STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

AFDELING MATHEMATISCHE STATISTIEK

Report S 364

A monotonicity property of the test for symmetry in a 2×2 table and the sign test

by

J. Oosterhoff.

November 1966

1. Introduction and results

Let $\underline{x}$[1] be a one-dimensional random variable. Define the three probabilities p_+, p_- and p_0 by $p_+ = P(\underline{x} > 0)$, $p_- = P(\underline{x} < 0)$, $p_0 = P(\underline{x} = 0)$, so that $p_+ + p_- + p_0 = 1$. A random sample of observations $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_N$ of $\underline{x}$ is given and the hypothesis to be tested is

$$G_0 : p_+ \leqslant p_-$$

against the alternative hypothesis

$$G_1 : p_+ > p_-.$$

In [2] it is shown that the uniformly most powerful unbiased (UMPU) test of G_0 against G_1 is the one-sided sign test, disregarding the observations equal to zero. If α is the prescribed size of the test and $\underline{n}$ the number of observations different from zero, this test can also be described as a conditional binomial test of size α of the hypothesis $p_+/p_- \leqslant 1$, against $p_+/p_- > 1$, given $\underline{n} = n$. To obtain the exact size α, this test procedure requires randomization in the boundary points of the conditional critical regions (if $n=0$, G_0 should be rejected with probability α).

We remark, that in practical applications this kind of randomization is often thought undesirable. HEMELRIJK [1] proved, that if no randomization is applied and the boundary points are included in the conditional acceptance regions, the power of this test is never smaller than the power of the test where the observations equal to zero are equally divided between the two classes $x > 0$ and $x < 0$.

The power of the test is usually expressed as a function of p_+/p_-. For a given alternative p_+/p_- the power also depends on $p=1-p_0$, the probability of the event $\underline{x} \neq 0$. For fixed p_+/p_- the power of the exact size-α test is obviously a strictly increasing function of p. However, in some situations alternatives $p_+-p_-=d$ may also be of interest.

---

1) Random variables will be distiguished from fixed numbers(e.g. from values they assume in an experiment) by underlining their symbols.

Now consider two characteristics A and B, which each member of a population may or may not possess, and denote the complement of A and B by $\bar{A}$ and $\bar{B}$ respectively. The probabilities of the four possible combinations can be displayed in a 2×2 table:

	B	$\bar{B}$	
A	p_{11}	p_{12}	P(A)
$\bar{A}$	p_{21}	p_{22}	P($\bar{A}$)
	P(B)	P($\bar{B}$)	

A random sample of size N is drawn from this population and we wish to test the hypothesis concerning the marginal distributions

H_0 : P(A) ≤ P(B) (or equivalently $p_{12} \leq p_{21}$)

against the alternative hypothesis

H_1 : P(A) > P(B) (or equivalently $p_{12} > p_{21}$)

Let $\underline{n}$ be the number of elements in the sample possessing the properties $A \cap \bar{B}$ or $\bar{A} \cap B$. Let $\underline{m}$ be the number of elements in the sample with the property $A \cap \bar{B}$. It is well known (cf [2], Ch.4) that the UMPU size-α test of H_0 against H_1 is given by the critical function (the probability with which H_0 should be rejected)

$$(1) \qquad \phi(m,n) = \begin{cases} 1 & \\ \gamma(n) & \text{if} \\ 0 & \end{cases} \quad \begin{matrix} m > c(n) \\ m = c(n) \\ m < c(n), \end{matrix}$$

where the arithmetical functions c(n) and γ(n), 0 ≤ γ(n) < 1, are determined by the relations

$$(2) \qquad E_0 \{\phi(\underline{m}, \underline{n}) \mid \underline{n} = n^*\} = \alpha, \quad n = 0, 1, \ldots, N$$

(E_0 denotes the expectation under the hypothesis $p_{12} = p_{21}$).

If we define

(3) $$p = p_{12} + p_{21}, \quad p_{12}^{*} = p_{12}/p,$$

the test given by (1) and (2) is the conditional binomial size-α test of the hypothesis $p_{12}^{*} \leq \frac{1}{2}$ against the alternative $p_{12}^{*} > \frac{1}{2}$, given $\underline{n} = n$. If $n = 0$, H_0 must be rejected with probability α. In fact this test is completely identical with the sign test described above, if we identify the occurrences of $A \cap \bar{B}$ and $\bar{A} \cap B$. Hence, if no randomization is used in the boundary points of the conditional critical regions, the property proved by HEMELRIJK also applies here.

