-X))\beta(\lambda(1-X)) = P\{\text{no } C_X \text{ in } \underline{w}_n+\underline{s}_n\} =$$

$$= P\{\text{no } C_X \text{ in } \underline{w}_n+\underline{s}_n \text{ and } (n+1)^{st} \text{ marked}\} +$$

$$+ P\{\text{no } C_X \text{ in } \underline{w}_n+\underline{s}_n \text{ and } (n+1)^{st} \text{ unmarked}\} =$$

$$= (1-X)\, P\{\underline{w}_{n+1} = 0\} + X\, P\{\text{no } C_X \text{ in } \underline{w}_{n+1}\} =$$

$$= (1-X)\, \gamma_n(\lambda)\beta(\lambda) + X\, \gamma_{n+1}(\lambda(1-X)).$$

Now take

$$(2.10)\qquad \gamma(\xi,z) = \sum_{n=1}^{\infty} \gamma_n(\xi) z^{n-1} \qquad (|z| < 1),$$

then from (2.9)

$$(2.11)\quad \gamma(\lambda(1-X),z) = \frac{X\,\gamma_1(\lambda(1-X)) - (1-X)\beta(\lambda)z\,\gamma(\lambda,z)}{X - z\,\beta(\lambda(1-X))}$$

or with $\xi = \lambda(1-X)$

$$(2.12)\qquad \gamma(\xi,z) = \frac{\beta(\lambda)z\,\gamma(\lambda,z) - (\frac{\lambda}{\xi}-1)\gamma_1(\xi)}{1 - \lambda\frac{1-z\beta(\xi)}{\xi}}.$$

The equation in ξ

$$(2.13)\qquad \xi - \lambda(1-z\beta(\xi)) = 0$$

has a unique solution $\delta(0,z)$ (Takács 1962) with Re $\delta(0,z) > 0$. Hence $\delta(0,z)$ is a zero of the numerator and

$$(2.14)\quad \gamma(\xi,z) = \frac{\left(\frac{\lambda}{\delta(0,z)} - 1\right)\gamma_1(\delta(0,z)) - (\frac{\lambda}{\xi}-1)\gamma_1(\xi)}{1 - \lambda\frac{1-z\beta(\xi)}{\xi}}.$$

If $p_{n,k}$ is the probability, that the n^{th} customer leaves an empty counter, and

$$(2.15)\qquad \begin{cases} p_n(X) = \sum_{k=0}^{\infty} p_{n,k}X^k & (|X| \leq 1), \\ p(X,z) = \sum_{n=1}^{\infty} p_n(X)z^{n-1} & (|z| < 1), \end{cases}$$

then for $0 \leq X \leq 1$

$$(2.16)\qquad p_n(X) = P\{\text{no } C_X \text{ in } \underline{w}_n + \underline{s}_n\} = \gamma_n(\lambda(1-X))\beta(\lambda(1-X))$$

or

$$(2.17)\qquad p(X,z) = \gamma(\lambda(1-X),z)\beta(\lambda(1-X)).$$

At $t=0$ a busy period starts with exactly one customer present, who only needs a service of duration s_1. The length of this busy period is $\underline{z}(s_1)$. If we take for s_1 a random variable $\underline{s}_1$ with distribution $B(s)$, we get the ordinary busy period with length $\underline{z}$. We write

$$(2.18) \qquad \begin{cases} \delta(\xi) = \mathcal{E}\, e^{-\xi \underline{z}} \\ \delta_{s_1}(\xi) = \mathcal{E}\, e^{-\xi \underline{z}(s_1)} \end{cases}$$

and wish to compute $\delta_n(\xi)$ and $\delta_{s_1,n}(\xi)$, which denote respectively the probability that no C_ξ catastrophe (method II) occurs during a busy period $\underline{z}$ in which exactly n customers are served and the same with $\underline{z}$ replaced by $\underline{z}(s_1)$. We follow Takács and make use of the fact that

$$(2.19) \qquad \underline{z}(s_1) = s_1 + \sum_{h=1}^{\underline{k}(s_1)} \underline{z}_h,$$

where $\underline{k}(s_1)$ is the number of arrivals during s_1 and $\underline{z}_1, \underline{z}_2, \ldots, \underline{z}_k$ are independent drawing of $\underline{z}$, given $\underline{k}(s_1) = k$. Hence

$$(2.20) \qquad \delta_{s_1}(\xi, X) = \sum_{n=1}^{\infty} \delta_{s_1,n}(\xi) X^n$$

satisfies (we use method I and II at the same time)

