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A production problem solved by Markov-programming

by

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1.1. Introduction.

A manufacturer produces and sells n different items 1, ...,n. Different items cannot be produced simultaneously. If a production-process is running, it is not possible to begin with another production. A productionprocess starts as soon as a decision is made to produce. At most M_i - s units of item i can be produced, if s units of item i are in stock. The number of units of time required to produce d units of item i is $T_{i,d}$. The production costs are $h_i(d)$. The setup cost for each productionrun of item i is K_i. Assume there are no costs and time attached to a change of production. The production-processes may succeed each other immediately. The cost of holding one unit of item i in stock for one unit of time is c_{i1}. Assume that the cost of storing a unit for any length of time is proportional to the time for which it is kept in stock. If the manufacturer cannot satisfy immediately the demand of a customer, then he makes emergency purchases to fulfil the demand. An emergency purchase costs the manufacturer c_{i2} per unit of item i. Each customer asks for one type of item. Assume that the customers, who ask for item i, arrive according to a Poisson process with rate λ_i *). Suppose that the Poisson processes are independent. Assume further that the random demands of the customers are independent. A customer, who asks for item i, buys k units with probability p_{ik} $(k = 0,1, \ldots,N_i)$. $\sum_{k=0}^{N_i} p_{ik} = 1$, $p_{i0} < 1$ $(i = 1, \ldots,n)$.

The manufacturer looks for a schedule of production for an infinite planning horizon, such that the expected average cost per unit of time is minimal.

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*) Equivalently can be stated that the customers arrive according to a Poisson process with rate λ, and that the demands of the customers are independent. Let q_{ik} be the probability that a customer demands k units of item i $(k = 0,1, \ldots,N_i)$. If $q_i = \sum_{k=0}^{N_i} q_{ik}$, then $\sum_{i=1}^{n} q_i = 1$. It can be proved that the customers, who ask for item i, arrive according to a Poisson process with rate λq_i. Further these Poisson processes are independent.

Let $(s_1, \ldots, s_n)$ correspond to the situation that s_i units of item $(i = 1, \ldots, n)$ are in stock and no production process is running. The manufacturer has to decide for each situation whether to begin a production of a certain item or not. He will have to develop a strategy z that specifies his decision for every possible situation. The strategy will be called optimal, if it minimizes the expected average cost per unit of time. Our task will be determine for each state $(s_1, \ldots s_n)$ a feasible decision $z(s_1, \ldots s_n)$. An optimal strategy may be obtained by a mathematical method, called Markov-programming. This method is a generalisation of the method of R.Howard and has been developed by G.de Leve in [1]. The manufacturers problem cannot be solved with Howards method*). Before presenting a review of the method and its application to the manufacturers problem, some numerical results will be given first.

1.2. Numerical results.

Each strategy z prescibes in every state $(i_1, \ldots, i_n)$ a decision $z(i_1, \ldots, i_n) = (0, \ldots, 0, d_k, 0, \ldots, 0)$. If $d_k > 0$ then in state $(i_1, \ldots, i_n)$ a decision is made to produce d_k units of item k. If $d_k = 0$ then nothing is done. The expected average cost per unit of time will denoted by r.

a) One item, each customer demands one unit.

$n = 1$; $M_1 = 4$; $\lambda_1 = 1$; $N_1 = 1$; $p_{11} = 1$; $T_{i,d} = 1$, $d = 1, \ldots, 4$; $K_1 = 3$; $c_{11} = 2$; $c_{12} = 16$.

a1) h (1) = 2; h (2) = 3,8; h (3) = 5,5; h (4) = 7.

Optimal strategy z:

z(0) = z(1) = 3

z(2) = z(3) = z(4) = 0

r = 8,49.

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*) If we assume that the durations of the productionprocesses are expo--nentially distributed, then Howards method can be applied in principe. The method of G.de Leve allows us to use arbitrary distributions for the durations of productionprocesses.

a2) $h(d) = 2d$.

Optimal strategy:

$z(0) = 3; z(1) = 2; z(2) = z(3) = z(4) = 0$.

$r = 8{,}59$.

b) <u>One item; each customer demands one or two units.</u>

$N_1 = 2$; $p_{11} = p_{12} = \frac{1}{2}$. The other data as in a).

b1) Optimal strategy:

$z(0) = 4$; $z(1) = 3$; $z(2) = z(3) = z(4) = 0$.

$r = 13{,}28$.

b2) The same optimal strategy as in b1) and $r = 13{,}51$.

c) <u>Two items; each customer demands one unit.</u>

$n = 2$; $N_i = 1$; $M_i = 3$; $\lambda_i = 1$; $p_{i1} = 1$; $T_{i,d} = 1$, $d = 1, \ldots, 3$; $K_i = 3$;

$c_{i1} = 2$; $c_{i2} = 16$, $i = 1,2$ and $h_1(d) = h_2(d)$.

c1) $h_1(1) = 2$; $h_1(2) = 3{,}8$; $h_i(3) = 5{,}5$.

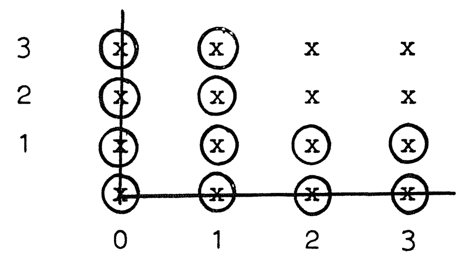
Optimal strategy:

$z(0,i) = (3,0)$, $i = 0,1, \ldots,3$; $z(i,0) = (0,3)$, $i = 1, \ldots,3$;

$z(1;i) = (2,0)$, $i = 1, \ldots,3$; $z(i;1) = (0,2)$, $i = 2,3$;

the other $z(i;j) = (0,0)$.

$r = 17{,}77$.



c2) $h_1(d) = 2d$.

The same optimal strategy and $r = 17{,}96$.

d) Two items, each customer demands either one or two units.

$N_1 = N_2 = 2$; $p_{ij} = \frac{1}{2}$, $i,j = 1,2$; the other data as in c).

d1) Optimal strategy is the same as in c1).
$r = 29{,}81$.

d2) Same optimal strategy as in c2).
$r = 30{,}05$.
Note. Suppose that in case a) after finishing a productionprocess at least τ units of time are needed to start a new production. Let i correspond here to the sitation that i units are in stock and at least τ units of time no productionprocess is running.
If we take the data of a) and $\tau = 1$, then

e) e1) Optimal strategy:
$z(0) = z(1) = 3; z(2) = z(3) = z(4) = 0$.
$r = 8{,}59$.
e2) Same optimal strategy, $r = 8{,}74$.

2. Markov-programming

Problems of the type to which Markov-programming can be applied are always related to some physical system. In our case the system comprises the inventory and the quantity in production.

At each point of time t the system is in some state x. In the mathematical model the state x is represented by a point in a finite dimensional Cartesian space. The set of all possible states will be called the state space X.

Besides deterministic transformations the state of the system may be subjected to random transitions. Owing to the latter transitions the system performs a random walk through the state space. In case no decisions are taken, this evolution is called the natural process.

A condition for application of Markov-programming is that for each initial state of the system the underlying natural process can be described by a stationary strong Markov-process.

A family of n-dimensional random vektors*) $\{\underline{x}_t,\ t \in T\}$ is called a Markov-process, it with probability one,

$$P\{\underline{x}_{t+s} \leq \lambda \,|\, \underline{x}_u,\ u \leq t\} = P\{\underline{x}_{t+s} \leq \lambda \,|\, \underline{x}_t\} \qquad (2.1)$$

for each $\lambda \in R^n$ and each s, $t \in T$, s > 0.

Roughly speaking: If we know the "present" then the additional knowledge of the "past" does not contribute any relevant information about the "future".

We use the term a stationary Markov-process, if the distribution of (1.1) does not depend on t. If the foregoing remains true when the arbitrary but fixed time t is replaced by a random variable $\underline{\tau}$, then the process is called a stationary strong Markov-process. The validity of (1.1) is only required for a random variable $\underline{\tau}$, determining the first entry time in a given closed set C in X (regardless the initial state of the system). Only closed sets C, are considered which satisfy certain regularity conditions, given in [1].

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*) Measurable functions on a probability space, which take their values in a n-dimensional Cartesian space R^n. Random vektors, called random variables if n = 1, are underlined.

In addition the following definitions are given with regard to a Markov-process with state space X. A subset S of X is called ergodic if the system remains with probability 1 in S as soon as it assumes a state of S. A ergodic set is called simple ergodic, if it contains no disjunct ergodic sets. The set T of states, which do not belong to any set from a given system of simple ergodic sets is called the transient set, if T does not contain a ergodic set.
A decomposition of the state space into simple ergodic sets and a transient set is not always unique. In this paper it is assumed that always a decomposition is given with disjunct simple ergodic sets.
We note that if the state space is finite or denumerable a decomposition can be given, such that an ergodic set S is simple ergodic if every state in S can be reached from every other state in S. The simple ergodic sets are then always disjunct.
In decisionproblems losses and gains play important roles. It is no restriction to consider only losses. In general the decisionmaker wants to influence the natural process by interventions, basically a finite number in each finite interval. He will try to prevent infavourable excursions through the state space. After an intervention the system may be transferred into some other state. Between two sucessive interventions the system is subject to the natural process. For that reason the natural process has to be defined for each initial state.
It is convenient to assume that at each point of time a decision is made. The decision will be primarily to decide whether to intervene or not secondly which intervention to choose. We call the decision not to intervene a null-decision.
We shall assume that in every state x exists a set D(x) of feasible decisions d. In many situations decisions result in a random transition in the state of the system. For that reason a decision is defined mathematically by the probability distribution of the state into which the system is transferred (by the decision !) A null-decision in state x can be interpreted as the probability distribution concentrated in x itself.
Decisions which lead to deterministic transitions are also defined by "concentrated" probability distributions but now in the new state.
The solution of the decisionproblem is given in the form of a strategy.

A strategy dictates at each point of time a feasible decision (including null-decisions) on the basis of available information. The result of the natural process and the extra transitions caused by the strategy is called the decisionprocess. We denote by Z the class of strategies z, which base their decisions on the present state only and add to each state x one and only one decision $d \in D(x)$.
Since we have only interventions and null-decisions each strategy $z \in Z$ partitions the state space into two disjunct sets, one denoted by A_z, comprising states in which always interventions are made, the other consisting of states in which always null-decisions are dictated.
From now on we shall consider only strategies $z \in Z$. Under some general conditions it can be shown this is no restriction. Further it can be proved under certain weak conditions that the decisionprocess, if a strategy $z \in Z$ is applied, is also a stationary strong Markov-process.
In order to find out which strategy is the best one we need a criterion. In most cases the random costs in a infinite period of time will be infinite with probability 1 for each strategy. Under rather weak conditions it can be proved that for each strategy $z \in Z$ the random average costs per unit time converge with probability 1 to a fixed value, when the system starts in a non-transient state and is considered during a time T with $T \to \infty$. As criterion for an optimal strategy we shall adopt the expected value of the average costs per unit time, when the system is considered for an infinite period of time.

The existence is assumed of a non-empty set A_0 with the following property. The set A_0 consists of states, where each $z \in Z$ dictates an intervention. Hence for each $z \in Z$,

$$A_z \supset A_0 . \qquad (2.2)$$

It is assumed that in the natural process from each initial state the set A_0 can be reached within a finite time with probability 1.

To each state x and decision $d \in D(x)$ two random walks, denoted by $\underline{W}^0$ and $\underline{W}^d$, can be assigned. During the walk $\underline{W}^d$ the decision d transfers the system to a random state $\underline{u}$.

From state $\underline{u}$ the system is subjected to the natural process and the walk ends as soon as a state of A_0 has been reached.
The functions $k_0\ (x)$ and $k_1\ (x;d)$ represent the expected costs incurred during $\underline{W}^0$ and $\underline{W}^d$ respectively. The functions $t_0(x)$ and $t_1\ (x;d)$ represent the expected duration of $\underline{W}^0$ and $\underline{W}^d$ respectively. We now define $k(x;d)$ and $t(x;d)$ by the difference in expected costs and expected duration of the walks $\underline{W}^d$ and $\underline{W}^0$. In formula:

$$k(x;d) = k_1\ (x;d) - k_0\ (x) \tag{2.3}$$
$$t(x;d) = t_1\ (x;d) - t_0\ (x)\ . \tag{2.4}$$

Note that for null-decisions $\underline{W}^d$ and $\underline{W}^0$ are indentical, and consequenty:

$$k(x;d) = t(x;d) = 0 \quad . \tag{2.5}$$

It follows from their definitions that $k(x;d)$ and $t(x;d)$ do not depend on any particular strategy. Hence we need only <u>once</u> to determine the $(x;d)$ - functions $k(x;d)$ and $t(x;d)$.

Let $\underline{I}_n$ $(n = 1,2, \dots)$ be the sequence of future interventionstates, if strategy $z \in Z$ is applied. The sequence $\{\underline{I}_n\ , n \geq 1\}$ constitutes a stationary Markov-process in A_z with discrete time parameter.
The probability distribution of $\underline{I}_n$, given the initial state x, will be denoted by

$$p^{(n)}(A;z;x) \qquad n = 1,2, \dots^{*)} \tag{2.6}$$

Consider an arbitrary but fixed strategy $z \in Z$. Suppose the Markov-process $\{\underline{I}_n\ , n \geq 1\}$ in A_z has m disjunct simple ergodic sets E_j. Choose in each E_j an arbitrary state e_j. Consider next the following functional equations in $r(z;x)$ and $c(z;x)$:

$$r(z;x) = \int_{A_z} r(z;I)\ p^{(1)}\ (dI;z;x) \tag{2.7}$$

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*) A is some Borelset in A_z. This distribution can be extended to the whole space by taking: $p^{(n)}\ (A;z;x) = p^{(n)}\ (A \cap A_z;z;x)$.

$$c(z;\ x) = k(x;\ z(x)) - r(z;\ x)\ t\ (x;\ z(x)) + \int_{A_z} c(z;\ I)\ p^{(1)}(dI;z;\ x) \qquad (2.8)$$

$$c(z;\ e_j) = 0, \quad j = 1, \ldots, m\ . \qquad (2.9)$$

The $k(x;\ z(x))$ and $t(x;\ z(x))$ are given functions [c.f. (2.3),(2.4)]. Note that for $x \notin A_z$ from (2.5) follows:

$$c(z;\ x) = \int_{A_z} c(z;\ I)\ p^{(1)}\ (dI;\ z;\ x)\ . \qquad (2.10)$$

It can be proved that $r(z;\ x)$ represents the average costs per unit time, if the initial state x belongs to a simple ergodic set, strategy z is applied and the system is considered for an infinite period of time. For states in the same simple ergodic set $r(z;\ x)$ has the same constant value. If x does not belong to any simple ergodic set $r(z;\ x)$ represents the expected average costs per unit time.
Because $r(z;\ x)$ is constant on a simple ergodic set, $r(z;\ x)$ indicates the most favourable simple ergodic set to start, if strategy z is applied. But the criterionfunction $r(z;\ x)$ does not indicate the most profitable initial state in that simple ergodic set. This state is determined by means of the $c(z;\ x)$. It can be proved that for two states x_1 and x_2 in the same simple ergodic set the difference in total expected costs is given by

$$c(z;\ x_1) - c(z;\ x_2)\ . \qquad (2.11)$$

A strategy z_0 is called optimal with respect to the class Z of strategies, if for each $x \in X$

$$r(z_0;\ x) = \min_{z \in Z} r(z;\ x)\ . \qquad (2.12)$$

With the aid of solutions $r(z;\ x)$ and $c(z;\ x)$ from (1.7),(1.8) and (1.9) it is possible to construct a better strategy. We shall illustrate this by giving an iterationprocedure which converges to an optimal strategy.

The iterationprocedure yields a sequence of strategies $z^{(i)}$ $(i = 1,2...)$ of which, under certain conditions, the following interesting properties can be proved:

a) $r(z^{(i)}; x) \geq r(z^{(i+1)}; x)$ (2.13)

b) $\lim\limits_{i\to\infty} r(z^{(i)}; x) = \min\limits_{z\in Z} r(z; x)$.

for each $x\in X$. Proofs and conditions are given in [1], and will be omitted here. We shall restrict ourselves to some definitions and an intuitive illustration of the procedure. First a bare catalogue of definitions.