Restricting the parameter space to the subspace satisfying $p_{12} \geq p_{21}$, H_0 is replaced by H_0' : $P(A) = P(B)$, equivalent with $p_{12} = p_{21}$. This hypothesis is known as the hypothesis of symmetry in a 2×2 table. The test is obviously not affected by this restriction.

The above test is sometimes called Mc NEMAR's test, since Mc NEMAR first advocated the use of this test in the social sciences, be it in a slightly different form. A detailed description is given in [3].

The power of the test may again be expressed as a function of p_{12}/p_{21}. For a fixed alternative p_{12}/p_{21} the power still depends on p and is in fact a strictly increasing function of p. However, in some applications, where the marginal probabilities are essential, the ratio p_{12}/p_{21} is irrelevant and one prefers to express the power of the test as a function of the marginal probabilities P(A) and P(B). If the events A and B are independent, $p_{12}/p_{21} = P(A) \{ 1-P(B) \} \ / \ P(B) \{ 1-P(A) \}$. This case was considered by WALD ([4], Ch. 6), who constructed a sequential test of H_0 against H_1, based on the test (1). However, if A and B are not independent, it is impossible to write p_{12}/p_{21} as a function of P(A) and P(B) alone, unless both p_{11} and p_{22} are known, a most unusual situation. Since $P(A)-P(B) = p_{12} - p_{21}$, it seems a reasonable approach in such cases to consider the power of the test for given $p_{12} - p_{21}$.

For the sign test this corresponds with the consideration of alternatives with fixed $p_+ - p_-$.

Consider therefore a fixed alternative

(4) $$P(A) - P(B) = p_{12} - p_{21} = d(> 0).$$

The definitions (3) and (4) imply $d \leq p \leq 1$. We want to investigate the behaviour of the unconditional power of the test (1) as a function of p for fixed d. For given $\underline{n} = n$ the alternative (4) corresponds to the alternative $p_{12}^* = \frac{1}{2} + d/2p$ in the conditional binomial test. Hence p_{12}^* decreases as p increases and the conditional power decreases as p increases for every fixed $n > 0$. However, as p increases larger values of $\underline{n}$ are more likely and hence the unconditional power increases as p increases for a fixed $p_{12}^* > \frac{1}{2}$. The following theorem shows that the first effect is in general more important than the second one.

<u>THEOREM</u> : For a fixed alternative (4) the unconditional power of the randomized size-α test defined by (1) and (2) is

(i) independent of p for all sample sizes N satisfying either $\alpha \leq 2^{-N}$ or $1 - \alpha \leq 2^{-N}$

(ii) a strictly decreasing function of p for all sample sizes N satisfying $2^{-N} < \alpha < 1 - 2^{-N}$

for all p in the interval $[d, 1]$.

The rather elaborate proof of this theorem will be given in section 2. The theorem also holds for the sign test if we replace the alternative (4) by $p_+ - p_- = d\ (> 0)$ and define $p = 1 - p_0$.

If one does not want to use a randomized test, the pairs (m, n) for which $0 < \phi(m, n) = \gamma(n) < 1$ may be included in the acceptance region of the test, resulting in a nonrandomized test with level of significance α, but size $\alpha' < \alpha$. The theorem is not necessarily true for this modified test, as is illustrated by the following example. Let α and N satisfy $2^{-N} \leq \alpha < 2^{-N+1}$. The critical region of the nonrandomized test now contains only one point, $(m, n) = (N, N)$. The power of this test against the alternative (4) is equal to $p^N(\frac{1}{2} + d/2p)^N = 2^{-N}(p + d)^N$, a strictly increasing function of p. However, as N tends to infinity and α and p remain bounded away from 0 and 1, the effect of the above modification ofthe test on the power becomes negligible. Hence we may expect that the power of the nonrandomized test roughly behaves like the power of the randomized test for moderate values of α and p and large N.

At the end of this report three tables are given, where the powers of the randomized and nonrandomized test are shown for N = 25, 100 and 1000, α = .05 and various values of p and d. The tables indicate that the influence of p on the power is rather important.