$$\begin{aligned}(2.21) \quad \delta_{s_1}(\xi, X) &= P\{\text{no } C_\xi \text{ in } \underline{z}(s_1), \text{ no marked customer served in } \underline{z}(s_1)\} = \\ &= \sum_{k=0}^{\infty} X\, P\{\text{no } C_\xi \text{ in } s_1, k \text{ arrivals in } s_1\} \delta(\xi, X)^k = \\ &= X \sum_{k=0}^{\infty} e^{-\xi s_1} e^{-\lambda s_1} \frac{(\lambda s_1)^k}{k!} \delta(\xi, X)^k = \\ &= X \exp(\xi + \lambda(1-\delta(\xi, X))) s_1,\end{aligned}$$

where

$$(2.22) \qquad \delta(\xi, X) = \sum_{n=1}^{\infty} \delta_n(\xi) X^n.$$

It follows from (2.21) that

$$(2.23) \qquad \delta(\xi, X) = \int_{0-}^{\infty} \delta_s(\xi, X)\, dB(s) = X\, \beta(\xi + \lambda(1-\delta(\xi, X))).$$

Takács has shown, that for given X with $|X| \leq 1$ and given ξ with Re $\xi \geq 0$ the equation for δ

(2.24) $$\delta = X\beta(\xi + \lambda(1-\delta))$$

has exactly one solution $\delta(\xi,X)$ with $|\delta(\xi,X)| \leq 1$. We have $\delta(\xi,1) = \delta(\xi)$.

Let $\varepsilon_n(\xi,X)$ be the probability, that in a busy period $\underline{z}$ at least n customers are served, no C_ξ happens during the first n servicetimes and no C_X remains at the counter at the n^{th} departure. The probability, that at least n+1 customers are served, the $(n+1)^{st}$ is unmarked, no C_ξ happens during the first n+1 servicetimes and no C_X remains at the counter at the $(n+1)^{st}$ departure is clearly $X\varepsilon_{n+1}(\xi,X)$ and is equal to the probability, that at the n^{th} departure at least one but no marked customer remains and no C_ξ has occurred till then and that in the next servicetime $\underline{s}_{n+1}$ no C_X arrives and no C_ξ occurs. This shows, that for $n > 0$

(2.25) $$X\,\varepsilon_{n+1}(\xi,X) = \{\varepsilon_n(\xi,X) - \varepsilon_n(\xi,0)\}\,\beta(\xi + \lambda(1-X)).$$

Because

(2.26) $$\varepsilon_1(\xi,X) = \beta(\xi + \lambda(1-X)),$$

(2.25) is also true for n=0 if we define $\varepsilon_0(\xi,X) = X$.

If now

(2.27) $$\varepsilon(\xi,X,z) = \sum_{n=1}^{\infty} \varepsilon_n(\xi,X)z^{n-1} \quad (|z| \leq 1),$$

then by (2.25)

(2.28) $$\varepsilon(\xi,X,z) = \frac{X - z\varepsilon(\xi,0,z)}{X - z\beta(\xi+\lambda(1-X))}\,\beta(\xi+\lambda(1-X)).$$

Because $X = \delta(\xi,z)$ is a zero of the denominator, it is also a zero of the numerator and therefore

(2.29) $$\varepsilon(\xi,X,z) = \frac{X - \delta(\xi,z)}{X - z\beta(\xi+\lambda(1-x))}\,\beta(\xi+\lambda(1-x)).$$

However, here the theory of analytic functions is not needed. For $\varepsilon_n(\xi,0)$ is the probability, that during $\underline{z}$ exactly n customers are served and no C_ξ occurs and so

$$(2.30)\qquad z\,\varepsilon(\xi,0,z) = \sum_{n=1}^{\infty} \varepsilon_n(\xi,0)z^n$$

is just the probability, that during $\underline{z}$ no C_ξ occurs and no C_z is served (method I with z instead of X) or

$$(2.31)\qquad z\,\varepsilon(\xi,0,z) = \delta(\xi,z).$$

Note that $\varepsilon_{n-1}(\xi,X)-\varepsilon_{n-1}(\xi,0)$ is the probability, that at the $(n-1)^{st}$ departure $(n \geq 1)$ at least one customer but no C_X remains and till then no C_ξ occurred. We have

$$(2.32)\qquad \sum_{n=1}^{\infty} \{\varepsilon_{n-1}(\xi,X)- \varepsilon_{n-1}(\xi,0)\} = X \frac{X-\delta(\xi)}{X-\beta(\xi+\lambda(1-X))} .$$