Let the mixed strategy $d.z$ with $z\in Z$ dictate the decision d in the initial state and then decisions in accordance with z. We define $r(d.z; x)$ and $c(d.z; x)$ by:

$$r(d.z; x) = E\{r(z; \underline{u})|d\} , \quad (2.15)$$

$$c(d.z; x) = k(x; d) - r(d.z; x)t(x; d) + E\{c(z; \underline{u})|d\} . \quad (2.16)$$

where $\underline{u}$ the random state is in which the system is transferred by the decision d in the initial state x. Note that the probability distribution of $\underline{u}$ is determined only by $d\in D(x)$. From the definitions it follows that for both null-decision and $d = z(x)$ we have,

$$r(d.z; x) = r(z; x), \; c(d.z; x) = c(z; x) . \quad (2.17)$$

Let the mixed strategy $A.z$ with $z\in Z$ interdict any intervention up to the moment that the system assumes a state in the closed A for the first time. From that time onwards decisions are made in accordance with z. We define $r(A.z; x)$ and $c(A.z; x)$ by,

$$r(A.z; x) = E\{r(z;\underline{v})|x,A\} , \quad (2.18)$$

$$c(A.z; x) = E\{c(z;\underline{v})|x,A\} . \quad (2.19)$$

where $\underline{v}$ is the first state of A taken on if x is the initial state. Note that the probability distribution of $\underline{v}$ depends only on the natural process, the set A and the state x. From the definitions it follows that for each state $x\in A$,

$$r(A.z; x) = r(z; x), \; c(A.z; x) = c(z; x) . \quad (2.20)$$

In order to gain an insight in the principle of solution, we consider the following problem: Suppose a decisionmaker has to make his decisions in accordance with a strategy z. In the initial state however he is free to choose a decision d. The decisionmaker certainly looks for that particular decision that makes the expected average costs per unit time as small as possible. Note that each drop in these costs leads to an infinite saving in an infinite period of time. If in the initial state x the decision $d \in D(x)$ is chosen the expected average costs per unit time are given by $r(d \circ z; x)$ [cf. (2.18)]. Hence, determine

$$\min_{d \in D(x)} r(d \circ z;\, x). \qquad (2.21)$$

Define

$$D_z(x) = \{d \mid d \in D(x),\ r(d \circ z;x) ; \min_{d^* \in D(x)} r(d^* \circ z;x)\}\ . \qquad (2.22)$$

In order to determine which d has to be chosen if $D_z(x)$ contains more than one decision, we note it is possible to prove that the difference in total expected costs of the mixed strategy $d \circ z$. and the strategy z is given by

$$c(d \circ z;x) - c(z;x)\ . \qquad (2.23)$$

A possible drop in these costs will be in general finite.
Consequently, first the d-function $r(d \circ z;x)$ is minimized with respect to $d \in D(x)$. If the minimizing set $D_z(x)$ contains more than one decision, (2.23) is minimized with respect to $d \in D_z(x)$, or what is equal,

$$\min_{d \in D_z(x)} c(d \circ z;x) \qquad (2.24)$$

is determined. If z(x) belongs to $D_z(x)$ and also minimizes $c(d \circ z;x)$, let the decision **z(x)** be chosen.
By this procedure a possible new decision is added to each state x.
We have then constructed a new strategy z_1.
The following important result can now be proved.

$$r(z_1;x) \leq r(z;x)\ . \qquad (2.25)$$

Hence we find a strategy z_1 at least as good as the strategy z.
It follows from (2.17) that every interventionstate of strategy z is also an interventionstate of z_1, hence

$$A_{z_1} \supset A_z . \qquad (2.26)$$

In other words by the foregoing procedure the decisionmaker may change but not cancel the intervention dictated by the orginal strategy z. This is the reason we need a second mechanism, which may cancel an intervention.

Consider the following problem. Suppose the decisionmaker has to make his decisions in accordance with a strategy z_1. But he is allowed to determine the point of time where upon the strategy comes into operation. This will be done by choosing a closed set A, the strategy z_1 comes into operation on the moment the system takes on a state of A. The decisionmaker certainly looks for a delay that decreases the expected costs per unit time. Hence sets A will be considered, which satisfy for each $x \in X$ [c.f.(2.18)],

$$r(A.z_1;x) < r(z_1;x) \quad . \tag{2.27}$$

It can be proved that the effect of a delay in the total expected costs is measured by

$$c(A.z_1;x) - c(z_1;x) \quad . \tag{2.28}$$

Consequently, sets A which satisfy for each $x \in X$

$$r(A.z_1;x) = r(z_1;x) \tag{2.29}$$

and $c(A.z_1;x) \stackrel{<}{=} c(z_1;x)$ (2.30)

will also be considered.

We denote by $\mathcal{X}_{z_1}$ the class of all closed sets A satisfying (2.27) or (2.29) and (2.30). From (2.20) it follows $A_{z_1} \in \mathcal{X}_{z_1}$.

Let $$A_{z_1}' \stackrel{def}{=} \bigcap_{A \in \mathcal{X}_{z_1}} A \quad . \tag{2.31}$$

If A_{z_1}' belongs to $\mathcal{X}_{z_1}$, then the set A_{z_1}' is the solution of the second decisionproblem.

It can be proved that the strategy z_2 defined by

$$z_2(x) = \begin{cases} z_1(x) & \text{if } x \in A_{z_1}' \\ \text{null-decision} & \text{otherwise} \end{cases} , \tag{2.32}$$

satisfies for each $x \in X$,

$$r(z_2;x) \leq r(z_1;x) \quad . \tag{2.33}$$

From the solution of the two foregoing decisionproblems, we can now deduce that a strategy $z_0 \in Z$ is optimal, if it has the following properties for each $x \in X$:

$$\min_{d \in D(x)} r(d.z_0;x) = r(z_0;x) \quad , \tag{2.34}$$

$$\min_{d \in D_z(x)} c(d.z_0;x) = c(z_0;x) \tag{2.35}$$

and $$A'_{z_0} = A_{z_0} \quad . \tag{2.36}$$

These formulas present us a direct approach, with which an optimal strategy may be determined.
We give now an iterationprocedure, which runs as follows:

Preparatory part.
Determine the $(x;d)$-functions $k(x;d)$ and $t(x;d)$.

Iterative approach.
Let $z^{(n-1)}$ be the strategy obtained at the $(n-1)^{th}$ cycle of the iterationprocedure.
1) Determine the functions $r(z^{(n-1)};x)$ and $c(z^{(n-1)};x)$ by using the functional equations (2.7),(2.8) and (2.9).
2a) Determine the functions $r(d.z^{(n-1)};x)$ and $c(d.z^{(n-1)};x)$ by using the relations (2.15) and (2.16).
b) Determine for each $x \in X$ the set $D_{z^{(n-1)}}(x)$ consisting of decisions $d \in D(x)$, which minimize $r(d.z^{(n-1)};x)$.
c) Minimize for each $x \in X$ the d-functions $c(d.z^{(n-1)};x)$ with respect to $d \in D_{z^{(n-1)}}(x)$.
d) Add to each x a solution of c). If $z^{(n-1)}(x)$ is a solution of d, this decision will be added to state x. (This instruction had been made in order to advance the convergence of the sequence of strategies $\{z^{(i)}, i \geq 1\}$). As soon as operation d) has been performed a new strategy $z_1^{(n-1)}$ has been constructed.

3) Determine the functions $r(z^{(n-1)}; x)$ and $c(z_1^{(n-1)}; x)$ by using the funtional equations (2.7),(2.8) and (2.9).

4) Determine the set $A'_{z_1}(n-1)$ [c.f.(2.31)]. The new strategy $z^{(n)}$is given by

$$z^{(n)}(x) = \begin{cases} z_1^{(n-1)}(x) & \text{if } x \in A'_{z_1}(n-1) \\ \text{null-decision} & \text{otherwise .} \end{cases}$$

End of the n^{the} cycle.

An optimal strategy has been reached if the strategies in two successive iterationcycles are identical.

Some notes

1) For any two states x, and x_2 belonging to the same simple ergodic set

$$r(z; x_1) = r(z; x_2) \quad . \tag{2.37}$$

If the Markov-process in A_z has only one simple ergodic set, then for each $x \in X$, $d \in D(x)$ and closed set A:

$$r(z; x) = r(d.z; x) = r(A.z; x) = r(z) . \tag{2.38}$$

2) The functions $r(d.z; x)$ and $c(z; x)$ are determined by functional equations. If these equations cannot be solved analytically they often can be solved by Monte Carlo methods.

3) The way in which the set A'_z can be determined depends heavily on the structure of the decisionproblem considered.

In the boundary points of A'_z if will be sometimes indifferent whether to intervene or not.

4) We shall give a variant of the foregoing iterationprocedure. The steps 3) and 4) can be replaced by other steps. For our problem the new iterationprocedure shall appear to be simpler from a computational point of view.

Let the strategy $z \in Z$ be given and let z_1 be determined in accordance with (2.21) and (2.22). We now introduce mixed strategies of the following type:

a) The mixed strategy of the form $(z_1)z$ dictating
   1) first an intervention in accordance with z_1
   2) then interventions in accordance with z.

If $\hat{z} = (z_1)z$, the x-functions $r(\hat{z};x)$ and $c(\hat{z};x)$ are defined by

$$r(\hat{z};x) \overset{def}{=} E\{r(z;\underline{u}|x;z_1)\} \qquad (2.39)$$

and

$$c(\hat{z};x) \overset{def}{=} E\{k(\underline{I}_1;z_1(\underline{I}_1)) - r(\hat{z};\underline{I}_1)t(\underline{I}_1;z_1(\underline{I}_1)) \Big| x;z_1\} +$$
$$+ E\{c(z;\underline{u})|x;z_1\} , \qquad (2.40)$$

where $\underline{u}$ is the state into which the system is transferred by the decision $z_1(\underline{I}_1)$, and $\underline{I}_1$ is the first state of A_{z_1} taken on by the system, if it starts in x and strategy $\hat{z}$ is applied.

From the definitions it follows that for $x \in A_{z_1}$ [c.f(2.15),(2.16)],

$$r(\hat{z};x) = \min_{d \in D(x)} r(d.z;x) \qquad (2.41)$$

$$c(\hat{z};x) = \min_{d \in D_z(x)} c(d.z;x) . \qquad (2.42)$$

b) The mixed strategy of the form $A.\hat{z}$, where A is a closed set in X. This strategy interdicts any intervention up to the moment that the system assumes a state of A for the first time. From that time onwards dicisions are made in accordance with $\hat{z}$.

The x-functions $r(A.\hat{z};x)$ and $c(A.\hat{z};x)$ are defined by

$$r(A.\hat{z};x) \overset{def}{=} E\{r(\hat{z};\underline{v})|x;A\} \qquad (2.43)$$

and

$$c(A.\hat{z};x) \overset{def}{=} E\{c(\hat{z};\underline{v})|x;A\} \qquad (2.44)$$

where $\underline{v}$ is the first state of A taken on if x is the initial state. Note that the probability distribution of $\underline{v}$ depends only on the natural process.

Let $\mathcal{G}_{\hat{z}}$ be the class of all closed sets A satisfying for each $x \in X$:

$$r(A.\hat{z};x) < r(\hat{z};x) \qquad (2.45)$$

or $\quad r(A.\hat{z};x) = r(\hat{z};x)$ and $c(A.\hat{z};x) \leq c(\hat{z};x)$. (2.46)

From (2.43) and (2.44) it follows $A_{z_1} \in \mathcal{G}_{\hat{z}}$.

Define

$$A'_{\hat{z}} = \bigcap_{A \in \mathcal{G}_{\hat{z}}} A \ . \qquad (2.47)$$

If $A'_{\hat{z}} \in \mathcal{G}_{\hat{z}}$ and we define the strategy z_3 by

$$z_3(x) = \begin{cases} z_1(x) & \text{if } x \in A'_{\hat{z}} \\ \text{null-decision otherwise} , \end{cases}$$

then $\quad r(z_3;x) \leq r(z_1;x) \leq r(z;x)$. (2.48)

Proofs are given in [1].

Second formulation of the iterationprocedure.

Replace in the foregoing iterationprocedure the steps 3) and 4) by:

3) Determine the functions $r(\hat{z}^{(n-1)};x)$ and $c(\hat{z}^{(n-1)};x)$.

4) Determine the set $A'_{\hat{z}}$ (n-1). The new strategy $z^{(n)}$ is given by

$$z^{(n)}(x) = \begin{cases} z_1^{(n-1)}(x) & \text{if } x \in A'_{\hat{z}} \\ \text{null-decision otherwise} \end{cases} . \qquad (2.49)$$

3. Application to the manufacturers problem.

In this section it is shown how the problem may be solved with Markov-programming. First we shall have to define in detail the state space, the natural process, the set of interventionstates and the set A_0. Next we treat a special case of the manufacturers problem in order to gain insight into the principle of solution. Then for the general case a probabilistic preparation is given. We treat then the general case and give some numerical examples. Finally some extensions are indicated and a discussion is given.

3.1. Definition of the state space

At each point of time the following information will be of interest.
(1) the inventory
(2) whether a productionprocess is running or not
(3) if a productionprocess is running, the item that is being produced and the productionsize. Further the time that the productionprocess is already running.

From the properties of the Poisson process it follows that the knowledge of the epochs of arrival of customers in the past does not contribute any relevant information.

We take as state space X the (2n + 1)-dimensional space consisting of the points:

(a) $((i),(0),0)$.

Where $(i) \overset{\text{def}}{=} (i_1, \ldots, i_n)$, i_k integer, $0 \leq i_k \leq M_k$, $k = 1, \ldots, n$. This state corresponds to the situation that i_k units of item $k (k = 1, \ldots, n)$ are in stock and no productionprocess is running

(b) $((i),(d),t)$.

Where $(d) \overset{\text{def}}{=} (0, \ldots, 0, d_s, 0, \ldots, 0)$, d_s integer, $1 \leq d_s \leq M_s - i_s$ and $0 \leq t \leq T_{s,d_s}$, $(s = 1, \ldots, n)$.

This state corresponds to the situation that i_k units of item k,

$(k = 1, \ldots, n)$ are in stock and a productionprocess is running since t units of time. Further d_s units of item s are being produced.

The natural process results from the passage of time and the demands of customers. The assumption that the customers arrive according to independent Poisson processes and the assumed independence of the demands together imply that the natural process is a stationary strong Markov-process.
The natural process can start in each state of the state space.
In the natural process no decisions are made, hence no production-process is started. On the other hand the natural process can start in a state which is of the form $((i),(d),t)$, i.e. a productionprocess is running. As soon as the productionprocess has been finished the produced item is added to the inventory.

Only in the states $((i),(0),0)$ feasible interventions are possible.
In the other states only null-decisions can be made.
Let X_0 be the subspace of the points $((i),(0),0)$. For simplicity of notation we denote $((i),(0),0)$ from hereon by (i).
Let $D(i)$ be the set of feasible decisions in state (i). Then,

$$D(i) = \{(d) \mid (d) = (0, \ldots, 0, d_s, 0, \ldots, 0);\ d_s \text{ integer},\ 0 \leq d_s \leq M_s - i_s \ (s = 1, \ldots, n)\}$$

For $(d) \in D(0)$ it is required $(d) \neq (0)$.

If in state (i) the feasible decision $(d) = (0, \ldots, 0, d_s, 0, \ldots, 0), d_s \neq 0$, has been made, the system is transferred instantly into state $((i),(d),0)$. In this state the production of d_s units of item s starts. The system runs next successively through the states $((\underline{i})_t,(d),t)$, $0 \leq t \leq T_{s,d_s}$. As soon as the production has been finished the system assumes the state $(\underline{y}) + (d)$. Where $(\underline{i})_t$ is the inventory if the productionprocess is running since t units of time, and $(\underline{y})$ is the inventory just before the end of production.
If in state (i) the null-decision has been made, the system remains in (i) until it is transferred in state $(\underline{j})$ by a random demand of a customer.