2. Proof of the theorem

The critical region of the test (1) is the union of the critical regions of the one-sided conditional binomial size-α tests for given $\underline{n} = n$, $n = 0, 1, \ldots, N$. Denote such a critical region by C_n. For n=0 this region is degenerate. For $n > 0$ a region C_n contains h+1 points $m = n-h, n-h+1, \ldots, n$ with positive probability, where $h \geq 0$ depends on n and the point $m = n-h$ is contained in C_n with probability $\gamma(n)$ satisfying

$$(5) \qquad 1 \geq \gamma(n) = \left[\alpha - 2^{-n} \sum_{j=n-h+1}^{n} \binom{n}{j}\right] / \binom{n}{h} > 0;$$

the h points $n-h+1, \ldots, n$ are contained in C_n with probability 1. We have

LEMMA : If for a given n ($1 \leq n < N$) the region C_n contains exactly h+1 points with positive probability, then C_{n+1} contains at least h+1 and at most h+2 points with positive probability.

Proof : Let C_n contain exactly h+1 points with positive probability. Then

$$\alpha = P_0(C_n) > P_0 \ (\underline{m} \geq n-h+1 \mid \underline{n}=n)$$

(P_0 denotes the probability under H_0) and hence

$$P_0(\underline{m} \geq n-h+2 \mid \underline{n}=n+1) < P_0(\underline{m} \geq n-h+1 \mid \underline{n}=n) < \alpha \ ,$$

i.e. C_{n+1} contains at least the h+1 points n-h+1, n-h+2, ..., n+1 with positive probability. On the other hand

$$\alpha = P_0(C_n) \leq P_0(\underline{m} \geq n-h \mid \underline{n}=n)$$

and hence

$$P_0(\underline{m} \geq n-h \mid \underline{n}=n+1) > P_0(\underline{m} \geq n-h \mid \underline{n}=n) \geq \alpha,$$

i.e. C_{n+1} contains at most the h+2 points n-h, n-h+1, ..., n+1 with positive probability.

Let k+1 ($k \geq 0$) be the number of points contained with positive probability in C_N. Then for $n > 0$ the regions $\{C_n\}$ can be grouped into k+1 mutually disjoint sets V_h, h= 0, 1, ..., k, where V_h is the collection of those conditional critical regions containing exactly h+1 points with positive probability. In view of the preceding lemma the index of the set V containing C_n is non-decreasing in n. Moreover, if we assign the degenerate region C_0 to the set V_0, none of the sets $V_0, V_1, \ldots, V_k$ is empty. As a result of these considerations we can define a unique set of integers $0 \leq n_1 < n_2 < \ldots < n_k < N$ such that

$$C_n \in V_h \quad \text{for} \quad n_h < n \leq n_{h+1} \tag{6}$$

for h= 1, 2, ..., k-1. If we define $n_0 = -1$ and $n_{k+1} = N$, then (6) also holds for h=0 and h=k, and the relations (5) and (6) yield

$$(7) \qquad 2^{-n}\sum_{j=0}^{h-1}\binom{n}{j} < \alpha \leq 2^{-n}\sum_{j=0}^{h}\binom{n}{j} \text{ for } n_h < n \leq n_{h+1},\ h=0,1,\dots,k.$$

The power of the test (1) against the alternative (4) can now be written as

$$\beta_N(d;\ p) =$$

$$= \sum_{h=0}^{k}\ \sum_{n=n_h+1}^{n_{h+1}} \binom{N}{n} p^n (1-p)^{N-n} \left\{ \sum_{m=n-h+1}^{n} \binom{n}{m}\left(\tfrac{1}{2}+\frac{d}{2p}\right)^m \left(\tfrac{1}{2}-\frac{d}{2p}\right)^{n-m} + \right.$$

$$\left. + \binom{n}{n-h}\gamma(n)\left(\tfrac{1}{2}+\frac{d}{2p}\right)^{n-h}\left(\tfrac{1}{2}-\frac{d}{2p}\right)^{h} \right\} =$$

$$= \sum_{h=0}^{k}\ \sum_{n=n_h+1}^{n_{h+1}} \binom{N}{n}(1-p)^{N-n}\left\{ 2^{-n}\sum_{m=n-h+1}^{n}\binom{n}{m}(p+d)^m(p-d)^{n-m} + \right.$$

$$(8) \qquad \left. + \left[\alpha - 2^{-n}\sum_{j=0}^{h-1}\binom{n}{j}\right](p+d)^{n-h}(p-d)^h \right\}.$$