Next consider the probability, that the n^{th} customer finds upon arrival $k > 0$ customers waiting (including the one being served) and that during the remaining servicetime of the one being served no C_ξ happens. Thus we wish to compute for $n \geq 2$ and $1 \leq k \leq n-1$

$$(2.33)\qquad P\{\underline{w}_{n-k} < \underline{y}_{n-k}+\ldots+\underline{y}_{n-1} < \underline{w}_{n-k}+\underline{s}_{n-k} < \underline{y}_{n-k}+\ldots+\underline{y}_{n-1}+\underline{x}\},$$

where all random variables are independent and

$$(2.34)\qquad P\{\underline{x} \leq x\} = 1-e^{-\xi x} \qquad (x \geq 0).$$

For fixed n and different k we must consider different $\underline{w}_{n-k}$, which means that a double generating function with respect to n and k must be introduced to handle these probabilities. It is easier to consider

$$(2.35)\qquad p_{n,k}(\xi) = P\{\underline{w}_n < \underline{y}_n+\ldots+\underline{y}_{n+k-1} < \underline{w}_n+\underline{s}_n < \underline{y}_n+\ldots+\underline{y}_{n+k-1}+\underline{x}\}$$

for $n \geq 1$ and $k \geq 1$, because then

$$(2.36)\qquad p_n(\xi,X) = \sum_{k=1}^{\infty} p_{n,k}(\xi)X^{k-1}$$

has a simple interpretation. For $(1-X)p_n(\xi,X)$ is just the probability, that independent tosses with a coin with probability

1-X for success lead to a first success at the $\underline{k}^{th}$ toss, that customer n+$\underline{k}$ arrives during $\underline{s}_n$ and that during the part of $\underline{s}_n$ remaining after the n+$\underline{k}^{th}$ arrival no C_ξ occurs. Equivalently we can consider a stationary Poisson process with parameter $\lambda(1-X)$ and ask for the probability, that the first event from this process after the n^{th} arrival occurs during $\underline{s}_n$ and that during the remainder of $\underline{s}_n$ no C_ξ occurs. Hence

$$(2.37)\qquad p_n(\xi,X) = P\{\text{no } C_{\lambda(1-X)} \text{ in } \underline{w}_n\} P\{\underline{v} < \underline{s}_n, \text{no } C_\xi \text{ in } \underline{s}_n - \underline{v}\} =$$
$$= \gamma_n(\lambda(1-x)) \int_0^\infty \int_0^s \lambda(1-X) e^{-\lambda(1-X)v} e^{-\xi(s-v)} dv\, dB(s) =$$
$$= \frac{\lambda(1-X)\gamma_n(\lambda(1-X))(\beta(\xi)-\beta(\lambda(1-X)))}{\lambda(1-X)-\xi},$$

where $\underline{v}, \underline{s}_n$ and the Poisson process with parameter ξ are independent and

$$(2.38)\qquad P\{\underline{v} \leq v\} = 1-e^{-\lambda(1-X)v} \qquad (v \geq 0).$$

For

$$(2.39)\qquad p(\xi,X,z) = \sum_{n=1}^{\infty} p_n(\xi,X) z^{n-1} \qquad (|z| < 1)$$

we find in this way

$$(2.40)\qquad p(\xi,X,z) = \frac{\lambda(1-X)(\beta(\xi)-\beta(\lambda(1-X)))}{\lambda(1-X)-\xi}\, \gamma(\lambda(1-X),z).$$

Wishart has computed

$$(2.41)\qquad \Pi(\xi,\eta,X) = \sum_{k=1}^{\infty} X^k \int_0^\infty \int_0^\infty \xi e^{-\xi x} e^{-\eta s} ds\; P\{\underline{k}(x)=k, \underline{s}(x) \leq s\}\, dx,$$

where at t=0 we have an empty counter, $\underline{k}(x)$ denotes the number of customers present at time x (including the one being served) and $\underline{s}(x)$ the at time x remaining servicetime of the one being served. Hence he found the probability, that at time $\underline{x}$ (for which (2.34) applies) at least one customer is present, that no C_X is present at that time and that no C_η happens during $\underline{s}(x)$. The probability, that at time $\underline{x}$ the counter is not empty is

$$(2.42)\quad \Pi(\xi,0,1) = \sum_{n=1}^{\infty} P\{\underline{u}_1+\underline{z}_1+\ldots+\underline{u}_n < \underline{x} < \underline{u}_1+\underline{z}_1+\ldots+\underline{u}_n+\underline{z}_n\} =$$

$$= \sum_{n=1}^{\infty} \frac{\lambda}{\lambda+\xi}\left(\delta(\xi)\,\frac{\lambda}{\lambda+\xi}\right)^{n-1}(1-\delta(\xi)) = \frac{\lambda(1-\delta(\xi))}{\xi+\lambda(1-\delta(\xi))},$$

where $\underline{x},\underline{u}_1,\underline{z}_1,\ldots$ are independent, the $\underline{u}_n$ have the arrival-interval distribution and the $\underline{z}_n$ the busy-period distribution.

