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Existence and uniqueness of a stationary distribution for a class of Markov chains on $(-\infty,+\infty)$

by

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1. The main theorem

Let $\{X_n, n \geq 0\}$ be a Markov chain on $(-\infty,+\infty)$ with constant transition probabilities given by

$$\left.\begin{aligned} P\{X_{n+1} &= \alpha x+1-\alpha \mid X_n = x\} = p \\ P\{X_{n+1} &= \beta x \mid X_n = x\} = q = 1-p \end{aligned}\right\} \quad \text{if } \alpha x+1-\alpha \neq \beta x$$

and

$$P\{X_{n+1} = \beta x \mid X_n = x\} = 1 \qquad \text{if } \alpha x+1-\alpha = \beta ,$$

where p, α and β are constant parameters with values strictly between 0 and 1, and let F_n denote the distribution function of X_n $(n \geq 0)$. Then the following is true:

A. The probability is one that after a finite number of steps the process will enter the interval $I = [0,1]$ never to leave it again, i.e.

$$P\Big(\bigcup_{n=0}^{\infty} \bigcap_{m=n}^{\infty} \{X_m \in I\} \Big) = 1 ;$$

B. There is a unique stationary distribution; its distribution function F is continuous and is the unique solution of the functional equation

$$F(x) = pF\left(\frac{x-1+\alpha}{\alpha}\right) + qF\left(\frac{x}{\beta}\right) \qquad (x \in (-\infty,+\infty))$$

under the sideconditions

$$F(x) = 0 \text{ for } x \leq 0 \quad \text{and } F(x) = 1 \text{ for } x \geq 1 ,$$

and

$$\lim_{n \to \infty} F_n(x) = F(x) \quad \text{uniformly in } x .$$

2. Proof of A

It is clear that $\alpha x+1-\alpha \in I$ and $\beta x \in I$ whenever $x \in I$. Thus the sequence of events $\{X_n \in I\}$ $(n \geq 0)$ is nondecreasing and hence

(1) $$\{X_n \in I\} = \bigcap_{m=n}^{\infty} \{X_m \in I\} \qquad (n \geq 0) ,$$

(2) $$P\left(\bigcup_{n=0}^{\infty} \bigcap_{m=n}^{\infty} \{X_m \in I\} \right) = \lim_{n \to \infty} P\{X_n \in I\}$$

and $P\{X_n \in I \mid X_0 = x\} = 1$ for $n \geq 0$, $x \in I$. A fortiori

(3) $$\lim_{n \to \infty} P\{X_n \in I \mid X_0 = x\} = 1 \qquad (x \in I).$$

To show that (3) remains valid without the restriction $x \in I$, we note that we may think of the process $\{X_n, n \geq 0\}$ as being determined by its initial state X_0 and a sequence of independent random variables $\{U_n, n \geq 0\}$ with

$$P\{U_n=1\} = 1 - P\{U_n=0\} = p \qquad (n \geq 0) ,$$

by means of the definition

$$X_{n+1} = \begin{cases} \alpha X_n+1-\alpha & \text{if } U_n = 1 \\ \beta X_n & \text{if } U_n = 0 \end{cases} \qquad (n \geq 0) .$$

Thus we can define two auxiliary processes by putting

$$\begin{aligned} & Y_0 = Z_0 = X_0 \\ & Y_{n+1} = Y_n , \quad Z_{n+1} = Z_n+1-\alpha \quad \text{if } U_n = 1 \\ & Y_{n+1} = \beta Y_n , \quad Z_{n+1} = Z_n \quad \text{if } U_n = 0 \end{aligned} \qquad (n \geq 0) .$$

Since $\alpha x+1-\alpha < x$ whenever $x > 1$, this definition and (1) imply

(4) $$P\{X_n \in I \mid X_0 = x\} \geq P\{Y_n \leq 1\} = \sum_{j \geq \nu} \binom{n}{j} q^j p^{n-j} \qquad (n \geq 0,\ x > 1)$$

where $\nu = -\frac{\log x}{\log \beta}$. Because both $\alpha x+1-\alpha > x+1-\alpha$ and $\beta x > x$ when $x < 0$, a similar argument shows that

$$(5)\quad P\{X_n \in I \mid X_0 = x\} \geq P\{Z_n \geq 0\} = \sum_{j \geq \nu'} \binom{n}{j} p^j q^{n-j} \qquad (n \geq 0,\ x < 0)$$

with $\nu' = -\frac{x}{1-\alpha}$.

From (3), (4) and (5) we obtain

$$(6)\qquad \lim_{n\to\infty} P\{X_n \in I \mid X_0 = x\} \qquad (x \in (-\infty,+\infty)),$$

and hence A follows from (2) and

$$P\{X_n \in I\} = \int P\{X_n \in I \mid X_0 = x\}\, dF_0(x) \qquad (n \geq 0),$$

(dominated convergence).

3. Proof of B

The essence of the proof is contained in Theorem 1 below, which is a special case of a more general theorem stated by DUBINS and FREEDMAN in [1] and proved in [2]. The proof given here is less general, since it was expressly constructed for the problem at hand.