Differentiating (8) with respect to p we obtain

$$\frac{\partial}{\partial p}\beta_N(d;p) =$$

$$= \sum_{h=0}^{k}\Big[- \sum_{n=n_h+2}^{n_{h+1}+1}\binom{N}{n}(1-p)^{N-n}\left\{ n\, 2^{-n+1}\sum_{m=0}^{h-1}\binom{n-1}{m}(p+d)^{n-m-1}(p-d)^m + \right.$$

$$\left. + n\left[\alpha - 2^{-n+1}\sum_{j=0}^{h-1}\binom{n-1}{j}\right](p+d)^{n-h-1}(p-d)^h \right\} +$$

$$+ \sum_{n=n_h+1}^{n_{h+1}}\binom{N}{n}(1-p)^{N-n}\left\{ 2^{-n}\sum_{m=0}^{h-1}\binom{n}{m}(n-m)(p+d)^{n-m-1}(p-d)^m + \right.$$

$$\left. + (n-h)\left[\alpha - 2^{-n}\sum_{j=0}^{h-1}\binom{n}{j}\right](p+d)^{n-h-1}(p-d)^h \right\} +$$

$$+ \sum_{n=n_h+1}^{n_{h+1}}\binom{N}{n}(1-p)^{N-n}\left\{ 2^{-n}\sum_{m=1}^{h-1}\binom{n}{m} m (p+d)^{n-m}(p-d)^{m-1} + \right.$$

$$(9) \qquad + h\left[\alpha - 2^{-n}\sum_{j=0}^{h-1}\binom{n}{j}\right](p+d)^{n-h}(p-d)^{h-1}\Big\}\Big] ,$$

where we have applied the identity $(N-n)\binom{N}{n} = (n+1)\binom{N}{n+1}$ in the terms corresponding with differentiation of the factors $(1-p)^{N-n}$ in β_N.

We shall investigate the sign of this derivative, assuming $d < p < 1$. The following well known identity will be needed in the sequel:

$$(10) \qquad \binom{r}{s} = \binom{r-1}{s} + \binom{r-1}{s-1}$$

for integer-valued s and all r.

Gathering the terms with $n=n_h+1$ in (9) and calling their sum $S(n_h+1)$, we have $(h=0, 1, \ldots, k)$

$$S(n_h+1) = \binom{N}{n_h+1}(1-p)^{N-n_h-1} \times$$

$$\times \Big\{ 2^{-n_h-1}\sum_{m=0}^{h-1}\binom{n_h+1}{m+1}(m+1)(p+d)^{n_h-m}(p-d)^m +$$

$$+(n_h-h+1)\left[\alpha - 2^{-n_h-1}\sum_{j=0}^{h-1}\binom{n_h+1}{j}\right](p+d)^{n_h-h}(p-d)^h +$$

$$+ 2^{-n_h-1}\sum_{m=0}^{h-2}\binom{n_h+1}{m+1}(m+1)(p+d)^{n_h-m}(p-d)^m +$$

$$+ h\left[\alpha - 2^{-n_h-1}\sum_{j=0}^{h-1}\binom{n_h+1}{j}\right](p+d)^{n_h-h+1}(p-d)^{h-1} +$$

$$- 2^{-n_h}\sum_{m=0}^{h-2}\binom{n_h+1}{m+1}(m+1)(p+d)^{n_h-m}(p-d)^m +$$

$$- (n_h+1)\left[\alpha - 2^{-n_h}\sum_{j=0}^{h-2}\binom{n_h}{j}\right](p+d)^{n_h-h+1}(p-d)^{h-1}\Big\} .$$

The first, third and fifth term within the braces cancel out except for a term with $m=h-1$. Writing

$$(p-d)^h = (p+d)(p-d)^{h-1} - 2d(p-d)^{h-1}$$

and combining the remaining terms we have

$$S(n_h+1) = \binom{N}{n_h+1}(1-p)^{N-n_h-1}(p+d)^{n_h-h}(p-d)^{h-1} \times$$

$$\times \left\{ 2^{-n_h-1}(p+d)\left[h\binom{n_h+1}{h} + (n_h+1)\left(-\sum_{j=0}^{h-1}\binom{n_h+1}{j} + 2\sum_{j=0}^{h-2}\binom{n_h}{j}\right)\right] + \right.$$

$$(11) \quad \left. - 2d(n_h-h+1)\left[\alpha - 2^{-n_h-1}\sum_{j=0}^{h-1}\binom{n_h+1}{j}\right]\right\} .$$