To find $\Pi(\xi,\eta,X)$ we have to replace the simple factor $P\{\underline{x} < \underline{z}\} = 1-\delta(\xi)$ of (2.42) by the more complicated

$$(2.43)\qquad P\{\underline{x} < \underline{z}, \text{no } C_X \text{ at } \underline{x}, \text{no } C_\eta \text{ during } \underline{s}(x)\},$$

where $\underline{x}$ and $\underline{z}$ start at time 0. Write $\underline{n}$ for the number of customers served in $\underline{z}$ and $\underline{t}'_n$ for the time of departure of the n^{th} customer ($n \geq 1$, $\underline{t}'_0 = 0$). Then (2.43) is equal to

$$(2.44)\quad \sum_{n=1}^{\infty} P\{\underline{t}'_{n-1} < \underline{x} < \underline{t}'_{n-1}+\underline{s}_n, \text{no } C_X \text{ at } \underline{x}, \text{no } C_\eta \text{ in } \underline{t}'_{n-1}+\underline{s}_n-\underline{x}\} =$$

$$= \sum_{n=1}^{\infty} P\{\underline{n} \geq n, \text{no } C_X \text{ at } \underline{t}'_{n-1}+, \text{no } C_\xi \text{ in } \underline{t}'_{n-1}\}\,\cdot$$

$$\cdot\, P\{\underline{x} < \underline{s}, \text{no } C_X \text{ in } \underline{x}, \text{no } C_\eta \text{ in } \underline{s}-\underline{x}\} =$$

$$= \sum_{n=1}^{\infty} (\varepsilon_{n-1}(\xi,X)-\varepsilon_{n-1}(\xi,0))\,\cdot$$

$$\cdot \int_0^{\infty}\int_0^{s} \xi e^{-\xi x} e^{-\lambda(1-X)x} e^{-\eta(s-x)}\,dx\,dB(s),$$

where we have used the notation of (2.32). But then

$$(2.45)\quad \Pi(\xi,\eta,X) = \frac{\lambda}{\xi+\lambda(1-\delta(\xi))}\cdot\frac{X(X-\delta(\xi))}{X-\beta(\xi+\lambda(1-X))}\cdot$$

$$\cdot\,\frac{\xi(\beta(\eta)-\beta(\xi+\lambda(1-X)))}{\xi+\lambda(1-X)-\eta}.$$

Example 3: Continuous aspects of queues

First consider the M/G/1 queue again. In Mathematical Reviews 22 (1961) 524-525 D.G. Kendall asked for a simple derivation of a formula due to Beneš. This was given by Takács and generalized by Beneš (cf. Beneš (1963), page 27). The present method leads to the following derivation.

We consider the virtual waitingtime $\underline{w}(t)$, which is the length of time needed at time t to finish work on the customers then present. Assume that

$$\underline{w}(0) = s_1 \tag{3.1}$$

is given, where s_1 is the (remaining) servicetime of the first customer, that at $\underline{y}_1$ the second customer arrives, etc. Then we have a first busy period $\underline{z}(s_1)$, followed by an idle period of length $\underline{u}_1$, a busy period $\underline{z}_1$, an idle period $\underline{u}_2$, etc., where $\underline{z}(s_1), \underline{u}_1, \underline{z}_1, \underline{u}_2, \ldots$ are independent random variables and the $\underline{u}_n$ have the arrivalinterval distribution. We want to find

$$\int_0^\infty \xi\, e^{-\xi x}\, P\{\underline{w}(x)=0 \mid \underline{w}(0) = s_1\}\, dx \tag{3.2}$$

or the probability, that at time $\underline{x}$ (where (2.34) applies), not depending on the waitingtime process, the counter turns out to be empty. But then we wish to compute (cf. (2.18))

$$\sum_{n=0}^{\infty} P\{\underline{z}(s_1) + \sum_{k=1}^{n} (\underline{u}_k+\underline{z}_k) < \underline{x} < \underline{z}(s_1) + \sum_{k=1}^{n} (\underline{u}_k+\underline{z}_k)+\underline{u}_{n+1}\} = \tag{3.3}$$

$$= \delta_{s_1}(\xi) \sum_{n=0}^{\infty} \left(\frac{\lambda}{\lambda+\xi}\, \delta(\xi)\right)^n \frac{\xi}{\lambda+\xi} = \frac{\xi\, \delta_{s_1}(\xi)}{\lambda+\xi-\lambda\delta(\xi)} \,.$$

The "superfluous" ξ in (3.2) (and hence in (3.3)) was added for the interpretation.