The class Z of strategies, which will be considered, consists of strategies z, which add to each state x a decision $z(x) \in D(x)$.
In the states $x \notin X_0$ only null-decisions can be made, and in the state (0) the decisionmaker has always to intervene. Hence for each $z \in Z$:

$$A_z \subset X_0 \quad \text{and} \quad (0) \in A_z. \tag{3.1.2}$$

This implies

$$A_0 = \bigcap_{z \in Z} A_z = \{(0)\}. \tag{3.1.3}$$

In order to determine the functions $r(z;x)$, $c(z;x)$, $r(d.z;x)$ and $c(d.z;x)$ we need the following definitions:
For $(d) \in D((i))$, $(d) > (0)$:

$q^{(d)}_{(i)(j)} \overset{\text{def}}{=}$ probability that (j) is the first future state in X_0 taken on by the system, if the initial state is (i), in which decision (d) has been made. (3.1.4)

and

$$q^{(0)}_{(i)(j)} \overset{\text{def}}{=} \begin{cases} 1 & \text{for } (j) = (i) \\ 0 & \text{otherwise.} \end{cases} \tag{3.1.5}$$

From the definition: $\sum_{(j) \in X_0} q^{(d)}_{(i)(j)} = 1.$ (3.1.6)

$p^z_{x,y} \overset{\text{def}}{=}$ probability that y is the first future interventionstate taken on by the system, if the initial state is x and strategy z is applied [c.f (2.6)]. (3.1.7)

Obviously,

$$\sum_{(j) \in A_z} p^z_{x,(j)} = 1, \quad x \in X. \tag{3.1.8}$$

It can be easily verified that for x ; ((i),(d),0), (d) > (0), $(d) \in D((i))$:

$$p^{z}_{x,(j)} = q^{(d)}_{(i)(j)} + \sum_{(h)\notin A_z} q^{(d)}_{(i)(h)} \, p^{z}_{(h)(j)} , \quad (j) \in A_z . \quad (3.1.9)$$

Let $\mathcal{P}_z$ be the matrix with elements $p^{z}_{(i)(j)}$, $(i),(j) \in A_z$.
Let the Markov-chain $\mathcal{P}_z$ have m disjunct simple ergodic sets E_j and let $(e_j) \in E_j$ be chosen arbitrary $(j = 1, \ldots, m)$.
The functional equations (27),(28) and (29) become:

$$r(z;(i)) = \sum_{(j)\in A_z} p^{z}_{(i)(j)} \, r(z;(j)), \quad (i) \in X_0, \quad (3.1.10)$$

$$c(z;(i)) = k((i);z(i)) - r(z;(i)) \, t((i);z(i)) + \sum_{(j)\in A_z} p^{z}_{(i)(j)} \, c(z;(j)), \quad (i) \in X_0, \quad (3.1.11)$$

$$c(z;(e_j)) = 0 , \quad j = 1, \ldots, m \quad (3.1.12)$$

$$r(z;x) = \sum_{(j)\in A_z} p^{z}_{x,(j)} \, r(z;(j)), \quad x \notin X_0 \quad (3.1.13)$$

$$c(z;x) = \sum_{(j)\in A_z} p^{z}_{x,(j)} \, c(z;(j)), \quad x \notin X_0 . \quad (3.1.14)$$

The relation (2.15) becomes:

$$r((d).z;(i)) = r(z;((i),(d),0)) . \quad (3.1.15)$$

Suppose $(d) \neq (0)$, then follows from (3.1.9)

$$r((d).z;\, i) = \sum_{(j)\in A_z} \Big\{ q^{(d)}_{(i)(j)} + \sum_{(h)\notin A_z} q^{(d)}_{(i)(h)} p^{z}_{(h)(j)} \Big\} r(z;(j)) =$$

$$= \sum_{(j)\in A_z} q^{(d)}_{(i)(j)} \, r(z;(j)) + \sum_{(h)\notin A_z} q^{(d)}_{(i)(h)} \sum_{(j)\in A_z} p^{z}_{(h)(j)} r(z;(j)) =$$

$$= \sum_{(j) \in A_z} q_{(i)(j)}^{(d)} \; r(z;(j)) + \sum_{(h) \notin A_z} q_{(i)(h)}^{(d)} \; r(z;(h)). \qquad (3.1.16)$$

Hence

$$r((d).z;(i)) = \sum_{(j) \in X_0} q_{(i)(j)}^{(d)} \; r(z;(j)). \qquad (3.1.17)$$

This formula is also true for $(d) = (0)$. In the same way,

$$c((d).z;(i)) = k((i);(d)) - r((d).z;(i)) \; t((i);(d)) +$$

$$+ \sum_{(j) \in X_0} q_{(i)(j)}^{(d)} \; c(z;(j)). \qquad (3.1.18)$$

For each $z \in Z$ the corresponding set A_z of interventionstates is contained in X_0. Thus we can restrict ourselves to sets A with $A \subset X_0$, when we have to determine the set $A_{\hat{z}}'$ or the set A_z'. [c.f. the definitions of the mixed strategies $A.\hat{z}$ and $A.z_1$]. It can be easily verified that (2.45) or (2.46) (respectively (2.29 or (2.30)) holds for each $x \in X$, if (2.45) or (2.46) (respectively (2.29) or (2.30)) holds for each $x \in X_0$.

In the states $x \notin X_0$ only null-decisions can be made. With the aid of the foregoing it is easily seen that we can restrict ourselves to the states of X_0, when we apply any of the two given iterationprocedures. If $x \notin X_0$, the knowledge of $r(z;x)$, $c(z;x)$ and the other related quantities is not needed for solving the problem.

In order to gain insight into our method for solving the problem we shall treat firstly the most simple case.

3.2. One item; each customer demands one unit.

The customers arrive in accordance with a Poisson process $\{\underline{w}(t), t \geq 0\}$ with rate $\lambda = \lambda_1$.

Two important properties of the Poisson process are

(a) The number of arrivals in any time interval of the lenght h has a Poisson distribution with mean λh. Hence,

$$P\{\underline{w}(t+h) - \underline{w}(t) = n\} = e^{-\lambda h} \frac{(\lambda h)^n}{n!}, \; n = 0,1,\ldots \qquad (3.2.1)$$

is the probability of n arrivals in a time interval of the length h,
(b) the length of the time interval from 0 up to the first arrival, and thereafter the intervals between successive arrivals, are independently distributed with the exponential density function

$$\lambda e^{-\lambda t} , \qquad (3.2.2)$$

Hence the length $\underline{t}_i$ of the time interval from 0 up to the $i^{\underline{th}}$ arrival is gamma distributed with density function

$$\frac{\lambda^i}{(i-1)!} t^{i-1} e^{-\lambda t} \qquad (3.2.3)$$

Note $\quad \underline{t}_i = \min \{t | \underline{w}(t) = i\}$, this implies

$$P\{\underline{t}_i \leq t\} = P\{\underline{w}(t) \geq i\} . \qquad (3.2.4)$$

Hence $$\int_0^t \frac{\lambda^i}{(i-1)!} x^{i-1} e^{-\lambda x} dx = \sum_{k=i}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} . \qquad (3.2.5)$$

The random variabele $\underline{t}_i$ has expectation

$$E\, \underline{t}_i = \frac{i}{\lambda} . \qquad (3.2.6)$$

The determination of the k(i;d) and t(i;d) functions.

For simplicity of notation we write $c_1, c_2, K, M_1, h(d)$ and T_d instead of $c_{11}, c_{12}, K_{11}, M_1, h_1(d)$ and $T_{1,d}$.

The walks $\underline{w}^0$ and $\underline{w}^d$ terminate both in state 0 ($A_0 = \{0\}$).

Note that the probability is zero, that the manufacturer has to make an emergency purchase for the last customer, in any walk because each customer demands one unit. Hence the only costs in the walk $\underline{w}^0$ are inventory costs.

Decision d made in the state i transforms the system to the state $[i,d,0]$. After this decision the walk $\underline{w}^d$ is subjected to the natural process until the state 0 is reached. If during the productionprocess a demand cannot be fulfilled from inventory, the manufacturer has to make an emergency purchase. In the walk $\underline{w}^d$ we also have inventory costs.

Let $\underline{v}(t)$ be the demand in a interval of the length t. Each customer demands one unit, hence $\underline{v}(t) = \underline{w}(t)$. If in state i has been decided to produce d units, let $\underline{y}$ be the inventory just before the end of production. It is easily verified,

$$k_0(i) = c_1 \sum_{k=1}^{i} E\underline{t}_k \tag{3.2.7}$$

and

$$t_0(i) = E\underline{t}_i \ . \tag{3.2.8}$$

When during a production the demand exceeds supply we have to make emergency purchases. After some reflections:

For $d > 0$,

$$k_1(i;d) = h(d) + K + c_1 \sum_{k=1}^{i} E\underline{t}_k + c_1 E \sum_{k=\underline{y}+1}^{\underline{y}+d} \underline{t}_k + $$

$$+ c_2 E\{(\underline{v}(T_d)-i)\mathcal{L}(\underline{v}(T_d)-i)\} , \tag{3.2.9}$$

and

$$t_1(i;d) = T + E\underline{t}_{\underline{y}} + d. \tag{3.2.10}$$

$$\text{Where } \mathcal{L}(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \leq 0. \end{cases} \tag{3.2.11}$$

From the well-known relation

$$E\underline{x} = E\{E(\underline{x} | \underline{y})\} \tag{3.2.12}$$

it follows

$$E \sum_{k=\underline{y}+1}^{\underline{y}+d} \underline{t}_k = \sum_{y=0}^{i} P\{\underline{y} = y\} \sum_{k=y+1}^{y+d} E\underline{t}_k. \tag{3.2.13}$$

Let a_k be the probability that during a production of d units the demand equals k, then

$$a_k = e^{-\lambda T_d} \frac{(\lambda T_d)^k}{k!} , \ k = 0,1, \ldots \tag{3.2.14}$$

Obviously,

$$P\{\underline{y} = y\} = a_{i-y} \quad , \ y = 1, \ldots, i$$

$$P\{\underline{y} = 0\} = \sum_{k=i}^{\infty} a_k \ . \tag{3.2.15}$$

After some calculations (use $E\underline{t}_k = \frac{k}{\lambda}$),

$$E \sum_{k=\underline{y}+1}^{\underline{y}+d} \underline{t}_k = \frac{1}{2\lambda} d(d+1) + \frac{d}{\lambda} \sum_{k=0}^{i-1} (i-k)a_k , \tag{3.2.15}$$

$$\begin{aligned} E\{\underline{v}(T_d - i)L(\underline{v}(T_d) - i)\} &= \sum_{k=i}^{\infty} (k-i)a_k = \\ &= \lambda T_d - i + \sum_{k=0}^{i-1} (i-k)a_k \end{aligned} \tag{3.2.16}$$

and

$$E\underline{t}_{\underline{y}+d} = \sum_{y=0}^{i} P\{\underline{y} = y\} \frac{y+d}{\lambda} = \frac{d}{\lambda} + \frac{1}{\lambda} \sum_{k=0}^{i-1} (i-k)a_k . \tag{3.2.17}$$

Hence for $d > 0$:

$$k(i;d) = k_1(i;d) - k_0(i) =$$

$$= h(d) + K + \frac{c_1}{2\lambda} d(d+1) + \left(\frac{c_1 d}{\lambda} + c_2\right) \sum_{k=0}^{i-1} (i-k)a_k + c_2 (\lambda T_d - i), \tag{3.2.18}$$

and

$$t(i;d) = t_1(i;d) - t_0(i) =$$

$$= T_d + \frac{d-i}{\lambda} + \frac{1}{\lambda} \sum_{k=0}^{i-1} (i-k)a_k . \tag{3.2.19}$$

$$k(i;d) = t(i;d) = 0 \quad \text{for } d = 0 . \tag{3.2.20}$$

<u>Determination of the probabilities q_{ij}^d and p_{ij}^z.</u>

From (3.2.15) it follows

$$q_{ij}^0 = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i, \end{cases} \tag{3.2.21}$$

and for $d \geq 1$, $i \geq 1$

$$q_{ij}^{d} = a_{i+d-j}\ ,\ j = d+1, \dots, d+i$$

$$q_{ij}^{d} = 1 - \sum_{k=0}^{i-1} a_k\ ,\ j = d$$

$$q_{ij}^{d} = 0 \qquad ,\ \text{the other } j, \tag{3.2.22}$$

and $$q_{0j}^{d} = \begin{cases} 1 & \text{if } j = d \\ 0 & \text{if } j \neq d \end{cases} \tag{3.2.23}$$

To determine the p_{ij}^{z} two cases are distinguished

(a) $i \notin A_z$,

(b) $i \in A_z$.

(a) Let k be the largest integer in A_z smaller than i, then

$$p_{ij}^{z} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \tag{3.2.24}$$

(b)

$$p_{ij}^{z} = q_{ij}^{z(i)} + \sum_{h \notin A_z} q_{ih}^{z(i)} p_{hj}^{z}\ ,\ j \in A_z \tag{3.2.25}$$

$$p_{ij}^{z} = 0 \qquad ,\ j \notin A_z\ . \tag{3.2.26}$$

Note that to each strategy z a unique integer $m_z \in A_z$ corresponds with the property $m_z + 1, \dots, M \notin A_z$.

If the strategy z has the additional property that $0, 1, \dots, m_z \in A_z$, then $\mathcal{P}_z = ((p_{ij}^{z}))$, $i, j \in A_z$ has only one simple ergodic set, because m_z can be reached from every state. In this case $r(z;x)$ does not depend on x.

<u>Determination of the set $A'_{\hat{z}}$ [c.f. 2.47].</u>

Obriously $r(A.\hat{z};i)$ and $c(A.\hat{z};i)$ are only defined if A contains an integer $\leq i$. We need to consider only sets $A \subset X_0$, because outside X_0 only null-decisions can be made. Each customer demands one unit, hence

$$r(A\circ\hat{z};i) = r(\hat{z};h) \ , \ c(A\circ\hat{z};i) = c(\hat{z};h), \tag{3.2.27}$$

if h is the largest integer in A smaller or equal to i.
In addition we note [c.f (2.39) and (2.40)] , that

$$r(\hat{z};j) = r(\hat{z};k) \ , \ j \notin A_{z_1} \tag{3.2.28}$$

and

$$c(\hat{z};j) = c(\hat{z};k) \ , \ j \notin A_{z_1} . \tag{3.2.29}$$

where k is the largest integer in A_{z_1} smaller than j.

With the aid of formula (3.2.27) we can give a simple procedure, which yields the set $A_{\hat{z}}'$.

This procedure runs as follows [c.f.(2.45) and (2.46)]:
Start with the set

$$H_1 = \{0\} .$$

If

$$r(\hat{z};0) > r(\hat{z};1)$$

or $r(\hat{z};0) = r(\hat{z};1)$ and $c(\hat{z};0)>c(\hat{z};1)$
then

$$H_2 \overset{\text{def}}{=} H_1 \cup \{1\} \ ,$$

otherwise

$$H_2 \overset{\text{def}}{=} H_1 .$$

Next for $j = 2, \ldots, M$: let h be the largest integer in H_j, if

$$r(\hat{z};h) > r(\hat{z};j)$$

or $r(\hat{z};h) = r(\hat{z};j)$ and $c(\hat{z};h)> c(\hat{z};j)$
then $H_{j+1} \overset{\text{def}}{=} H_j \cup \{j\}$,

otherwise $H_{j+1} \overset{\text{def}}{=} H_j$.

From the stucture of this procedure it follows immediately that $H_{M+1} \in \mathcal{G}_{\hat{z}}$ and that H_{M+1} encloses each set $A \in \mathcal{G}_{\hat{z}}$, hence $A_{\hat{z}}' = H_{M+1}$.
The construction of $A_{\hat{z}}'$ and the relation $A_{z_1} \subset A_{\hat{z}}'$,

imply that $H_{j+1} = H_j$ if $(j) \notin A_{z_1}$. Hence for the determination of $A'_{\hat{z}}$ we have only to consider the states of A_z and we have only to know $r(\hat{z};i)$ and $c(\hat{z};i)$ for $i \in A_{z_1}$. And these quantities are given by (2.41) and (2.42).

Determination of an optimal strategy

We shall use the second formulation of the iterationprocedure.
For simplicity of notation we write r_i, c_i, k_i, t_i and p_{ij} instead of $r(z;i)$, $c(z;i)$, $k(i;z(i))$, $t(i;z(i))$ and ${}^z p_{ij}$.

The iterationprocedure runs as follows:

Prepatory part.