The first term within the braces in (11) is zero, since from (10)

$$(n_h+1)\left(-\sum_{j=0}^{h-1}\binom{n_h+1}{j} + 2\sum_{j=0}^{h-2}\binom{n_h}{j}\right) =$$

$$=(n_h+1)\left(-\sum_{j=0}^{h-1}\binom{n_h}{j} - \sum_{j=1}^{h-1}\binom{n_h}{j} + 2\sum_{j=0}^{h-2}\binom{n_h}{j-1}\right) = -(n_h+1)\binom{n_h}{h-1} =$$

$$= -h\binom{n_h+1}{h} .$$

From (7) we derive that the second term within the braces in (11) is strictly negative if $n_h > h-1$ and zero if $n_h = h-1$ (the definition of $n_0, n_1, \ldots, n_{k+1}$ implies that $n_h \geq h-1$). Hence

$$(12) \qquad S(n_h+1) = 0 \quad \text{if} \quad n_h = h-1$$

$$\qquad\qquad < 0 \quad \text{if} \quad n_h > h-1 \quad , \quad h = 0, 1, \ldots, k.$$

Next consider the terms in (9) with n satisfying $n_h+2 \leq n \leq n_{h+1}$ and call their sum $S(n_h+2, n_{h+1})$, $h=0, 1, \ldots, k$. Such a sum is void if and only if $n_{h+1} = n_h+1$. Suppose $n_{h+1} > n_h+1$. Then

$$S(n_h+2, n_{h+1}) = \sum_{n=n_h+2}^{n_{h+1}} \binom{N}{n}(1-p)^{N-n} \times$$

$$\times \left\{ 2^{-n}\sum_{m=0}^{h-2}(p+d)^{n-m-1}(p-d)^m \left[-2n\binom{n-1}{m} + (n-m)\binom{n}{m} + (m+1)\binom{n}{m+1}\right] + \right.$$

$$(13) \qquad +(p+d)^{n-h-1}(p-d)^h \left[-h\alpha+2^{-n+1} n \sum_{j=0}^{h-1} \binom{n-1}{j} - 2^{-n}(n-h) \sum_{j=0}^{h-1} \binom{n}{j} \right] \Big\} .$$

It is easily verified that the first term within the braces is zero. The second term may be written as

$$(14) \qquad (p+d)^{n-h}(p-d)^{h-1} h \left[\alpha - 2^{-n} \sum_{j=0}^{h} \binom{n}{j} \right],$$

and the third term within the braces in (13) is equal to (apply (10))

$$(p+d)^{n-h-1}(p-d)^h \Big[-h\alpha + 2^{-n} h \sum_{j=0}^{h-1} \binom{n}{j} + 2^{-n} n \Big(2 \sum_{j=0}^{h-1} \binom{n-1}{j} + $$
$$- \sum_{j=0}^{h-1} \binom{n-1}{j} - \sum_{j=1}^{h-1} \binom{n-1}{j-1}\Big) \Big] =$$

$$(15) \qquad = - (p+d)^{n-h-1}(p-d)^h\, h \left[\alpha - 2^{-n} \sum_{j=0}^{h} \binom{n}{j} \right] .$$

It follows from (7) that the expression between square brackets in (14) and (15) is nonpositive, therefore the sum of (14) and (15) is also nonpositive. Hence

$$(16) \qquad S(n_h+2,\ n_{h+1}) = 0 \quad \text{if} \quad n_{h+1} = n_h+1 \ (h = 1, 2, \dots, k) \text{ or if } h=0$$

$$\leq 0 \quad \text{if} \quad n_{h+1} > n_h+1 \ (h = 1, 2, \dots, k) .$$

We consider the case $n_k = k-1 \geq 0$ somewhat more closely. In view of (6) $n_k=k-1$ implies

$$P_0\ (\underline{m} \geq 1 \mid \underline{n} = n_k+1 = k) < \alpha .$$

Hence for $k \geq 1$

$$P_0\ (\underline{m} \geq 2 \mid \underline{n}=k+2) \leq P_0(\underline{m} \geq 1 \mid \underline{n} = k) < \alpha,$$