In Runnenburg (1958) a formula due to Takács and connecting

$$\int_0^\infty e^{-\xi w}\, d_w\, P\{\underline{w}(t) \leq w\} \tag{3.4}$$

and

$$P\{\underline{w}(x) = 0\} \tag{3.5}$$

was derived by interpretation. We reproduce this interpretation here in order to show that it contains the basic idea for more general results, which we derive afterwards.

Consider the M/G/1 queue and start with an empty counter at t=0. The probability, that no C_ξ occurs (method II) during the time the counter is busy with work due to customers arriving in $[0,t]$, is equal to

(3.6) $$\pi_1 = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \beta(\xi)^n = \exp -\lambda t(1-\beta(\xi)),$$

because during each servicetime the counter has to work with probability $\beta(\xi)$ no C_ξ occurs. Now the probability, that no C_ξ occurs in $[0, t+\underline{w}(t)]$ is

(3.7) $$\pi_2 = e^{-\xi t} \int_{0-}^{\infty} e^{-\xi w} d_w P\{\underline{w}(t) \leq w\} .$$

Then $\pi_1 - \pi_2$ is the probability, that no C_ξ occurs during the time the counter is working on customers arriving in $[0,t]$ and that the first catastrophe of the Poisson process with parameter ξ happens before t at a time $\underline{x}$ (and hence at a moment the counter is idle), while during the work originating in $[\underline{x}, t]$ no C_ξ occurs. But then

(3.8) $$\pi_1 - \pi_2 = \int_0^t \xi e^{-\xi x} P\{\underline{w}(x)=0\} \exp -\lambda(t-x)(1-\beta(\xi))dx$$

and Takács result has been obtained.

We get a much more complicated problem (Takács (1962), page 36, equation (13)) if we assume, that the M/G/1 queue is replaced by this variant: arrivals occur as before in a Poisson stream with parameter λ and service is in the order of arrival, but now batches of size m are served simultaneously. Hence servicing only starts when at least m customers are present. The n^{th} batch has servicetime $\underline{s}_n$ with distribution B(s) and $\underline{y}_1, \underline{s}_1, \underline{y}_2, \underline{s}_2, \ldots$ are again independent random variables. Let $\underline{w}(t)$ be the length of time the last customer of the last complete batch present at time t will remain at the counter after time t (for waiting and servicing). Let $\underline{r}(t)$ be the number of customers present at time t, who do not yet make a complete batch (hence $0 \leq \underline{r}(t) < m$). Let

(3.9) $$P\{\underline{w}(0) \leq w,\ \underline{r}(0) = r\}$$

be given for $0 \leq r < m$ and all real w. We wish to derive Takács relation for

(3.10) $$\int_{0-}^{\infty} e^{-\xi t} d_w P\{\underline{w}(t) \leq w,\ \underline{r}(t) = r\} .$$

As the probability, that no C_ξ occurs during the time the counter is working on complete batches of customers present at time s (with $0 \leq s < t$) or arriving during $(s,t]$ and that there is no C_X in the final incomplete batch present at time t, under the condition $\underline{w}(s) = x$, $\underline{r}(s) = j$ and exactly $mn-j+r$ (with $n \geq 0$ and $0 \leq r < m$) customers arrive in $(s,t]$, is equal to

$$(3.11) \qquad e^{-\xi x} \beta(\xi)^n X^r,$$

we find for the unconditional probability, that no C_ξ occurs during the time the counter is working on complete batches present at time s or arriving during $(s,t]$ and that no C_X is present at time t in the last incomplete batch

$$(3.12) \quad \pi_1(s,t) = \sum_{j=0}^{m-1} \sum_{n=0}^{\infty} \sum_{r=0}^{m-1} \beta(\xi)^n X^r e^{-\lambda(t-s)} \cdot \left(\frac{\partial}{\partial \lambda(t-s)}\right)^j \frac{(\lambda(t-s))^{mn+r}}{(mn+r)!} \int_{0-}^{\infty} e^{-\xi x} d_x \, P\{\underline{w}(s) \leq x, \underline{r}(s) = j\}$$

or

$$(3.13) \qquad \pi_1(s,t) = \sum_{j=0}^{m-1} p_{j,t-s}(\xi, X) \gamma_j^*(\xi, s),$$

where

$$(3.14) \begin{cases} \gamma_j^*(\xi,s) = \displaystyle\int_{0-}^{\infty} e^{-\xi x} \, d_x \, P\{\underline{w}(s) \leq x, \underline{r}(s) = j\}, \\ p_{j,t-s}(\xi, X) = \dfrac{1}{m} \dfrac{X^m - \beta(\xi)}{\beta(\xi)} \cdot \displaystyle\sum_{h=0}^{m-1} \dfrac{\{\varepsilon_h \beta(\xi)^{1/m}\}^{j+1}}{X - \varepsilon_h \beta(\xi)^{1/m}} e^{-\lambda(t-s)(1-\varepsilon_h \beta(\xi)^{1/m})}. \end{cases}$$