Determine the functions $k(i;d)$ and $t(i;d)$ [c.f.(3.2.18),(3.2.19)].

Iterative approach.

Let $z^{(n-1)}$ be the strategy obtained at the $(n-1)^{th}$ cycle.
Suppose $\mathcal{P}_{z^{(n-1)}}$ has m disjunct simple ergodic sets E_k. Choose an arbitrary state $e_k \in E_k$ $(k = 1, \ldots, m)$
Solve the following linear equations in the unknown r_i and c_i:

$$r_i = \sum_{j=0}^{M} p_{ij}\, r_j \qquad (i = 0, \ldots, M)$$

$$c_i = k_i - r_i\, t_i + \sum_{j=0}^{M} p_{ij}\, c_j \quad (i = 0, \ldots, M)$$

$$c_{e_k} = 0, \ (k = 1, \ldots, m). \tag{3.2.40}$$

Note $p_{ij} = k_j = t_j = 0$ if $j \notin A_{z^{(n-1)}}$.

2)a) Using the solutions r_i and c_i of 1), determine for each i the set S(i) of decisions d, which minimize

$$\sum_{j=0}^{M} q_{ij}^{d}\, r_j. \tag{3.2.41}$$

(b) Minimize for each i the d-function

$$k(i;d)-\{\sum_{j=0}^{M} q_{ij}^{d} r_j\}t(i;d) + \sum_{j=0}^{M} q_{ij}^{d} c_j \tag{3.2.42}$$

subject to $d \in S(i)$.

(c) Add to each i a solution of b). If $z^{(n-1)}(i)$ is a solution of b), this decision will be added to state i. As soon as operation c) has been performed a new strategy $z_1^{(n-1)}$ has been constructed.

d) Use the r_i and c_i of 1). If $i \in A_{z_1}(n-1)$, set

$$\rho_i = \min_{d \in D(i)} \sum_{j=0}^{M} q_{ij}^{d} r_j \tag{3.2.43}$$

and

$$\gamma_i = \min_{d \in D_z(i)} \{k(i;d) - \rho_i t(i;d) + \sum_{j=0}^{M} q_{ij}^{d} c_j\} \tag{3.2.44}$$

3). The set $A_{\hat{z}}'(n-1)$ is determined as follows:

$$\text{Let} \qquad H_1 = \{0\} \; . \tag{3.2.45}$$

Do for j = 1, ...,M-1 the following procedure:

Let h be the largest integer in H_j .

If $j \in A_{z_1}(n-1)$ and in addition

$$\rho_h > \rho_j$$

$$\text{or} \qquad \rho_h = \rho_j \text{ and } \gamma_h > \gamma_j \tag{3.2.46}$$

$$\begin{aligned} &\text{then} \quad H_{j+1} \overset{\text{def}}{=} H_j \cup \{j\}, \\ &\text{otherwise} \quad H_{j+1} \overset{\text{def}}{=} H_j \; . \end{aligned} \tag{3.2.47}$$

Finally

$$A_{\hat{z}}'(n-1) = H_M. \tag{3.2.48}$$

4) The new strategy $z^{(n)}$ is given by

$$z^{(n)}(i) = \begin{cases} z_1^{(n-1)}(i) & \text{if } i \in A_{\hat{z}}'(n-1) \\ 0 & \text{otherwise} \end{cases} .$$

When the new strategy is identical with the one from the $(n-1)^{\underline{th}}$ cycle, it is an optimal strategy. The iterationprocedure converges in a finite number of steps.

Remarks

1) If $\mathcal{P}_z$ has only one simple ergodic set, then $r(z;i)$ does not depend on i. Hence $r(z;i) = r(d.z;i) = r(A.\hat{z};i) = r(z)$. $\mathcal{P}_z$ has only one simple ergodic set if $0,1, \ldots, m_z \in A_z$.

2) The number of equations of (3.2.40) is determined by the number of points in A_z and the number of simple ergodic sets of $\mathcal{P}_z$. For $i \notin A_z$ the $r(z;j)$ and $c(z;i)$ are lineair combinations of the $r(z;j)$ and $c(z;j)$, $j \in A_z$.

3) If we assume that the duration of a production of d units is a random variabele $\underline{T}_d$, the problem can be solved in the same manner. If $F_d(t)$ the distribution function of $\underline{T}_d$ is, then we have to replace

$$a_k = e^{-\lambda T_d} \frac{(\lambda T_d)^k}{k!} \tag{3.2.49}$$

by

$$a_k^* = \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^k}{k!} \, dF_d(t) \,. \tag{3.2.50}$$

Note $\sum_{k=1}^{\infty} k \, a_k^* = \lambda \, E \, \underline{T}_d$. Further we have to replace in the formulas for the k- and t-functions T_d by $E \, \underline{T}_d$.

Example 1.

Numerical data: $M = 4; \lambda = 1; K = 3; c_1 = 2; c_2 = 16; T_{1,d} = 1$, $d = 1, \ldots 4$;
$h(1) = 2; h(2) = 3{,}8; h(3) = 5{,}5; h(4) = 7$.

From a table of the Poisson distribution:
$a_0 = 0{,}368$; $a_1 = 0{,}368$; $a_2 = 0{,}184$; $a_3 = 0{,}061$.

The probabilities q_{ij}^d can be calculated easily with the aid of formulas (3.2.20),(3.2.21) and (3.2.22).

From (3.2.18) and (3.2.19) follows after some calculations:

i \ d	1	2	3	4
0	23	28,80	36,50	46
1	13,62	20,16	28,60	
2	10,87	18,88		
3	11,43			

The function k(i;d).

i \ d	1	2	3	4
0	2	3	4	5
1	1,37	2,37	3,37	
2	1,10	2,10		
3	1,02			

The function t(i;d).

Iterationprocedure.

1) Start with the strategy

$$z = (z(0),z(1),z(2),z(3),z(4)) = (4,3,2,0,0) \;.$$

$$\mathcal{P}_z = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \end{array} .$$

The state $m_z = 2$ can be reached from every state, hence $\mathcal{P}_z$ has only one simple ergodic set. Choose $e_1 = 2$.

Because $r(z;i)$ does not depend on i, we need to solve only the equations :

$$c_0 = c_2 + 46 - 5r$$

$$c_1 = c_2 + 28,60 - 3,375$$

$$c_2 = c_2 + 18,88 - 2,105$$

$$c_2 = 0$$

$$c_3 = c_2$$

$$c_4 = c_2 \;.$$

Solution:

$r = 8,99; c_0 = 1,05; c_1 = -1,70; c_2 = c_3 = c_4 = 0$.

2) Because $r(z;i)$ does not depend on $i, r(z;i) = r$, we have to minimize for each i (3.2.42) subject to $d \in D(i)$.

b) State i	Alternative d	Test quantity $k(i;d) - rt(i;d) + \sum_{j=0}^{4} q_{ij}^{d} c_j$.
0	1	3,32
	2	1,83
	3	0,54 ←
	4	1,05
1	0	-1,70
	1	0,23
	2	-1,15
	3	-1,70 ←
2	0	0
	1	0,53
	2	0 ←
3	0	0 ←
	1	2,03

c) $z_1 = (3,3,2,0,0)$.

d) $\rho_0 = \rho_1 = \rho_2 = 8{,}99$. $\gamma_0 = 0{,}54; \gamma_1 = -1{,}70; \gamma_2 = 0$.

3) From

$$\gamma_1 < \gamma_0 < \gamma_2 \quad ,$$

it follows

$$A_{\hat{z}} = \{0,1\}.$$

4) New strategy $z = (3,3,0,0,0)$.

<u>End 1^{the} cycle</u> .

1) $z = (3,3,0,0,0)$. $\mathcal{P}_z = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $\mathcal{P}_z$ has only one simple ergodic set.

Choose $e_1 = 1$. We have to solve the following lineair equations :

$$c_0 = c_1 + 36{,}50 - 45$$

$$c_1 = c_1 + 28{,}60 - 3{,}375$$

$$c_1 = 0$$

$$c_2 = c_3 = c_4 = c_1 .$$

Solution:

$r = 8{,}49$; $c_0 = 2{,}55$; $c_1 = c_2 = c_3 = c_4 = 0$.

2)b) We have to minimize for each i (3.2.42) subject to $d \in D(i)$.
After some calculations,

c) $z_1 = (3,3,0,0,0)$, and hence

$\rho_0 = \rho_1 = 8{,}49$. $\gamma_0 = 2{,}55$; $\gamma_1 = 0$.

3) From $\gamma_1 < \gamma_0$, it follows

$$A_{\hat{z}}' = \{0,1\} = A_{z_1} .$$

We can now conclude that $z = (3,3,0,0,0)$ is an optimal strategy, because

$$\min_{d \in D(i)} r(d.z;i) = r(z;i)$$

$$\min_{d \in D_z(i)} c(d.z;i) = c(z;i)$$

and

$$A_{\hat{z}}' = A_z .$$

If we had applied the other iterationprocedure, we find this optimal strategy also after two cycles.

Remark that if the optimal strategy (3,3,0,0,0) is applied, and once an intervention has been made, the next interventions are all made in state 1. The random time between two interventions is given by

$\underline{t}_{\underline{y} + 3-1}$.

It is easily verified,

$$E\, \underline{t}_{\underline{y} + 2} = E\, \frac{\underline{y} + 2}{\lambda} = 2{,}368 .$$

and (3.2.51)

$$\text{var}\ \underline{t}_{\underline{y}+2} = E\,\frac{\underline{y}+2}{\lambda^2} + \text{var}\,\frac{\underline{y}+2}{\lambda} = 2{,}603 \ .$$

Remark. Suppose that the duration of each production has an exponential distribution with parameter 1. Hence $E\,\underline{T}_{1,d} = 1$, $d = 1, \ldots, 4$. It is easily verified [c.f.(3.2.50)]

$$a_k^* = \frac{1}{2^k + 1}\ , \quad k = 0, 1, \ldots \tag{3.2.52}$$

After some calculations :

i \ d	1	2	3	4
0	23	28,8	36,5	46
1	16	22,8	31,5	
2	13,5	21,8		
3	13,25			

The function $k(i;d)$.

1	2	3	4
2	3	4	5
1,5	2,5	3,5	
1,25	2,25		
1,125			

The function $t(i;d)$.

The strategy (3,3,0,0,0) is also optimal in this example.
$r = 9$; $c_0 = 0{,}5$; $c_1 = c_2 = c_3 = c_4 = 0$.

Example 2.
We take $h(d) = 2\,d$ and the other data as in a).

We start with the strategy $z = (3,3,0,0,0)$. After some calculations,
$r = 8{,}64$; $c_0 = 2{,}46$; $c_1 = c_2 = c_3 = c_4 = 0$.

Ater two iterationcycles we find the optimal strategy $z = (3,2,0,0,0)$, with $r = 8{,}59$; $c_0 = 2{,}64$; $c_1 = c_2 = c_3 = c_4 = 0$.

When we assume that the duration of each production is a random variable, which is exponentially distributed with parameter 1, then $z = (3,2,0,0,0)$ is optimal and $r = 9{,}2$; $c_0 = 0{,}2$ and $c_i = 0$, $i = 1, \ldots, 4$.

3.3. Probabilistic preparation.

First some problems will be treated, which are connected with the determination of the k- and t- functions and the $q^{(d)}_{(i)(j)}$.

Let $\{\underline{w}(t),\ t\geq 0\}$ be a Poisson process with rate λ. Suppose this Poisson process represents the arrivals of customers. Let $\{\underline{y}_k\}_{k=1}^{\infty}$ be a sequence of mutually independent random variables with the common distribution ,

$$P\{\underline{y}_k = j\} = p_j,\ j = 0, \ldots, N. \quad \sum_{j=0}^{N} p_j = 1. \tag{3.3.1}$$

Assume that $\underline{y}_k$ represents the demand of the $k^{\underline{th}}$ customer.

Let $\{\underline{w}(t),\ t\geq 0\}$ and $\{\underline{y}_k,\ k\geq 1\}$ be mutually independent stochastic processes.

Let
$$\underline{v}(t) \overset{def}{=} \sum_{k=1}^{\underline{w}(t)} \underline{y}_k. \tag{3.3.2}$$

The random variable $\underline{v}(t)$ can be interpreted as the total demand in any interval of time of the length t.
The generating function of $\underline{y}_k$ is defined by

$$f(s) = \sum_{n=0}^{N} p_n s^n \tag{3.3.3}$$

and
$$g_t(s) = \sum_{n=0}^{\infty} P\{\underline{w}(t) = n\} s^n = e^{-\lambda t(1-s)} \tag{3.3.4}$$

is the generating function of $\underline{w}(t)$. The generating function

$$h_t(s) = \sum_{n=0}^{\infty} P\{\ \underline{v}\ (t) = n\}\ s^n \tag{3.3.5}$$

of $\underline{v}(t)$ satisfies the relation [4]:

$$h_t(s) = g_t(f(s)). \tag{3.3.6}$$

Set

$$a_n(t) = P\{\underline{v}(t) = n\},\ n \geqslant 0 \tag{3.3.7}$$

then

$$\sum_{n=0}^{\infty} a_n(t)s^n = e^{-\lambda t(1-\sum_{n=0}^{N} p_n s^n)} \tag{3.3.8}$$

Hence

$$a_n(t) = e^{-\lambda t(1-p_0)} \sum_{k_1=0}^{\left[\frac{n}{N}\right]} \sum_{k_2=0}^{\left[\frac{n-Nk_1}{N-1}\right]} \cdots \sum_{k_{N-1}=0}^{\left[\frac{n-\sum_{i=1}^{N-2}(N-i+1)k_i}{2}\right]} \alpha(k_1, \ldots, k_{N-1}), \tag{3.3.9}$$

where $[x]$ is the largest integer smaller or equal to x and

$$\alpha(k_1, \ldots, k_{N-1}) = \prod_{j=1}^{N-1} \frac{(\lambda t p_{N-j+1})^{k_j}}{k_j!} \frac{(\lambda t p_1)^{n-\sum_{j=1}^{N-1}(N-j+1)k_j}}{(n-\sum_{j=1}^{N-1}(N-j+1)k_j)!}\ . \tag{3.3.10}$$

From the well-known formula $E\,\underline{v}\,(t) = h_t'\,(1)$ it follows that

$$E\,\underline{v}(t) = \lambda t \sum_{n=1}^{N} np_n\ . \tag{3.3.11}$$

$$\text{Let } \underline{t}_k \overset{\text{def}}{=} \min\ \{t \mid \underline{v}(t) \geq k\},\ k = 1,2\ldots \tag{3.3.12}$$

We may interprete $\underline{t}_k$ as the interval from 0 up to the epoch, on which the $k^{\underline{th}}$ unit is demanded.

Let $F_k(t)$ be the distributionfunction of $\underline{t}_k$, then

$$F_k(t) = P\{\underline{t}_k \leq t\} = P\{\ \underline{v}(t) \geq k\} = 1- \sum_{n=0}^{k-1} a_n\ (t). \tag{3.3.13}$$

Let F(x) be the distribution function of a non-negative random variable $\underline{x}$, then [4] , $E(\underline{x}) = \int_0^\infty (1 - F(x))\,dx$, hence

$$E\underline{t}_k = \sum_{n=0}^{k-1} \int_0^\infty a_n(t)\,d\,t \quad . \tag{3.3.14}$$

This implies the recursion formula ($\underline{t}_0 \equiv 0$),

$$E\underline{t}_k = E\underline{t}_{k-1} + \int_0^\infty a_{k-1}(t)\,d\,t, \; k = 1,2, \ldots \tag{3.3.15}$$

Lemma 3.3.1 Let $\underline{x}_1, \ldots, \underline{x}_n$ be mutually independent random variables and let $S_k(x)$ be the distribution function of $\underline{x}_k$.
Suppose there does not exist a real c with $P\{\underline{x}_i = c\} > 0$ and $P\{\underline{x}_j = c\} > 0$ for some i and j, $i \neq j$. Assume $E\underline{x}_k$ is finite ($k = 1, \ldots, n$).
Then

$$E \max(\underline{x}_1, \ldots, \underline{x}_n) = \sum_{i=1}^{n} E\left[\underline{x}_i \prod_{j \neq i} S_j(\underline{x}_i)\right] . \tag{3.3.16}$$

Proof. We consider n = 2, for higher n the proof is the same.
There does not exist a real c with $P\{\underline{x}_1 = c\} > 0$ and $P\{\underline{x}_2 = c\} > 0$,

hence

$$\max(\underline{x}_1, \underline{x}_2) = \underline{x}_1 \underline{\chi}_{\{\underline{x}_2 \leq \underline{x}_1\}} + \underline{x}_2 \underline{\chi}_{\{\underline{x}_1 \leq \underline{x}_2\}}, \tag{3.3.17}$$

where $\underline{\chi}_A$ is a random variable which is equal one on A and zero otherwise.
The independence of $\underline{x}_1$ and $\underline{x}_2$ implies

$$E \max(\underline{x}_1, \underline{x}_2) = \int_{-\infty}^{+\infty} x S_2(x)\,d S_1(x) + \int_{-\infty}^{+\infty} x S_1(x)\,d S_2(x). \tag{3.3.18}$$

The following is a preparation for the determination of the $p^{z}_{(i)(j)}$ and the set $A_{\hat{z}}'$.