i.e. C_{k+2} contains more than k+1 points with positive probability. Therefore $n_k=k-1 \geq 0$ implies $N < k+2$ for otherwise C_N would contain more than k+1 points with positive probability, contradicting the definition of k. If N=k, $n_{k+1}=n_k+1$ and $S(n_k+2,\ n_{k+1})=0$. If N=k+1, $S(n_k+2,\ n_{k+1}) = 0$ if and only if $\alpha= 1+2^{-N}$(cf.(15)). Thus we have proved

$$(17)\qquad S(n_k+2,\ n_{k+1}) \begin{cases} = 0 & \text{if } n_k=k-1 \geq 0 \text{ and } N=k \\ & \text{or if } n_k=k-1 \geq 0,\ N=k+1 \text{ and } \alpha=1-2^{-N} \\ < 0 & \text{if } n_k=k-1 \geq 0,\ N=k+1 \text{ and } \alpha<1-2^{-N}. \end{cases}$$

Since

$$\frac{\partial}{\partial p}\beta_N(d;\ p) = \sum_{h=0}^{k} S(n_h+1) + \sum_{h=0}^{k} S(n_h+2,\ n_{h+1}),$$

we have shown (cf.(12), (16) and (17))that

$$\frac{\partial}{\partial p}\beta_N\ (d;\ p) \leq 0$$

with equality if and only if one of the following conditions is fulfilled

(a) $k=0$

(b) $k > 0$, $n_k=k-1$ and either $N=k$ or $\alpha=1-2^{-N}$.

Since (a) is equivalent with $\alpha \leq 2^{-N}$ and (b) is equivalent with $\alpha \geq 1-2^{-N}$, the proof of the theorem is complete.

References

[1] HEMELRIJK, J (1953), A theorem on the sign test when ties are present, Indag. Math. 14, No. 3.

[2] LEHMANN, E.L.(1959), Testing statistical hypotheses, J. Wiley & Sons, New York.

[3] SIEGEL, S. (1956) , Nonparametric statistics for the behavioral sciences,McGraw Hill, New York.

[4] WALD, A (1947) , Sequential analysis, J. Wiley & Sons, New York.

TABLE 1

Power of the UMPU test of $P(A) \leq P(B)$; $N=25$, $\alpha=.05$.
The entries on every second line are the powers of the nonrandomized test.

d \ p	.1	.15	.2	.25	.3	.35	.4	.5	.75	1
.1	.376	.318	.275	.243	.220	.203	.189	.169	.141	.126
	.098	.121	.127	.125	.120	.116	.113	.111	.101	.064
.2			.800	.671	.585	.523	.475	.406	.308	.259
			.579	.493	.427	.380	.348	.310	.244	.154
.3					.967	.886	.818	.711	.541	.447
					.909	.797	.716	.616	.466	.306

TABLE 2

Power of the UMPU test of $P(A) \leq P(B)$; $N=100$, $\alpha=.05$.
The entries on every second line are the powers of the nonrandomized test.

d \ p	.02	.05	.1	.15	.2	.3	.4	.5	.75	1
.02	.284	.205	.150	.127	.114	.100	.091	.086	.079	.074
	.051	.086	.086	.085	.080	.074	.070	.067	.062	.066
.05		.782	.472	.357	.295	.230	.195	.173	.142	.126
		.564	.348	.277	.232	.185	.159	.142	.117	.115
.1			.992	.859	.739	.576	.475	.408	.311	.259
			.976	.801	.675	.515	.421	.359	.270	.241

TABLE 3

Power of the UMPU test of $P(A) \leq P(B)$; $N=1000$, $\alpha=.05$.
The entries on every second line are the powers of the nonrandomized test.

d \ p	.005	.01	.02	.05	.1	.2	.3	.5	.75	1
.005	.777	.471	.295	.173	.126	.098	.087	.078	.072	.069
	.560	.347	.232	.142	.106	.087	.079	.072	.067	.064
.01		.990	.737	.408	.259	.174	.143	.115	.100	.092
		.971	.673	.359	.227	.157	.130	.107	.094	.087
.02			~ 1	.891	.640	.409	.312	.226	.180	.156
			~ 1	.865	.603	.382	.292	.214	.171	.148