Here we have written ε_h for $e^{\frac{2\pi i}{m} h}$ and made use of

$$(3.15) \qquad \sum_{n=0}^{\infty} \frac{z^{mn+r}}{(mn+r)!} = \frac{1}{m} \sum_{h=0}^{m-1} \varepsilon_h^{-r} e^{\varepsilon_h z},$$

which holds for all complex z and integer r and m with $0 \leq r < m$.

The probability, that the first C_ξ occurs after $t+\underline{w}(t)$ and that there is no C_X in the final incomplete batch present at time t, is

(3.16) $$\pi_2(t) = e^{-\xi t} \sum_{r=0}^{m-1} X^r \int_{0-}^{\infty} e^{-\xi w} d_w P\{\underline{w}(t) \leq w, \underline{r}(t)=r\} = e^{-\xi t} \sum_{r=0}^{m-1} \gamma_r^*(\xi, t) X^r.$$

Finally the probability, that the first catastrophe from the Poisson process with parameter ξ occurs before t while the counter is unoccupied and no C_ξ happens after that moment during the time the counter is working on complete batches of customers arriving before t and that there is no C_X in the final incomplete batch present at time t, is

(3.17) $$\pi_3(t) = \sum_{j=0}^{m-1} \int_0^t \xi e^{-\xi u} P\{\underline{w}(u)=0, \underline{r}(u)=j\} p_{j,t-u}(\xi, X) du.$$

From the definition of $\pi_1(s,t)$, $\pi_2(t)$ and $\pi_3(t)$ it is clear that

(3.18) $$\pi_1(0,t) = \pi_2(t) + \pi_3(t).$$

Hence we have indeed connected $\sum_{r=0}^{m-1} \gamma_r^*(\xi,t)X^r$ and $\sum_{j=0}^{m-1} P\{\underline{w}(u) = 0, \underline{r}(u) = j\}X^j$ as we set out to do.

The idea of the foregoing proofs of this example is in fact sufficient to derive a general formula due to Beneš. Suppose that k(t) is an ordinary nondecreasing stepfunction (continuous from the right), which has a finite number of jumps of finite height in any interval $[0,t]$ with $0 < t < \infty$. Let k(t) for each $t > 0$ describe the total servicetime needed to serve all customers arriving in the interval $[0,t]$ for a first come, first served queue at one counter. Hence at each jump of k(t) a customer arrives at the counter, who needs a service of length denoted by the height of that particular jump. From such a nonstochastic k(t) we can construct the virtual waitingtime w(t), which is also a well defined nonstochastic function of t. Let p(x,0) be 1 if w(x) = 0 and 0 otherwise. Then the probability, that no C_ξ occurs during the time in which the server is busy with customers entering in (0,t), is

(3.19) $$e^{-\xi k(t)}.$$

The probability, that no catastrophe occurs before $t+w(t)$, is

$$(3.20) \qquad e^{-\xi(t+w(t))}$$

and the probability, that the first catastrophe occurs at some time $\underline{x} \leq t$ at which the counter is idle, while after that no C_ξ occurs during the time the counter is busy with customers arriving in $[\underline{x},t]$, is

$$(3.21) \qquad \int_0^t \xi\, e^{-\xi x}\, p(x,0) e^{-\xi(k(t)-k(x))} dx.$$

Hence

$$(3.22) \qquad e^{-\xi k(t)} = e^{-\xi(t+w(t))} + \int_0^t \xi\, e^{-\xi x} p(x,0) e^{-\xi(k(t)-k(x))} dx$$

or Theorem 3.1 from Beneš (1963), page 38 has been obtained.