Customers, who ask for item i, arrive according to a Poisson process $\{\underline{w}_i(t), t \geq 0\}$ with rate λ_i. A customer, who asks for item i, demands k units with probability p_{ik}. The Poisson processes $\{\underline{w}_i(t), t \geq 0\}$, $i = 1, \ldots n$ are mutually independent, hence the customers arrive according to a Poisson process $\{\underline{w}(t), t \geq 0\}$ with rate λ.

Where

$$\underline{w}(t) = \sum_{i=1}^{n} \underline{w}_i(t) \text{ and } \lambda = \sum_{i=1}^{n} \lambda_i . \tag{3.3.19}$$

The interval from 0 up to arrival of the first customer, and thereafter the intervals between the successive arrivals of customers are independently distributed random variables with the common exponential density

$\lambda e^{-\lambda t}$. Hence $P\{\text{a customer asks for item } i\} = \int_0^\infty P\{\underline{w}_i(t) = 1 | \underline{w}(t) = 1\} \lambda e^{-\lambda t}\, dt = \frac{\lambda_i}{\lambda}$. (3.3.20)

This formula implies

Lemma 3.3.2 Let q_{ik} be the probability that a customer demands k units of item i. Then,

$$q_{ik} = p_{ik} \frac{\lambda_i}{\lambda}, \; k = 0, \ldots, N_i \;, \; i = 1, \ldots, n. \tag{3.3.21}$$

Each customer asks for one type of item. Let the state $(i) \in X_0$ be given. We now introduce the set

$$V(i) \overset{\text{def}}{=} \{(j) | (j) = (i_1, \ldots, i_{h-1}, j_h, i_{h+1}, \ldots, i_n), \; j_h < i_h, \; i_h - j_h \leq N_h, \; h = 1, \ldots, n\} \tag{3.3.22}$$

and the probability

$f_{(i)(j)} \overset{\text{def}}{=}$ probability that (j) is the first state of V(i) taken on by the system, if it starts in (i) and is subjected to the natural process. (3.3.23)

From the definition:

$$f_{(i)(j)} = 0, \text{ if } (j) \notin V(i), \text{ and} \tag{3.3.24}$$

$$\sum_{(j) \in V(i)} f_{(i)(j)} = 1 . \tag{3.3.25}$$

Lemma 3.3.3. Let $(j) \in V(i)$, then

$$f_{(i)(j)} = \begin{cases} \dfrac{1}{1-c_{(i)}} \; q_{h,i_h - j_h} & \text{if } j_h \neq 0 \\[2ex] \dfrac{1}{1-c_{(i)}} \displaystyle\sum_{t = i_h}^{N_h} q_{ht} & \text{if } j_h = 0, \end{cases} \qquad (3.3.26)$$

where

$$c_{(i)} = \sum_{r=1}^{n} q_{r0} + \sum_{\substack{r=1 \\ i_r = 0}}^{n} \sum_{t=1}^{N_r} q_{rt} \,. \qquad (3.3.27)$$

<u>Proof.</u> $f_{(i)(j)} = \sum_{m=1}^{\infty} P$ {the first m-1 customers demand each 0 units or ask for an item, which is not in stock and owing to the demand of the $m^{\underline{th}}$ customer the state (j) is reached}. (3.3.28)

The demands are mutually independent, hence

$$f_{(i)(j)} = \begin{cases} \displaystyle\sum_{m=1}^{\infty} c_{(i)}^{(m-1)} \; q_{h,i_h - j_h} & \text{if } j_h \neq 0, \\[2ex] \displaystyle\sum_{m=1}^{\infty} c_{(i)}^{(m-1)} \sum_{t = i_h}^{N_h} q_{ht} & \text{if } j_h = 0. \end{cases} \qquad (3.3.29)$$

Let A be a subset of X_0. We now introduce,

$\beta_{(j)(k)A} \overset{\text{def}}{=}$ probability that (k) is the first state of A taken on by the system, if it starts in (j) and is subjected to the natural process.

From the definition:

$\beta_{(j)(k)A} = 0$, if $(k) \notin A$ or $(k) \leq (j)$ is not true.*) (3.3.31)

$\beta_{(k)(k)A} = 1$, if $(k) \in A$.

---

*) $(k) \leq (j)$ means $k_s \leq j_s$ $s = 1, \ldots, n$, and $(k) < (j)$ means $(k) \leq (j)$ and $(k) \neq (j)$.

The recursionformula

$$\beta_{(i)(k)A} = \sum_{(j)\in V(i)} f_{(i)(j)} \cdot \beta_{(j)(k)A}, \quad (i) \neq (k) \tag{3.3.32}$$

can be easily verified.

3.4. Manufacturers problem (continuation of section 3.1).

Customers, who ask for item r arrive according to a Poisson process $\{\underline{w}_r(t),\ t \geq 0\}$ with rate $\lambda_r (r = 1, \ldots, n)$. Let $\underline{v}_r(t)$ be the total demand for item r in an interval of the lenght t.
Let $a_{r,n}(t)$ be the probability that n units of item r are demanded in an interval of the lenght t. Let $\underline{t}_{r,k}$ be the lenght of the interval from 0 up to the epoch, on which the $k^{\underline{th}}$ unit of item r is demanded. The random variable $\underline{v}_r(t)$ is given by (3.3.2), if $\underline{w}(t) = \underline{w}_r(t)$ and $P\{\underline{y}_k = j\} = p_{rj}$ is taken. The formula (3.3.9) gives the probability $a_{r,n}(t)$ by taking $\lambda = \lambda_r$, $N = N_r$, $p = p_{rj}$. The probability distribution of $\underline{t}_{r,k}$ is given by (3.3.13) by taking $a_n(t) = a_{r,n}(t)$.
The Poisson processes $\{\underline{w}_r(t),\ t \geq 0\}$, $r = 1, \ldots, n$ are mutually independent and the random demands are mutually independent. Hence $\underline{v}_1(t), \ldots, \underline{v}_n(t)$ are mutually independent and $\underline{t}_{1,k_1}, \ldots, \underline{t}_{n,k_n}$ are mutually independent for each $(k_1, \ldots, k_n)$.

Determination of k- and t- functions.

The only costs in the walk $\underline{W}^0$ are storage costs and possibly costs of emergency purchases. Set

$$\underline{m}((i)) = \max_{h = 1, \ldots, n} \{\underline{t}_{h,i_h}\}. \tag{3.4.1}$$

It is easily verified that *)

$$k_0((i)) = \sum_{r=1}^{n} c_{r1}\, E \sum_{k=1}^{i_r} \underline{t}_{r,k} + \sum_{r=1}^{n} c_{r2}\, E\{\underline{v}_r(\underline{m}((i))) - i_r\} \tag{3.4.2}$$

and

$$t_0((i)) = E\,\underline{m}((i)).$$

*) From (3.3.13) and (3.4.1) it follows that $\underline{v}_r(\underline{m}((i)) \geq i_r$ with probability one.

Assume $(d) = (0, \ldots, 0, d_s, 0, \ldots, 0)$, $d_s \neq 0$. Let $(\underline{y})$ be the inventory, just before the end of the production of d_s units of item s, if the inventory was given by (i) at the start of production. Set

$$\underline{m}((\underline{y}) + (d)) = \max_{h = 1, \ldots, n} \{\underline{t}_{h,\underline{y}_h + d_h}\} . \tag{3.4.4}$$

It is easily verified that

$$\begin{aligned} k_1((i);(d)) = {} & h_s(d_s) + K_s + \sum_{r=1}^{n} c_{r1}\, E(\sum_{k=1}^{i_r} \underline{t}_{r,k}) + \\ & + \sum_{r=1}^{n} c_{r2}\, E\{(\underline{v}_r(T_{s,d_s}) - i_r)\,\ell\,(\underline{v}_r(T_{s,d_s}) - i_r)\} + \\ & + \sum_{r=1}^{n} c_{r2}\, E\{\underline{v}_r(\underline{m}((\underline{y}) + (d))) - (\underline{y}_r + d_r)\}. \end{aligned} \tag{3.4.5}$$

and

$$t_1((i);(d)) = T_{s,d_s} + E\,\underline{m}((\underline{y}) + (d)). \tag{3.4.6}$$

The expected demand for item r in any interval of the lenght t is given by [c.f. (3.3.11)],

$$E\underline{v}_r(t) = \lambda_r t \sum_{k=1}^{N_r} kp_{rk} . \tag{3.4.7}$$

From the wellknown formula

$$E\,\underline{u} = E\{E(\underline{u})\,|\underline{x}_1, \ldots, \underline{x}_m)\} , \tag{3.4.8}$$

it follows that

$$E\underline{v}_r(\underline{m}((i))) = \lambda_r E\underline{m}((i)) \sum_{k=1}^{N_r} kp_{rk} , \tag{3.4.9}$$

and

$$E\underline{v}_r(\underline{m}((\underline{y}) + (d))) = \sum_{y} P\{(\underline{y}) = (y)\}\, E\underline{v}_r(\underline{m}((y) + (d))). \tag{3.4.10}$$

It is easily seen that*) $P\{(\underline{y}) = (y)\} = q^{(d)}_{(i),(y) + (d)}$, hence

$$E\underline{v}_r(\underline{m}((\underline{y}) + (d))) = \sum_{(j)\in X_0} q^{(d)}_{(i)(j)}\, E\underline{v}_r\,(\underline{m}((j))). \tag{3.4.11}$$

---

*) $P\{(\underline{y}) = (y)\} = \prod_{r=1}^{n} P\{\underline{y}_r = y_r\}$.

The quantity $\underline{Em}((j))$ can be calculated by means of (3.3.13) and (3.3.15). Further,

$$E\{(\underline{v}_r(T_{s,d_s}) - i_r) L (\underline{v}_r(T_{s,d_s}) - i_r)\} = \sum_{k=i_r}^{\infty} (k-i_r) a_{r,k}(T_{s,d_s}) =$$

$$= \lambda_r T_{s,d_s} \sum_{k=1}^{N_r} k p_{rk} - i_r + \sum_{k=0}^{i_r-1} (i_r-k) a_{r,k}(T_{s,d_s}), \qquad (3.4.12)$$

and

$$E\underline{y}_r = \sum_{k=0}^{i_r-1} (i_r-k) a_{r,k}(T_{s,d_s}). \qquad (3.4.13)$$

After some calculations:

$$k((i);(d)) = k_1((i);(d)) - k_0((i)) =$$

$$= h_s(d_s) + K + \sum_{r=1}^{n} c_{r2}\{\lambda_r T_{s,d_s} \sum_{k=1}^{N_r} k p_{r,k} - i_r - d_r\} +$$

$$+ c_{s1}\{ \sum_{k=i_s}^{\infty} a_{s,k}(T_{s,d_s}) \sum_{m=1}^{d_s} E\underline{t}_{s,m} + \sum_{k=0}^{i_s-1} a_{s,k}(T_{s,d_s}) \times$$

$$\times \sum_{m=i_s-k+1}^{i_s-k+d_s} E\underline{t}_{s,m}\} + \sum_{r=1}^{n} c_{r2}\, \lambda_r \sum_{k=1}^{N_r} k p_{r,k} \times$$

$$\times \left(\sum_{(j)} q_{(i)(j)}^{(d)} \underline{Em}((j)) - \underline{Em}((i)) \right). \qquad (3.4.14)$$

and

$$t((i);(d)) = t_1((i);(d)) - t_0((i)) =$$

$$= T_{s,d_s} + \sum_{(j)} q_{(i)(j)}^{(d)} \underline{Em}((j)) - \underline{Em}((i)). \qquad (3.4.15)$$

Determination of the $q^{(d)}_{(i)(j)}$ and $p^{z}_{(i)(j)}$.

Let $(d) = (0, \ldots, 0, d_s, 0, \ldots, 0)$.

1 . $d_s = 0$

$$q^{(d)}_{(i)(j)} = \begin{cases} 1 & \text{if } (j) = (i) \\ 0 & \text{otherwise} \end{cases} . \qquad (3.4.16)$$

2 . $d_s \neq 0$.

Each customer asks for one item, and the demands are mutually independent, hence

for $i_s \geq 1, 1 + d_s \leq j_s \leq i_s + d_s$, $0 \leq j_r \leq i_r, r \neq s$:

$$q^{(d)}_{(i)(j)} = a_{s,i_s + d_s - j_s} \prod_{\substack{r \neq s \\ j_r \neq 0}} a_{r,i_r - j_r} \cdot \prod_{\substack{r \neq s \\ j_r = 0}} \left(\sum_{m = i_r}^{\infty} a_{r,m} \right), \qquad (3.4.17)$$

and for $j_s = d_s, 0 \leq j_r \leq i_r, r \neq s$:

$$q^{(d)}_{(i)(j)} = \sum_{m = i_s}^{\infty} a_{s,m} \prod_{\substack{r \neq s \\ j_r \neq 0}} a_{r,i_r - j_r} \prod_{\substack{r \neq s \\ j_r = 0}} \left(\sum_{m = i_r}^{\infty} a_{r,m} \right), \qquad (3.4.18)$$

$$q^{(d)}_{(i)(j)} = 0 \qquad \text{otherwise} . \qquad (3.4.19)$$

Where $a_{i,j} = a_{i,j}(T_{s,d_s})$. (3.4.20)

The $p^{z}_{(i)(j)}$ can be determined by means of the $q^{d}_{(i)(j)}$ and

$\beta_{(i)(j)A_z}$ [c.f.(3.3.30)].

1 . $(i) \notin A_z$, then

$$p^{z}_{(i)(j)} = \beta_{(i)(j)A_z}. \qquad (3.4.21)$$

2 . $(i) \in A_z$, then

$$p^{z}_{(i)(j)} = q^{z(i)}_{(i)(j)} + \sum_{\substack{(k)\\(j)<(k)\\(k)\notin A_z}} q^{z(i)}_{(i)(k)}\, p^{z}_{(k)(j)},\quad (j)\in A_z \quad , \qquad (3.4.22)$$

$$p^{z}_{(i)(j)} = 0 \quad , \quad \text{otherwise.} \qquad (3.4.23)$$

<u>Determination of the set $A_{\hat{z}}'$.</u>

We need to consider only sets $A\subset X_0$. From the definitions (2.43),(2.44) and (3.3.30) it follows

$$r(A.\hat{z};(i)) = \sum_{(j)\leq(i)} \beta_{(i)(j)A}\, r(\hat{z};(j)) \qquad (3.4.24)$$

and

$$c(A.\hat{z};(i)) = \sum_{(j)\leq(i)} \beta_{(i)(j)A}\, c(\hat{z};(j)). \qquad (3.4.25)$$

If $(i)\in A$, then

$$r(A.\hat{z};(i)) = r(\hat{z};(i)) \text{ and } c(A.\hat{z};(i)) = c(\hat{z};(i)). \qquad (3.4.26)$$

The set $A_{\hat{z}}'$ can be determined analogously like in section 3.2.
We start the set $H = \{(0)\}$. The set H may be enlarged by the following procedure. Starting in state (0), we go from state to state in X_0, such that we have visited every state (j) with (j) < (i), when we enter state (i). If

$$\sum_{(j)<(i)} \beta_{(i)(j)H}\, r(\hat{z};(j)) > r(\hat{z};(i))$$

or

$$\sum_{(j)<(i)} \beta_{(i)(j)H}\, r(\hat{z};(j)) = r(\hat{z};(i))$$

and $$\sum_{(j)<(i)} \beta_{(i)(j)H}\, c(\hat{z};(j)) > c(\hat{z};(i)), \qquad (3.4.27)$$

then state (i) is added to H, otherwise not.
When we have visited every state of X_0, then the obtained set H is the set $A_{\hat{z}}'$. This follows immediately from the construction of H.