Let $\mathcal{B}$ be the set of all bounded real-valued functions on $(-\infty,+\infty)$ and let

$$\mathcal{D} = \{g \in \mathcal{B} \mid g(x) = 0 \text{ for } x < 0,\ g(x) = 1 \text{ for } x \geq 1\}.$$

For $g \in \mathcal{B}$ and $A \subset (-\infty,+\infty)$ we write

$$\|g\|_A = \sup_{x\in A} |g(x)|,$$

and

$$\|g\| = \|g\|_{(-\infty,+\infty)}.$$

It is easy to see that the operator T, defined on $\mathcal{B}$ by

$$Tg(x) = pg\left(\frac{x-1+\alpha}{\alpha}\right) + qg\left(\frac{x}{\beta}\right) \qquad (g \in \mathcal{B},\ x \in (-\infty,+\infty)),$$

is a linear operator which maps $\mathcal{B}$ into itself in such a way that

(7) $$Tg \in \mathcal{D} \quad (g \in \mathcal{D})$$

and

(8) $$\|Tg\| \leq \|g\| \quad (g \in \mathcal{B}) .$$

<u>Theorem 1</u>: There exist an integer $N \geq 1$ and a number $\rho \in (0,1)$, such that

(9) $$\|T^N g_1 - T^N g_2\| \leq \rho \cdot \|g_1 - g_2\| \text{ for all } g_1, g_2 \in \mathcal{D} .$$

<u>Proof</u>: The definitions of T and $\mathcal{D}$ imply that, for all $g_1, g_2 \in \mathcal{D}$ and all $x > 0$,

$$\|Tg_1 - Tg_2\|_{(-\infty, 1-\alpha)} \leq q \cdot \|g_1 - g_2\|$$

$$\|Tg_1 - Tg_2\|_{[x,\infty)} \leq p \cdot \|g_1 - g_2\| + q \cdot \|g_1 - g_2\|_{[x/\beta,\infty)} .$$

Because of (7) and (8), iteration gives

$$\|T^n g_1 - T^n g_2\|_{(-\infty, 1-\alpha)} \leq q \cdot \|g_1 - g_2\|$$

and

$$\|T^n g_1 - T^n g_2\|_{[\beta^n,\infty)} \leq (1-q^n) \cdot \|g_1 - g_2\|$$

for all $g_1, g_2 \in \mathcal{D}$ and $n \geq 1$. Hence (9) holds with

$$N = \min \{n \mid \beta^n \leq 1-\alpha\} , \qquad \rho = \max (q, 1-q^N) .$$

<u>Theorem 2</u>: The restriction of T on $\mathcal{D}$ has a unique fixed point F. This function F is a continuous distribution function and $\lim_{n\to\infty} \|T^n g - F\| = 0$ for all $g \in \mathcal{D}$.

<u>Proof</u>: Let $g \in \mathcal{D}$ and let N and ρ be such that (9) holds ($N \geq 1$, $0 < \rho < 1$). We have then, by virtue of (7), (8) and (9),

$$\|T^{n+m} g - T^n g\| \leq \rho^{\left[\frac{n}{N}\right]} \cdot \|T^m g - g\| \leq 2 \cdot \|g\| \cdot \rho^{\left[\frac{n}{N}\right]} \quad (n,m \geq 0) .$$

It follows that the sequence of functions $\{T^n g, n \geq 0\}$ is uniformly Cauchy convergent and hence converges uniformly to a limit F, necessarily an element of $\mathcal{D}$. The invariance of F under T follows from the uniform convergence of $T^{n+1}g$ to F and the continuity of the operator T implied by (8).

If $f \in \mathcal{D}$ is a fixed point of T, then we have

$$||f - F|| = ||T^N f - T^N F|| \leq \rho . ||f - F|| ,$$

and hence $||f - F|| = 0$, i.e. $f = F$, so that F is the unique fixed point of T in $\mathcal{D}$.

To show that F is a continuous distribution function we take $g \in \mathcal{D}$ to be a continuous distribution function. Then, by the definition of T, the same is true for all $T^n g$ $(n \geq 1)$, hence also for their uniform limit F, and the proof is complete.

Let us now return to the Markov chain $\{X_n, n \geq 0\}$. Its definition implies that $F_{n+1} = TF_n$ and hence $F_n = T^n F_0$ $(n \geq 0)$. Consequently B follows immediately from Theorem 2, except for the assertion that F_n converges uniformly to F in case $F_0 \notin \mathcal{D}$, i.e. in case $P\{X_0 \in I\} < 1$. To show the validity of this assertion, let $\varepsilon > 0$. In view of A and (2) there is then an integer $m \geq 0$, such that $P\{X_m \in I\} > 1-\varepsilon$.

Putting

$$\hat{F}_n(x) = P\{X_n \leq x \mid X_m \in I\} \qquad (n \geq m,\ x \in (-\infty,+\infty)) ,$$

we have, for all $n \geq m$ and all $x \in (-\infty,+\infty)$,

$$F_n(x) = P\{X_m \in I\} . \hat{F}_n(x) + P\{X_m \notin I, X_n \leq x\} ,$$

and hence

$$||F_n - F|| \leq ||F_n - \hat{F}_n|| + ||\hat{F}_n - F|| < ||\hat{F}_n - F|| + \varepsilon \qquad (n \geq m) .$$

Since $\hat{F}_m \in \mathcal{D}$ and $\hat{F}_{m+k} = T^k F_m$ $(k \geq 0)$, Theorem 2 implies that $\hat{F}_n$ converges uniformly to F as $n \to \infty$, and thus we obtain

$$\limsup_{n \to \infty} ||F_n - F|| < \varepsilon .$$

Since $\varepsilon > 0$ but otherwise arbitrary, the assertion follows.

References:

[1] L.E. DUBINS and D.A. FREEDMAN, Random distribution functions, to be published in: Proceedings of the Fifth Berkeley Symposium on Probability and Statistics.

[2] L.E. DUBINS and D.A. FREEDMAN, Invariant probabilities for certain Markov processes, to be published in Ann. Math. Statist.