Example 4: The Bloemena-Le Gall queue

Customers arrive in a Poisson stream with parameter λ at a counter, the first one arriving at $t=0$ at an empty counter. They are served in the order in which they arrive in batches of size m. However, if at the end of a servicetime less than m customers are present but at least one or if a customer finds an empty counter, the incomplete batch does not h ve to wait till it becomes complete, but it is served at once as if it were complete. The arrival-intervals $\underline{y}_1, \underline{y}_2, \ldots$ and the servicetimes $\underline{s}_1, s_2, \ldots$ are again independent random variables, the latter having distribution function $B(s)$ with Laplace-Stieltjes transform $\beta(\xi)$. The n^{th} customer arrives at $\underline{t}_n = \underline{y}_1 + \ldots + \underline{y}_{n-1}$ (with $\underline{t}_1=0$). This queue was considered in the stationary situation in Bloemena (1958) and for general arrivalinterval distribution in Le Gall (1962), page 267.

If $p_{n,k}$ is the probability, that at the n^{th} departure of a batch exactly k customers remain at the counter, then

$$(4.1) \qquad p_n(X) = \sum_{k=0}^{\infty} p_{n,k} X^k$$

is the probability, that no C_X remains at the n^{th} departure (method I), where

(4.2) $$p_1(X) = \beta(\lambda(1-X)).$$

As

(4.3) $$\sum_{j=0}^{m-1} p_{n,j} + \sum_{k=0}^{\infty} p_{n,m+k} X^k$$

is the probability, that no C_X are waiting immediately after the $(n+1)^{st}$ service has started, while $\beta(\lambda(1-X))$ is the probability, that during the $(n+1)^{st}$ service no C_X arrives, we have for $n \geq 1$

(4.4) $$p_{n+1}(X) = \left(\sum_{j=0}^{m-1} p_{n,j} + \sum_{k=0}^{\infty} p_{n,m+k} X^k \right) \beta(\lambda(1-X)).$$

Now with

(4.5) $$\begin{cases} \bar{p}_j(z) = \sum_{n=1}^{\infty} p_{n,j} z^{n-1} & (|z| < 1), \\ p(X,z) = \sum_{n=1}^{\infty} p_n(X) z^{n-1} & (|z| < 1) \end{cases}$$

we find from (4.4) with (4.2)

(4.6) $$p(X,z) = \frac{X^m + z \sum_{j=0}^{m-1} (X^m - X^j) \bar{p}_j(z)}{X^m - z\beta(\lambda(1-X))} \beta(\lambda(1-X)).$$

As the equation in X

(4.7) $$X^m - z\beta(\lambda(1-X)) = 0$$

has exactly m roots $X_0(z), X_1(z), \ldots, X_{m-1}(z)$ with $|X_j(z)| < 1$, we can determine the $\bar{p}_j(z)$ from the condition that the $X_j(z)$ are also roots of the numerator.

If we multiply both sides of (4.6) with $1-z$ and take $z \uparrow 1$, we obtain (3.6) from Bloemena (1958).

Let $r_{n,k}$ be the probability, that the n^{th} customer is the last one of a batch and that at his departure exactly k customers remain at the counter. Then for $n \geq 1$

(4.8) $$r_n(X) = \sum_{k=0}^{\infty} r_{n,k} X^k$$

is the probability, that the n^{th} customer is the last of a batch and that during his stayingtime no C_X arrives. The probability, that this event happens and moreover the $(n+1)^{st}$ customer is marked, is equal to the probability, that the n^{th} customer is the last of a batch, that the $(n+1)^{st}$ customer is marked and that the $(n+1)^{st}$ customer does not arrive in the waitingtime of the n^{th} customer. Hence this probability is

(4.9) $$(1-X)r_{n,0} .$$

Now $r_n(X)-(1-X)r_{n,0}$ is the probability, that the n^{th} customer is the last of a batch, that the $(n+1)^{st}$ customer is unmarked and that during the stayingtime of the n^{th} customer no C_X arrives. But this is just the probability, that the $(n+1)^{st}$ customer is unmarked, that he is the first customer of a batch and that during his waitingtime no C_X arrives. Let $q_{n,k}$ be the probability, that the n^{th} customer is the first one of a batch and that during his waitingtime exactly k customers arrive. Then

(4.10) $$r_n(X) = (1-X)r_{n,0}+X\, q_{n+1}(X),$$

where

(4.11) $$q_n(X) = \sum_{k=0}^{\infty} q_{n,k}\, X^k,$$

with

(4.12) $$q_1(X) = 1.$$

We also have for $n \geqslant 1$

(4.13) $$r_n(X) = \left(\sum_{j=0}^{m-1} q_{n-j,j} + \sum_{k=m}^{\infty} q_{n-m+1,k} X^{k-m+1} \right) \beta(\lambda(1-X)),$$

because with probability $q_{n-j,j}$ the n^{th} customer is the last one of a batch of j+1 and no customers arrive during his waitingtime (for $0 \leqslant j < m$) and with probability $q_{n-m+1,k}$ he is the last one of a batch of m and exactly k-m+1 customers arrive during his waitingtime (for $k \geqslant m$). In particular we find (take X=0 in (4.13))