The relation $A_{z_1} \subset A_{\hat{z}}'$ implies that the test (3.4.26) turns out to be negative for states $(i) \notin A_{z_1}$. Hence we need only to test (3.4.26) for states of A_{z_1}. In addition we remark that for $(i) \in A_{z_1}$ the quantities $r(\hat{z};(i))$ and $c(\hat{z};(i))$ are given by (2.41) and (2.42).

Determination of an optimal strategy.

The iterationprocedure, which converges to an optimal strategy in a finite number of steps, can be given analogously as in section 3.2.

3.5. Examples.

a) One item: each customer demands at most two units.

Suppose the following numerical data are given:

$$M_1 = 4;\ T_{1,d} = 1,\ d = 1, \ldots, 4;\ K_1 = 3;\ c_{11} = 2; c_{12} = 16;\ \lambda_1 = 1;$$
$$p_{11} = p_{12} = \tfrac{1}{2};\ h_1(1) = 2;\ h_1(2) = 3{,}8;\ h_1(3) = 5{,}5;\ h_1(4) = 7. \tag{3.5.1}$$

From (2.3.10) it follows:

$$a_{1,0}(t) = e^{-\lambda t};\ a_{1,1}(t) = p_{11}\lambda t e^{-\lambda t};$$
$$a_{1,2}(t) = \left(\frac{p_{11}^2}{2} + p_{12}\right)(\lambda t)^2 e^{-\lambda t};\ a_{1,3}(t) = \left(\frac{p_{11}^3}{6} + \frac{p_{11}p_{12}^2}{2}\right)(\lambda t)^3 e^{-\lambda t}. \tag{3.5.2}$$

After some calculations [c.f.(3.4.14) and (3.4.15)]:

i \ d	1	2	3	4
0	39	39,80	48	54,25
1	19,79	23,35	31,63	
2	15,56	23,44		
3	12,41			

The function k(i;d).

1	2	3	4
2	2,50	3,25	3,88
1,19	1,78	2,48	
1,05	1,64		
0,79			

The function t(i;d).

Iterationprocedure:

1) Start with $z = (3,3,0,0,0)$. $\mathcal{D}_z$ has only one simple ergodic set. Choose $e_1 = 1$. After some calculations:
$r = 13{,}44$; $c_0 = 5{,}80$; $c_1 = 0$; $c_2 = \frac{1}{2}c_0 + \frac{1}{2}c_1$; $c_3 = \frac{1}{4}c_0 + \frac{3}{4}c_1$; $c_4 = \frac{3}{8}c_0 + \frac{5}{8}c_1$.

2) After some calculations,

$z_1 = (4,3,0,0,0)$, with $c(\hat{z};0) = 0{,}63$; $c(\hat{z};1) = 0$, hence

3) $A_{z_1} = A_{\hat{z}}'$.

4) New strategy $z = (4,3,0,0,0)$. End $1^{\underline{th}}$ cycle.

1) $\mathcal{D}_z$ corresponding to $z = (4,3,0,0,0)$ has only one simple ergodic set. Choose $e_1 = 1$. After some calculations,

$r = 13{,}28$; $c_0 = 4{,}24$; $c_1 = 0$; $c_2 = \frac{1}{2}c_0$; $c_3 = \frac{1}{4}c_0$; $c_4 = \frac{3}{8}c_0$.

2) After some calculations,

$z_1 = (4,3,0,0,0)$ with $c(\hat{z};0) = 4{,}24$ and $c(\hat{z};1) = 0$

3) $A_{\hat{z}}' = A_{z_1}$, hence an optimal strategy is given by

$$z = (4,3,0,0,0). \tag{3.5.3}$$

When we take $h_1(d) = 2d$, then $(4,3,0,0,0)$ remains an optimal strategy with $r = 13{,}51$; $c_0 = 4{,}64$; $c_1 = 0$; $c_2 = \frac{1}{2}c_0$; $c_3 = \frac{1}{4}c_0$; $c_4 = \frac{3}{8}c_0$.

b) Two items; each customer demands one unit.

Suppose the following numerical data are given:

$$M_i = 3;\ T_{i,d} = 1,\ d = 1, \ldots, 3;\ c_{i1} = 2;\ c_{i2} = 16;\ K_i = 3;$$

$$p_{i1} = 1;\ h_i(1) = 2;\ h_i(2) = 3{,}8;\ h_i(3) = 5{,}5 \quad \text{for } i = 1,2. \tag{3.5.4}$$

The symmetry in the data implies:
$k((i,j);(d,0)) = k((j,i);(0,d))$, $t((i,j);(d,0)) = t((j,i);(0,d))$.

After some calculations:

(i)(j) \ d	1	2	3
(0,0)	55	76,8	100,50
(0,1)	28,89	47,74	69,97
(0,2)	11,61	24,58	43,12
(0,3)	1,69	8,03	21,43
(1,0)	35,51	57,80	
(1,1)	24,32	44,50	
(1,2)	13,45	27,96	
(1,3)	2,86	13,49	
(2,0)	28,85		
(2,1)	23,62		
(2,2)	16,63		
(2,3)	9,92		

The function $k((i,j);(d,0))$.

1	2	3
2	3	4
1,18	2,09	3,05
0,64	1,37	2,21
0,33	0,85	1,53
1,37	2,37	
1,02	1,94	
0,66	1,43	
0,40	0,97	
1,10		
0,95		
0,68		
0,36		

The function $t((i,j);(d,0))$.

Iterationprocedure.

1) Start with the strategy z, which is given by $z(0,0) = z(0,1) = 3,0)$; $z(1,0) = (0,3)$ and the other $z(i,j) = (0,0)$. $\mathcal{P}_z$ has only one simple ergodic set. Choose $e_1 = (1,0)$. It is easily seen that the symmetry in z implies $c_{ij} = c_{ji}$.

After some calculations,

$r = 22,94$; $c_{00} = 8,74$; the other $c_{ij} = 0$.

2) After some calculations we find the strategy z_1 with $z_1(i,j) = z_1\ (j,i)$ and further $z_1(0,0) = z_1(0,1) = z_1(1,1) = z_1(1,2) =$

$= z_1(1,3) = (2,0)$; $z_1(0,2) = z_1(0,3) = (3,0)$ and the other $z_1(i,j) = (0,0)$. Further,

$c(\hat{z};(i,j)) = c(\hat{z};(j,i))$; $c(\hat{z};(0,0)) = -7,98$; $c(\hat{z};(0,1)) = -0,21$; $c(\hat{z};(0,2)) = -7,58$; $c(\hat{z};(0,3)) = -13,67$; $c(\hat{z};(1,1)) = -0,01$; $c(\hat{z};(1,2)) =$ $= -4,58$; $c(\hat{z};(1,3)) = -8,67$.

3) It is easily verified that

$$A_{\hat{z}}' = A_{z_1} \setminus \{(1,1),(1,3),(3,1)\}.$$

4) New strategy z: $z(i,j) = z(j,i)$; $z(0,0) = z(0,1) = z(1,2) = (2,0)$; $z(0,2) = z(0,3) = (3,0)$ and the other $z(i,j) = (0,0)$.
End $1^{\underline{th}}$ cycle.
1) $\mathcal{P}_z$ corresponding to the strategy given in 4) has only one simple ergodic set. Choose $e_1 = (1,2)$. It is easily seen that $c_{ij} = c_{ji}$.
After some calculations,
$r = 18{,}02$; $c_{00} = 20{,}90$; $c_{01} = 8{,}92$; $c_{02} = -1{,}84$; $c_{03} = -9{,}53$; $c_{12} = 0$;

$c_{11} = c_{01}$; $c_{13} = \frac{1}{2}c_{03} + \frac{1}{2}c_{12}$; $c_{22} = c_{12}$; $c_{23} = c_{33} = \frac{1}{4}c_{03} + \frac{3}{4}c_{12}$.
2) After some calculations we find the strategy z_1 with $z_1(i,j) = z_1(j,i)$ and $z_1(0,0) = z_1(0,1) = z_1(0,2) = z_1(0,3) = (3,0)$; $z_1(1,1) = z_1(1,2) = z_1(1,3) = (2,0)$ and the other $z(i,j) = (0,0)$.
Further
$c(\hat{z};(0,0)) = 18{,}89$; $c(\hat{z};(0,1)) = 7{,}23$; $c(\hat{z};(0,2)) = -1{,}84$;
$c(\hat{z};(0,3)) = -9{,}53$; $c(\hat{z};(1,1) = 5{,}95$; $c(\hat{z};(1,2)) = 0$;
$c(\hat{z};(1,3)) = -5{,}87$ and $c(\hat{z};(i,j)) = c(\hat{z};(j,i))$.
3) It is easily verified that

$$A_{\hat{z}}' = A_{z_1}.$$

4) New strategy z equals the strategy given in 2).
End $2^{\underline{th}}$ cycle.

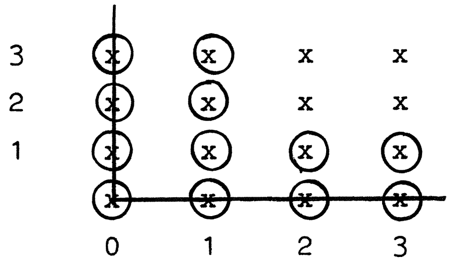
1) $\mathcal{P}_z$ corresponding to this strategy has only one simple ergodic set.
Choose $e_1 = (1,2)$. Obviously $c_{ij} = c_{ji}$.
After some calculations:
$r = 17{,}77$; $c_{00} = 19{,}50$; $c_{01} = 7{,}26$; $c_{02} = -2{,}13$; $c_{03} = -9{,}91$; $c_{11} = 6{,}04$; $c_{12} = 0$; $c_{13} = -6{,}09$; $c_{22} = c_{12}$; $c_{13} = \frac{1}{2}c_{03} + \frac{1}{2}c_{12}$; $c_{23} = c_{33} = \frac{1}{4}c_{03} + \frac{3}{4}c_{12}$.

2) We find that strategy z_1 equals strategy z, hence $c(\hat{z};(i,j)) = c(\hat{z};(i,j)) = c_{ij}$ for $(i,j) \in A_{z_1}$.

3) It is easily verified

$$A_{\hat{z}}' = A_{z_1},$$ hence

the strategy, given in 2) of the $2^{\underline{th}}$ cycle, is optimal.



The optimal strategy, $r = 17{,}77$.

When we take $h_1(d) = h_2(d) = 2\ d$, this strategy remains optimal and $r = 17{,}96$; $c_{ij} = c_{ji}$; $c_{00} = 19{,}52$; $c_{01} = 7{,}38$; $c_{02} = -1{,}96$; $c_{03} = -9{,}65$; $c_{11} = 6{,}02$; $c_{12} = 0$; $c_{13} = -6{,}03$.

c) Two items; each customer demands at most two units.

Suppose the same numerical data as in example b) are given, except $p_{ij} = \frac{1}{2}$, $i, j = 1{,}2$.

After some calculations:

(i,j) \ d	1	2	3
(0,0)	87,00	99,80	126
(0,1)	47,83	58,42	81,86
(0,2)	34,87	54,38	63,80
(0,3)	18,19	22,12	39,18
(1,0)	48,20	65,97	
(1,1)	32,22	47,58	
(1,2)	24,35	37,34	
(1,3)	14,00	23,06	
(2,0)	42,60		
(2,1)	31,20		
(2,2)	24,97		
(2,3)	16,45		

The function $k(i,j;(d,0))$.

1	2	3
2	2,5	3,25
1,18	1,64	2,33
0,91	1,38	1,95
0,57	0,90	1,44
1,18	1,78	
0,85	1,39	
0,69	1,18	
0,47	0,88	
1,05		
0,81		
0,08		
0,50		

The function $t((i,j);(d,0))$.

We find the same optimal strategy as in example c) and $r = 29{,}81$; $c_{ij} = c_{ji}$; $c_{00} = 20.7$; $c_{01} = 5{,}07$; $c_{02} = -0{,}12$; $c_{03} = -8{,}38$; $c_{11} = 3{,}40$; $c_{12} = 0$; $c_{13} = -5{,}54$; $c_{22} = c_{12}$; $c_{23} = \frac{1}{4}c_{13} + \frac{1}{4}c_{03} + \frac{1}{4}c_{22} + \frac{1}{4}c_{12}$; $c_{33} = \frac{1}{4}c_{31} + \frac{1}{4}c_{32} + \frac{1}{4}c_{13} + \frac{1}{4}c_{23}$.

When we take $h_1(d) = h_2(d) = 2\,d$, we find the same optimal strategy, with

$r = 30{,}05$; $c_{ij} = c_{ji}$; $c_{00} = 20{,}68$; $c_{01} = 5{,}17$; $c_{02} = 0{,}03$; $c_{03} = -8{,}15$; $c_{11} = 3{,}38$; $c_{12} = 0$; $c_{ij} = -5{,}50$.

3.6. Generalisations.

A)

We shall assume that several items can be produced simultaneously. It is assumed that for any feasible production of several items, each item is finished at the same moment. Only one production process can be running. A production of d_k units of item k , $k = 1, \ldots, n$ will be called a production (d). When the inventory is given by (i) and it has been decided for a production (d), then $i_k + d_k \leq M_k$, $k = 1, \ldots, n$ is the only restriction imposed on (d). In addition more restrictions (e.g items i_1 and i_2 cannot produced simultaneously) may be of course imposed on (d).
It is assumed that the duration of a production (d) is a random variable $\underline{T}_{(d)}$. Further we suppose that after the end of production (d), there is an idle time $\underline{\tau}_{(d)}$, during which no production can be started. Let K be the setup cost for each production and let $h((d))$ be the cost of a production (d). Let the other data of the manufacturers problem remain unchanged. It is assumed that the collections of random variables $\{\underline{T}_{(d)}\}$, $\{\underline{\tau}_{(d)}\}$ and the random demands of the customers are mutually independent. This generalized problem can be solved analogously as the former problem.

Only a few points need some modification.

We take as state space the (2n+3)-dimensional space consisting of the points:

(a) $((i),(0),0,0)$.

This state corresponds to the situation that the inventory is given by (i) and a production may be started.

(b) $((i),(d),t,0)$.

This state corresponds to the situation that the inventory is given by (i) and a production (d) is running since t units of time. The range of t is determined by the random time of production $\underline{T}_{(d)}$.

(c) $((i),0,(d),t)$.

This state corresponds to the situation that the inventory is given by (i), and no production is running and no production may be started. The last production is a production (d) and has been finished since t units of time. The range of t is determined by the idle time $\underline{\tau}_{(d)}$.

Let X_0 be the subset of X consisting of the points $((i),(0),0,0)$. For simplicity of notation we denote $((i),(0),0,0)$ by (i). Let D(i) be the set of feasible decisions in (i). It is required $i_k + d_k \leq M_k$, $k = 1, \ldots, n$ and $(0) \notin D(0)$. In the states $x \notin X_0$ only null-decisions can be made.

The results of section 3.1. can be simply transmitted. As far as section 3.4. concerns only the calculation of the k-and t-functions and the $q^{(d)}_{(i)(j)}$ changes slightly. The rest can simply transmitted to the generalized problem.

Let $(\underline{y})$ be the inventory just before production (d) stops, if the system starts in state (i), in which the feasible decision $(d) \neq (0)$ has been made. It can be easily verified [c.f section 3.4], that

$$k((i);(d)) = h((d)) + K + \sum_{r=1}^{n} c_{r1} \sum_{k=\underline{y}_r+1}^{\underline{y}_r+d_r} E\,\underline{t}_{r,k} +$$

$$+ \sum_{r=1}^{n} c_{r2}\, E\{(\underline{v}_r(\underline{T}_{(d)}) - i_r)\,\mathcal{L}(\underline{v}_r(\underline{T}_{(d)}) - i_r)\} +$$

$$+ \sum_{r=1}^{n} c_{r2}\Big[\lambda_r \sum_{k=1}^{N_r} kp_{rk}\, E \max_{h=1,\ldots,n} \{\underline{t}_{h}, \underline{y}_h + d_h, \underline{\tau}_{(d)}\} +$$

$$- (E\underline{y}_r + d_r)\Big] + \sum_{r=1}^{n} c_{r2}\Big\{\lambda_r \sum_{k=1}^{N_r} kp_{rk}\, E \max_{h=1,\ldots n} \{\underline{t}_{h}, i_h\} +$$

$$- i_r\Big]. \tag{3.6.1}$$

and

$$t((i);(d)) = E\, \underline{T}_{(d)} + E \max_{h=1,\ldots,n} \{\underline{t}_{h}, \underline{y}_h + d_h, \underline{\tau}_d\} +$$

$$- E \max_{h=1,\ldots,n} \{\underline{t}_{j}, i_h\}. \tag{3.6.2}$$

<u>Formulas for the $q^{(d)}_{(i)(j)}$.</u>

Because the customers, who ask for item i, arrive according to a Poisson process, and addition these processes are mutually independent, we can give analytical formulas for the $q^{(d)}_{(i)(j)}$. For simplicity of notation we give these formulas only for the simple case $n = 1$.
Let

$$a_j \overset{\text{def}}{=} \int_0^\infty a_{1,j}(t) d\, P\{\underline{T}_d \le t\},\ b_j \overset{\text{def}}{=} \int_0^\infty a_{1,j}\ (t) d\, P\{\underline{\tau}_d \le t\}. \tag{3.6.3}$$

Then

1) $d = 0$. $$q^d_{ij} = \begin{cases} 1 & , \text{if } j = i, \\ 0 & , \text{if } j \ne i. \end{cases} \tag{3.6.4}$$

2) $d \ne 0$.
If $d \ge 1, i \ge 1$,

$$q^d_{i,i+d-h} = \sum_{j=0}^{h} a_j b_{j-h}\ ,\ h = 0, \ldots, i-1. \tag{3.6.5}$$

If $d \geq 1, i \geq 0$,

$$q^d_{i,d-h} = \sum_{j=0}^{i-1} a_j b_{i+h-j} + \sum_{j=i}^{\infty} a_j b_h \ , \ h = 0, \ldots, d-1. \quad (3.6.6)$$

$$q^d_{i,0} = \sum_{j=0}^{i-1} a_j \sum_{k=i+d-j}^{\infty} b_k + \sum_{j=i}^{\infty} a_j \sum_{k=d}^{\infty} b_k \ , \quad (3.6.7)$$

$$q^d_{ij} = 0 \quad , \quad \text{otherwise}. \quad (3.6.8)$$

The formulas can be generalized.

Example.
Suppose the following numerical data are given:
$n = 1;\ M_1 = 4;\ \underline{T}_d = \underline{\tau}_d = 1,\ d = 1, \ldots, 4;\ c_{11} = 2;\ c_{12} = 16;\ K = 3;$

$h_1(1) = 2;\ h_1(2) = 3{,}8;\ h_1(3) = 5{,}5;\ h_1(4) = 7;\ \lambda_1 = 1;\ p_{11} = 1.$

After some calculations.

i \ d	1	2	3	4
0	28,89	30,46	36,87	46,04
1	17,96	21,34	28,86	
2	13,17	19,47		
3	12,33			

The function k(i;d).

i \ d	1	2	3	4
0	2,37	3,10	4,02	5
1	1,64	2,44	3,38	
2	1,25	2,14		
3	1,08			

The function t(i;d).

The strategy $z = (3,3,0,0,0)$ is optimal, with $r = 8{,}58$; $c_0 = 2{,}57$; $c_i = 0$, $i = 1, \ldots, 4$. This strategy remains optimal when we take $h_1(d) = 2d$, and then $r = 8{,}73$; $c_0 = 2{,}40$; $c_i = 0$, $i = 1, \ldots, 4$.

Note. We may assume that costs $k(i,j)$ are involved, when production (d_j) succeeds production (d_i). The state space becomes then more complex, but the problem can be solved analogously.

B)

Up to now we have supposed that each customer asks for one type of item. This assumption and the assumed independence of the demands unables us to give analytical formulas for the quantities $E \max_{h = 1, \ldots n} \{\underline{t}_{h,i_h}\}$ and $q^{(d)}_{(i)(j)}$. We shall now drop the assumption that each customer asks for one type of item.*) Suppose that the customers arrive according to a Poisson process with rate λ. Let the demands be mutually independent and identically distributed. Let $p_{(v)}$ be the probability that a customer demands simultaneously v_k units of item k for k = 1, ...n. Where $0 \leq v_k \leq N_k$, k = 1, ...n and $\sum_{(v)} p_{(v)} = 1$.

Thus a customer may ask for more than one type of item. The formulation of the problem implies that customers, who ask for item i arrive according the <u>same</u> Poisson process for i = 1, ...,n.

Hence $\{\underline{v}_i(t)\}$ are generally mutually dependent, and their dependence implies that the $\underline{t}_{h,i_h}$ (h = 1, ...,n) are mutually dependent. Where $\underline{v}_h(t)$ is the demand for item h in an interval of the lenght t, and $\underline{t}_{h,i_h}$ is the epoch on which the i_h $\underline{\text{th}}$ unit of item h is demanded.

Nevertheless this generalized problem can be solved analogously as the former problems.

Let p(i,k) be the probability that the demand of a customer for item i equals k, then

$$p(i,k) = \sum_{(v), v_i = k} p_{(v)} \tag{3.6.9}$$

It is easily seen that in the formulas of the k- and t-functions we have to replace p_{ik} by p(i,k).

The troubles are caused by the calculation of the quantities

$E \max_{h = 1, \ldots, n} \{\underline{t}_{h,i_h}\}$ and $q^{(d)}_{(i)(j)}$.

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*) Remember that the independence of the demands is essential for applying Markov-programming.

Simulation seems to be the only way to calculate these quantities if $\underline{v}_i(t)$, $i = 1,\dots,n$ are not mutually independent. This is no serious objection, because we need only once for all to determine these quantities.

For the rest only the calculation of the probabilities $f_{(i)(j)}$ changes slightly [c.f.(3.3.29)]. Let the state (i) be given, we introduce now

$$V(i) \overset{\text{def}}{=} \{(j) \mid (j) < (i),\ i_k - j_k \leq N_k,\ k = 1, \dots,n\}. \qquad (3.6.10)$$

The definition of $f_{(i)(j)}$ is of course the same. It can be easily verified

$$f_{(i)(j)} = \frac{\sum\limits_{(v)\in W} p_{(v)}}{1 - \sum\limits_{\substack{(v) \\ v_i = 0, i = 1, \dots,s}} p_{(v)}} \quad , (j)\in V(i),(j) = (j_1, \dots,j_s,0, \dots,0), \; j_k \neq 0, \; k = 1, \dots,s \qquad (3.6.11)$$

Where $W = \{(v) \mid v_k = i_k - j_k,\ k = 1, \dots,s;\ v_k \geq i_k,\ k > s\}$ (3.6.12).

C).

If demand exceeds supply, the excess demand has been thought of as satisfied immediately through emergency purchases. Another interpretation of excess demand, which leads to the same model is to consider such demands as lost sales. A second possible way of treating excess demand is to allow for deferring this demand to a later time. This makes it necessary for the current stock level variable(s) to assume both negative and positive values. We shall now treat a simple case of the manufacturers problem with this interpretation of excess demand.

We assume that only one item is sold. Assume that for each production the production time *) is T units of time.

---

*) The production time may be taken random.

Suppose that after a production there is no obliged idle time, hence the productions may succeed each other immediately. Unfilled demand, which arises during a production, is backlogged to the end of that production as far the production size it permits. The unfilled demand, which eventually remains, is backlogged to the end of the next production and is then filled. The same is done for unfilled demand, which arises on a moment that no production is running. We assume that the production capacity this allows. If no production is running and the stock is smaller than an integer b_0 (b_0 may negative) we have to start a new production. The setup costs for each production are K. The production costs of p units are given by h(p), and it is assumed that h(p) is lineair from a given c_0 ($c_0 \geq 1$), hence

$$h(p) = a\,p\ ,\ p \geq c_0. \qquad (3.6.13)$$

There are penalty costs, when demand cannot be fulfilled immediately. For each unit these costs are a function f(t) of the time t of subsequent delivery.

The assumptions about the storage costs and the behavions of the customers are as in the introduction.

We note that in the long-run each demanded unit will be produced and delivered. Hence it does not matter when we reduce h(p) with ap. (Interpretation: a subsidy a is given on each unit). For every strategy the corresponding expected costs per unit of time will then change with a same amount*). Suppose,

$$h(p) = 0,\ \text{if } p \geq c_0. \qquad (3.6.14)$$

We take as state space the 3-dimensional space consisting of the points:

a) (i,0,0). The stock is i and no production is running. Where $i \leq M$ and i is an integer.

b) (i,p,t). The stock is i and a production of p units is running since t units of time.

The state (i,0,0) will be denoted by i. Let $X_0 = \{i \mid i \leq M\}$.

In states $x \notin X_0$ only null-decision can be made.

*) With $a\lambda \sum_{k=1}^{N} k p_k$.

When in state i has been decided to produce p units, we shall denote this decision by d, where

$$d = p + i. \qquad (3.6.15)$$

For each $d \in D(i)$ it is required $d \leq M$, and in addition we require

$$d \geq \max(0, b_0) \quad \text{if} \quad i < b_0. \qquad (3.6.16)$$

Obviously,

$$A_0 \subset \{i \mid i < b_0\}^{*)} \qquad (3.6.17)$$

When the system takes on a state i of A_0 and $i < 0$, we may assume that in the natural process in this state automatically a production of i units is started.

We call such a production a natural production. The production time of each natural production is T units of time and the production-and setup-costs are zero. Further we assume that during a natural production no customers arrive. As soon as a natural production is finished the existing shortage is filled with the produced units. When a natural production is started in the course of a walk $\underline{w}^0$ or $\underline{w}^d$, it is assumed that the walk considered ends as soon as the natural production is finished.

These assumptions**) can be made, because in each decisionprocess considered no natural productions happen. In each state of A_0 the decisionmaker has to start a "non-natural" production for every strategy $z \in Z$. This production involves of course production costs. With a production we shall mean a non-natural production.

When we want to determine $k(i;d)$ for a state $i < 0$, we do not need to consider the penalty costs of the i units. When we should consider these costs, they belong both to the costs of the walk $\underline{w}^0$ and the walk $\underline{w}^d$. The difference between the costs of these two walks is $k(i;d)$. We are only interested in this difference, therefore for any walk, which starts in a state $i < 0$ we shall leave out of consideration the penalty costs of the i units.

---

*) $A_0 = \{i \mid i < 0\}$ may be advantageous if $b_0 \geq 0$ and $N \geq b_0 + 1$

**) These assumptions will simplify the derivation of formulas for the k-and t-functions.

However when during a walk units are demanded, which cannot be delivered immediately, the costs of subsequent delivery of these units belong to the costs of the walk considered.
We recall that for each production the production time is T units of time. Further $h(p) = 0$, if $p \geq c_0$ where p is the production size.
From the above considerations it follows easily that for every feasible d [c.f (3.6.15),(3.6.16)]:*)

$$k(i;d) = k(m_0;d)\ ,\quad i < m_0 \tag{3.6.18}$$

and

$$t(i;d) = k(m_0;d)\ ,\quad i < m_0 \tag{3.6.19}$$

Where

$$m_0 \overset{\text{def}}{=} \min\,(-c_0, b_0-1) \text{ if } b_0 < 0,\ \min\,(b_0-c_0,-1) \text{ otherwise.} \tag{3.6.20}$$

For each strategy $z \in Z$ the corresponding Markov-chain $\{\underline{I}_n\}_{n=1}^{\infty}$ of future interventionstates has only one simple ergodic set, because from every state in A_z each state $i \leq 0$ can be reached. Hence for each $z \in Z$ the function $r(z;x)$ does not depend on x,

$$r(z;x) = r(d.z;x) = r(A.z;x) = r(z). \tag{3.6.21}$$

It is easily seen that**) [c.f(2.16)]:

$$c(d.z;i) = k(i;d) - t(i;d) + Ec(z;\underline{j}), \tag{3.6.22}$$

where $\underline{j}$ is the first future interventionstate in A_z if in the initial state i decision d had been made.
The distribution of $\underline{j}$ depends only on d; with (3.6.19) and (3.6.20) it follows:

$$c(d.z;i) = c(d.z;m_0)\ ,\ i < m_0. \tag{3.6.23}$$

In step 2 of the iterationprocedure the strategy z_1 is determined by means of the test quantity $c(d.z;i)$, because $r(d.z;i) = r(z)$.
With the aid of (3.6.23) we can conclude it is no restriction to consider only strategies $z \in Z$ with

$$z(i) = z(m_0)\ ,\ i < m_0 \tag{3.6.24}$$

There is an optimal strategy $z \in Z$, which satisfies (3.6.24).

---

*) Take $A_0 = \{i \mid i \leq \min\,(b_0-1,-1)\}$.

**) Note that each decision d leads to a deterministic transition.

Let Z^* be the class of strategies $z \in Z$ with $z(i) = z(m_0)$, $i < m_0$.
From now on we only consider strategies $z \in Z^*$.
Using a same argument as above,

$$c(z;i) = c(z;m_0)\ ,\ i < m_0. \tag{3.6.25}$$

Consider the equation:

$$c(z;i) = k(i;z(i)) - r(z)t(i;z(i)) + \sum_{j \in A_z} p^z_{ij}\ c(z;j). \tag{3.6.26}$$

From (3.6.25) it follows,

$$\sum_{j \in A_z} p^z_{ij}\ c(z;j) = \sum_{j > m_0} p^z_{ij}\ c(z;j) +$$

$$+ \left(1 - \sum_{j > m_0} p^z_{ij}\right) c(z;m_0). \tag{3.6.27}$$

It is now easily seen that in applying the iterationprocedure we can restrict ourselves to the states $m_0, \ldots, M$. From now on we only consider these states.
We can transmit the other results of section 3.1. with a slight modification.
For $i \geq m_0$, $j > m_0$ the definition of q^d_{ij} is according to (3.1.4) and

$$q_{i,m_0} \overset{\text{def}}{=} 1 - \sum_{j > m_0} q^d_{ij}. \tag{3.6.28}$$

Let a_k be the probability that the demand in the production time equals k, then:

$$q^d_{ij} = a_{d-j}\quad ,\quad j > m_0. \tag{3.6.29}$$

For $i \geq m_0$, $j > m_0$ the definition of p^z_{ij} is according to (3.1.7)

and

$$p_{i,m_0}{}^z \overset{\text{def}}{=} 1 - \sum_{j > m_0} p^z_{ij}. \tag{3.6.30}$$

The other results of section 3.1 can be simply transmitted. To determine p^z_{ij} if $i \notin A_z$ we have introduced in section 3.2 the set $V(i)$ and the probabilities f_{ij} and β_{ijA}.

We redefine:

$$V(i) = \{j \mid m_0 < j < i, \quad i - j \leq N\} . \tag{3.6.31}$$

The definition of f_{ij} is the same if $j > m_0$ and

$$f_{i,m_0} = 1 - \sum_{j > m_0} f_{ij}. \tag{3.6.32}$$

If $A \subset \{m_0, \dots, M\}$ and $m_0 \in A$ the definition of β_{ijA} is the same for $j > m_0$ and

$$\beta_{im_0A} \overset{\text{def}}{=} 1 - \sum_{j > m_0} \beta_{ijA}. \tag{3.6.33}$$

Lemma 3.3.3 and the recursionformula (3.3.32) remain true. As far as section 3.4 concerns only the calculation of the k- and t-functions changes. The rest can be simply transmitted. (To determine $A_{\hat{z}}'$ we start with $H = \{m_0, \dots, b_0 - 1\}$).

Determination of the k- and t-functions.

Customers arrive according to a Poisson process with rate λ and each customer demands k units with probability p_k $(k = 0, \dots, N)$. The demands of the customers are mutually independent. Hence the demands in disjunct intervals are mutually independent. The demand in any interval of the lenght t is a random variable $\underline{v}(t)$ and $a_n(t) = P\{\underline{v}(t) = n\}$.

Theorem 6.1.