(4.14) $$r_{n,0} = \beta(\lambda) \sum_{j=0}^{m-1} q_{n-j,j} ,$$

and (by combining (4.10) and (4.13)) for $n \geq 1$

$$(4.15) \qquad X\, q_{n+1}(X) = \sum_{j=0}^{m-1} q_{n-j,j} \{\beta(\lambda(1-X))-(1-X)\beta(\lambda)\} + \sum_{k=m}^{\infty} q_{n-m+1,k}\, X^{k-m+1}\beta(\lambda(1-X)).$$

This could have been deduced by interpretation and is left as an exercise for the reader. If now we introduce

$$(4.16) \qquad \begin{aligned} \overline{q}_j(z) &= \sum_{n=1}^{\infty} q_{n,j}\, z^{n-1} \qquad (|z|<1), \\ q(X,z) &= \sum_{n=1}^{\infty} q_n(X) z^{n-1} \qquad (|z|<1), \end{aligned}$$

we find from (4.15) (using $q_{n,j} = 0$ for $n<1$ and (4.12))

$$(4.17) \quad q(X,z)(1-(X^{-1}z)^m\beta(\lambda(1-X))) = 1+X^{-1}\{\beta(\lambda(1-X))-(1-X)\beta(\lambda)\}z \sum_{j=0}^{m-1} \overline{q}_j(z)z^j + - (X^{-1}z)^m\beta(\lambda(1-X)) \sum_{j=0}^{m-1} \overline{q}_j(z)X^j .$$

Again the $\overline{q}_j(z)$ with $0 \leq j < m$ can be found by using the m roots $X_o(z^m), X_1(z^m), \dots, X_{m-1}(z^m)$ with $|X_j(z^m)| < 1$ of the equation in X

$$(4.18) \qquad X^m - z^m\beta(\lambda(1-X)) = 0.$$

From $q(X,z)$ the $r_n(X)$ may be found.

Now write $\gamma_{n,1}(\xi)$ for the probability, that the n^{th} customer is the first of a batch and that during his waiting-time no C_X arrives (take $\xi = \lambda(1-X)$). Because

$$(4.19) \qquad \gamma_{n,1}(\xi) = \gamma_{n,1}(\lambda(1-X)) = q_n(X),$$

we have found the relation

$$(4.20) \qquad \gamma_1(\xi, z) = q(X,z)$$

for

$$(4.21) \qquad \gamma_1(\xi, z) = \sum_{n=1}^{\infty} \gamma_{n,1}(\xi) z^{n-1}.$$

Next consider

$$\gamma_n(\xi) = \mathcal{E} e^{-\xi \underline{w}_n}, \tag{4.22}$$

where $\underline{w}_n$ is the waitingtime of the n^{th} customer. Clearly

$$\gamma_n(\lambda(1-X)) - \left(\sum_{j=0}^{m-1} q_{n-j,j} + \sum_{k=m}^{\infty} q_{n-m+1,k} X^{k-m+1} \right) \tag{4.23}$$

is the probability, that the n^{th} customer is <u>not</u> the last one of a batch and that during his waitingtime no C_X arrives at the counter. This is equal to the probability, that the $(n+1)^{st}$ customer is unmarked, that he is <u>not</u> the first of a batch and that during his waitingtime no C_X arrives. For this last probability we may write

$$X \gamma_{n+1}(\lambda(1-X)) - X q_{n+1}(X). \tag{4.24}$$

We introduce further

$$\gamma(\xi,z) = \sum_{n=1}^{\infty} \gamma_n(\xi) z^{n-1} \tag{4.25}$$

and note that $\gamma_1(\xi) = 1$.

We may combine (4.10), (4.13), (4.23) and (4.24) for $n \geq 1$ to

$$\gamma_n(\xi) - X \gamma_{n+1}(\xi) = \left(\frac{1}{\beta(\xi)} - 1 \right) r_n(X) + (1-X) r_n(0). \tag{4.26}$$

This equation is the analogue of equation (4.9) in Bloemena (1958), where only the stationary situation is considered. Finally

$$(z-X)\gamma(\xi,z) = X\left\{(X^{-1}z)^m - 1\right\} q(X,z) + \tag{4.27}$$

$$+ z \sum_{j=0}^{m-1} \bar{q}_j(z) z^j - X(X^{-1}z)^m \sum_{j=0}^{m-1} \bar{q}_j(z) X^j.$$

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