Let $t^{(k,n)}$ be the interval from 0 up to the epoch, on which the $k^{\underline{th}}$ unit is demanded, given that in $(0,T]$ n units are demanded, then for $k = 1, \dots, n$:

$$F_k(t) = P\{\underline{t}^{(k,n)} \leq t\} = \begin{cases} \dfrac{1}{a_n(T)} \displaystyle\sum_{j=k}^{n} a_j(t)\, a_{n-j}(T-t), & 0 \leq t \leq T \\ 0 & \text{, otherwise.} \end{cases} \tag{3.6.34}$$

Proof

$$P\{\underline{t}^{(k,n)} \leq t\} =$$

$$= P\{\text{demand in } (0,t] \leq k \mid \text{demand in } (0,T] = n\} =$$

$$= \frac{1}{a_n(T)} \sum_{j=k}^{n} P\{\text{demand in } (0,t] = j, \text{ demand in } (t,T] = n-j\} =$$

$$= \frac{1}{a_n(T)} \sum_{j=k}^{n} a_j(t)\, a_{n-j}(T-t).$$

Corollary: $$E\underline{t}^{(k,n)} = \int_0^T tF_k'(t)dt = \int_0^T (1-F_k(t))dt. \qquad (3.6.35)$$

In particular, if each customer demands one unit:

$$E\underline{t}^{(k,n)} = \frac{kT}{n+1} \qquad (3.6.36)$$

We shall now give formulas for the k- and t-functions (only states $i \geq m_0$ are considered).
To avoid notational complexities we take

$$b_0 \geq 0 \qquad (3.6.37)$$

Take

$$A_0 = \{i \mid i \leq -1\}\ . \qquad (3.6.38)$$

As soon as a state i of A_0 is assumed, a natural production of i units starts, the production time is T units of time and the production costs are zero.
If a walk $\underline{w}^0$ starts in a state $i \geq 0$ a natural production starts as soon as the $(i+1)^{\underline{th}}$ unit is demanded. The production size equals the shortage.
Hence, for $i \geq m_0$

$$k_0(i) = c_1 \sum_{k=1}^{i} E\underline{t}_k + f(T)\, E\,(\underline{v}(\underline{t}_{i+1}) - i^+) \qquad (3.6.39)$$

and

$$t_0(i) = E\underline{t}_{i+1} + T. \qquad (3.6.40)$$

Where

$$i^+ \underset{=}{\text{def}} \begin{cases} i & \text{if} \quad i > 0 \\ 0 & \text{if} \quad i \leq 0 \end{cases} \qquad (3.6.41)$$

and

$$\underline{t}_k \equiv 0, \text{ if } k \leq 0. \tag{3.6.42}$$

Let in state i a decision d with d > i has been made.
(d = i+ production size). The units, to be produced, cause storage costs if the demand in the production time is smaller than d. Hence the expected storage costs corresponding to the walk $\underline{w}^d$ are given by,

$$c_1\left\{\sum_{k=1}^{i} E\underline{t}_k + \sum_{k=0}^{i^+} a_k \sum_{j=i^+-k+1}^{d-k} E\underline{t}_j + \sum_{k=i^++1}^{d-1} a_k \sum_{j=1}^{d-k} E\underline{t}_j\right\}. \tag{3.6.43}$$

When units, demanded during the production, cannot be delivered immediately, they are subsequent delivered as far as it is possible at the end of the production. If after the production the stock is negative a natural production starts. If the stock is non-negative a natural production begins as soon as the stock becomes negative. Hence the expected penalty costs are given by,

$$\sum_{k=i^++1}^{d} a_k \sum_{j=i^++1}^{k} Ef(T-\underline{t}^{(j,k)}) + \sum_{k=d+1}^{\infty} a_k \left\{\sum_{j=i^++1}^{d} Ef(T-\underline{t}^{(j,k)}) + \sum_{j=d+1}^{k} Ef(2T-\underline{t}^{(j,k)})\right\} +$$

$$+ \sum_{k=0}^{d} a_k f(T) E\{\underline{v}(\underline{t}_{d-k+1}) - (d-k)\}. \tag{3.6.44}$$

Hence, for $i \geq m_0$, $d > i$ and $d \in D(i)$:

$$k_1(i;d) = h(d-i) + K + (3.6.43) + (3.6.44), \tag{3.6.45}$$

and

$$t_1(i;d) = T + \sum_{k=0}^{d} a_k E\underline{t}_{d-k+1} + T. \tag{3.6.46}$$

Application.

Assume that each customer demands one unit hence

$$a_k = e^{-\lambda T}\frac{(\lambda T)^k}{k!}, \quad k \geq 0, \tag{3.6.47}$$

and

$$\underline{v}(\underline{t}_k) = \begin{cases} k & , k > 0 \\ 0 & , k \leq 0. \end{cases} \tag{3.6.48}$$

Suppose

$$f(t) = c_2 t, \tag{3.6.49}$$

hence

$$E\underline{t}^{(j,k)} = \frac{c_2\, jT}{k+1} \quad . \tag{3.6.50}$$

Assume

$$c_0 = 1, \tag{3.6.51}$$

then the reduced h(d) is given by,

$$h(d) = 0, \quad d \geq 0 \quad . \tag{3.6.52}$$

After some calculations,*) we find for $i \geq 0$ and $d > i$:

$$k(i;d) = K + \frac{c_1}{\lambda}\{\tfrac{1}{2}(d-i)(d-i+1)\ S(d-1) +$$

$$+ \sum_{k=1}^{d-i-1} \tfrac{1}{2}(k^2-k-2k(d-i))a_{i+k} + (d-i)(iS(i)-\lambda TS(i-1))\} +$$

$$+ \frac{c_2}{2\lambda}\{(\lambda T)^2(1-S(i-2)) - 2i\lambda T(1-S(i-1)) +$$

$$+ (i^2+i)(1-S(i))\} +$$

$$+ c_2 T\{\lambda T(1-S(d-1)) - (d+1)(1-S(d))\}, \tag{3.6.53}$$

*) Use $ka_k = \lambda T a_{k-1}$ and $\sum_{k=0}^{\infty} a_k = 1.$

and

$$t(i;d) = T - \frac{1}{\lambda} \{S(d-1) + i + 1 - (d+1)S(d)\}. \tag{3.6.54}$$

Where

$$S(i) = \sum_{k=0}^{i} a_k. \tag{3.6.55}$$

Further,

$$\begin{aligned} k(-1;d) &= k(0;d) + c_2T, \ d \geq 1 \\ k(-1,0) &= K + c_2T(\tfrac{3}{2}\lambda T + a_0), \end{aligned} \tag{3.6.56}$$

and

$$\begin{aligned} t(-1;d) &= t(0;d) + \frac{1}{\lambda}, \ d \geq 1 \\ t(-1,0) &= T + \frac{a_0}{\lambda}. \end{aligned} \tag{3.6.57}$$

Numerical example.

Suppose the following numerical data are given:
$M = 4$; $\lambda = 1$; $T = 1$; $K = 3$; $c_1 = 2$; $c_2 = 16$.

From a table of the Poisson distribution:
$a_0 = a_1 = 0.368$; $a_2 = 0.184$; $a_3 = 0.061$; $a_4 = 0.015$.

After some calculations:

i \ d	0	1	2	3	4
-1	33.256	29.393	30.315	34.060	40.013
0		13.393	14.315	18.060	24.013
1			7.693	11.438	17.391
2				7.573	13.526
3					9.105

The function k(i;d).

i \ d	0	1	2	3	4
-1	1.368	2.104	3.023	4.004	5.001
0		1.104	2.023	3.004	4.001
1			1.023	2.004	3.001
2				1.004	2.001
3					1.001

The function $t(i;d)$.

Iterationprocedure.

1^e. Start with $z = (z(-1), \ldots, z(4)) = (3,3,3,2,3,4)$.

Choose $e_1 = 1$. We have to solve:

$c_{-1} = 0.019c_{-1} + 0.061c_0 + 0.92c_1 + 34.06 - 4.004r$

$c_0 = 0.019c_{-1} + 0.061c_0 + 0.92c_1 + 18.06 - 3.004r$

$c_1 = 0.019c_{-1} + 0.061c_0 + 0.92c_1 + 11.438 - 2.004r$

$c_1 = 0$, $c_2 = c_3 = c_4 = c_1$.

Solution: $r = 5.835$; $c_{-1} = 10.951$; $c_0 = 0.787$; $c_i = 0, i = 1, \ldots, 4$.

2^e. When we minimize the test quantity $c(d.z;i)$ we find $z_1 = (4,4,4,2,3,4)$, and $c(\hat{z};-1) = 10.888$; $c(\hat{z};0) = 0.723$;

$c(\hat{z};1) = -0.063$.

3^e. From $c(\hat{z};-1) < c(\hat{z};0) < c(\hat{z};1)$ it follows

$A_{\hat{z}}' = A_{z_1}$.

4^e. $z = (4,4,4,2,3,4)$.

1^e. Choose $e_1 = 2$. We have to solve:

$c_{-1} = 0.004\, c_{-1} + 0.015c_0 + 0.981c_1 + 40.013 - 5.001r$

$c_0 = 0.004c_{-1} + 0.015c_0 + 0.981c_1 + 24.013 - 4.001r$

$c_1 = 0.004c_{-1} + 0.015c_0 + 0.981c_1 + 17.392 - 3.001r$

$c_1 = 0, c_2 = c_3 = c_4 = c_1.$

Solution: $r = 5.814$; $c_{-1} = 10.993$; $c_0 = 0.807$; $c_i = 0, i = 1, \ldots, 4.$

2e. When we minimize the test quantity $c(d.z;i)$ we find $z_1 = z$.

3e. $A_z' = A_z$.

Hence $z = (4,4,4,2,3,4)$ is optimal. The strategy, which prescribes to produce 4-i units when the inventory falls below 2 , is an optimal strategy.

D.

Also other possibilities of treating excess demand can be considered. A possibility is, that some of the excess demand is defferred to a later period, while the remaining excess demand represents lost sales or is satisfied by emergency purchases. Under some additional restrictions simple cases of the manufacturers problem may be solved with Markov-programming if this interpretation of excess demand is used.

Litterature.

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Appendix.

The formula [c.f (3.49), p. 40]

$$E\underline{v}_r(\underline{m}((i)) = \lambda_r^{*} E\underline{m}((i)), \tag{1}$$

where

$$\lambda_r^{*} = \lambda_r \sum_{k=1}^{\infty} k p_{r,k} \tag{2}$$

is true, but the statement that (1) follows directly from the relations $E\underline{x} = E\{E(\underline{x}|\underline{y})\}$ and $E\underline{v}_r(t) = \lambda_r^{*} t$ is not correct.

We shall prove (1) by induction. In addition we shall find results, which simplify greatly the determination of $E\underline{m}((i))$.

We shall use the theorem of total expectation

$$E\underline{x} = \int_{-\infty}^{+\infty} E(\underline{x}|\underline{y} = y\} \, dP\{\underline{y} \leq y\}. \tag{3}$$

Let

$$t(i,k) \overset{def}{=} E\underline{t}_{i,k} \text{ and } v_i(j,k) \overset{def}{=} E\underline{v}_i(\underline{t}_{j,k}), \quad i,j = 1, \ldots, n. \tag{4}$$

Under the condition that $\underline{t}_{\beta,k} = t$, the random variable $\underline{v}_\alpha(\underline{t}_{\beta,k})$ has the same distribution as $\underline{v}_\alpha(t)$, if $\alpha \neq \beta$.

Hence, (use (3))

$$v_\alpha(\beta,k) = \int_0^{\infty} \lambda_\alpha^{*} t \, dP\{\underline{t}_{\beta,k} \leq t\} = \lambda_\alpha^{*} t(\beta,k), \quad \alpha \neq \beta. \tag{5}$$

This formula is also true if $\alpha = \beta$, what we shall prove.

Let $\underline{u}_\alpha$ be the waiting time to the arrival of the first customer, who asks for item α. Obviously $\underline{u}_\alpha$ has an exponential distribution with parameter λ_α. Given that this customer demands i units, where $0 \leq i \leq k-1$, $\underline{t}_{\alpha,k}$ has the same distribution as $\underline{u}_\alpha + \underline{t}_{\alpha,k-i}$. Given that the demand of that first customer exceeds k-1 the random variable $\underline{t}_{\alpha,k}$ is equal to $\underline{u}_\alpha$. Hence applying (3):

---

*) This conclusion is false if $\alpha = \beta$.

Lemma 1.

$$t(\alpha,k) = \sum_{i=0}^{k-1} p_{\alpha,i}\ t(\alpha,k-i) + \frac{1}{\lambda_\alpha} . \qquad (6)$$

On a similiar way it is proved,

Lemma 2.

$$v_\alpha(\ ,k) = \sum_{i=0}^{k-1} p_{\alpha,i}\ v_\alpha(\alpha,k-i) + \sum_{i=0}^{\infty} ip_{\alpha,i}. \qquad (7)$$

Lemma 3.

$$v_\alpha(\alpha,k) = \lambda_\alpha^{\ *} t(\alpha,k). \qquad (8)$$

Proof $\sum_{i=0}^{\infty} ip_{\alpha,i} = \frac{\lambda_\alpha^{*}}{\lambda_\alpha}$ [c.f (2)] (9)

The lemma follows now by induction from the lemmas 1 and 2.

Let for $\alpha \neq \beta$, $k\ ,\ m \geq 1$

$$t(\alpha,k;\beta,m) \overset{def}{=} E\max(\underline{t}_{\alpha,k}, \underline{t}_{\beta,m}) \qquad (10)$$

and

$$v(\alpha,k;\beta,m) \overset{def}{=} E\underline{v}_\alpha(\max(\underline{t}_{\alpha,k},\underline{t}_{\beta,m})). \qquad (11)$$

Let $\underline{u}_{\alpha,\beta}$ be equal to the waiting time to the arrival of the first customer, who asks for item α or item β ($\alpha \neq \beta$).
Let $q(\alpha,i)$ (respectively $q(\beta,i)$) be equal to the probability that this customer asks i units of item α (respectively item β).
Obviously, [c.f. lemma 3.3.2]

$$E\underline{u}_{\alpha,\beta} = \frac{1}{\lambda_\alpha + \lambda_\beta} \qquad (12)$$

$$q(\alpha,i) = p_{\alpha,i}\ \frac{\lambda_\alpha}{\lambda_\alpha + \lambda_\beta} . \qquad (13)$$

and

$$q(\beta,i) = p_{\beta,i}\ \frac{\lambda_\beta}{\lambda_\alpha + \lambda_\beta} \qquad (14)$$

Using the theorem of total expectation it follows

Lemma 4.

$$t(\alpha,k;\beta,m) = \sum_{i=0}^{k-1} q(\alpha,i)t(\alpha,k-i;\beta,m) + \sum_{i=k}^{\infty} q(\alpha,i)t(\beta,m) +$$

$$+ \sum_{j=0}^{m-1} q(\beta,j)t(\alpha,k;\beta,m-j) + \sum_{j=m}^{\infty} q(\beta,j)t(\alpha,m) +$$

$$+ \frac{1}{\lambda_\alpha + \lambda_\beta}, \tag{15}$$

and

$$v(\alpha,k;\beta,m) = \sum_{i=0}^{k-1} q(\alpha,i)v(\alpha,k-i;\beta,m) + \sum_{i=k}^{\infty} q(\alpha,i)v_\alpha(\beta,m) +$$

$$+ \sum_{j=0}^{m-1} q(\beta,j)v(\alpha,k;\beta,m-j) + \sum_{j=m}^{\infty} q(\beta,j)v_\alpha(\alpha,k) +$$

$$+ \sum_{i=0}^{\infty} iq(\alpha,i). \tag{16}$$

From

$$\sum_{i=0}^{\infty} iq(\alpha,i) = \frac{\lambda_\alpha^*}{\lambda_\alpha + \lambda_\beta} \tag{17}$$

and the relations (5) and (8) it follows by induction:

Lemma 5.

$$v_\alpha(\alpha,k;\beta,m) = \lambda_\alpha^* t(\alpha,k;\beta,m), \ \alpha \neq \beta.$$

Continuing in the same way, it is seen that the formula

$$\underline{E}\underline{v}_r(\underline{m}((i))) = \lambda_r^* \underline{E}\underline{m}((i))$$

can be proved by induction.

The general form of the recursionformula for $\underline{E}\underline{m}((i))$ will be obvious.