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Normal approximations to the Poisson distribution

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NORMAL APPROXIMATIONS TO THE POISSON DISTRIBUTION *)

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SUMMARY

This paper considers improvements, some known and some possibly new, upon the classical approximation to the Poisson distribution. The emphasis is on approximations which are simple enough for hand calculation, and accurate near the customary significance levels. The evaluation of accuracy and simplicity is to some extent a matter of taste.

Let F_λ denote the (cumulative) Poisson distribution function with parameter λ (see section 1). In the equation $F_\lambda(k) = P$ one can try to

1) find P from k and λ (distribution problem, section 1),
2) find λ from k and P (confidence problem, section 2),
3) find k from λ and P (fractile problem, section 3).

In all three cases, the transcendental equation admits no explicit solution, but asymptotic expansions for the solution exist. They are used in the Appendix for a comparison of approximate solutions based on the normal distribution. The paper briefly reports on numerical investigations of accuracy.

A general advice for the three problems mentioned above is given in Table 1, 2 and 3 respectively. The choice of an approximation will depend on the desired accuracy and the available computational facilities.

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1. THE DISTRIBUTION PROBLEM

Throughout this paper, λ is a positive real number, k a nonnegative integer and P a real number with $0 < P < 1$. Let F_λ denote the Poisson distribution function:

$$F_\lambda(k) = \sum_{j=0}^{k} e^{-\lambda} \lambda^j/j! \ , \qquad (1)$$

and let Φ denote the unit normal distribution function:

$$\Phi(u) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{u} \exp\,(-\tfrac{1}{2}t^2)dt \ . \qquad (2)$$

As tables of F_λ are not always immediately accessible, and as interpolation in these tables is cumbersome, our goal is to find simple functions u of k and λ such that $F_\lambda(k) \simeq \Phi(u)$. We shall use u as a general notation for such a deviate, and v, w, w_1 etc. for special cases. The simplest example is the standardized variable $w_1 = (k-\lambda)\lambda^{-\frac{1}{2}}$, to which one usually adds a continuity correction:

$$w = (k + \tfrac{1}{2} - \lambda)\ \lambda^{-\frac{1}{2}}. \qquad (3)$$

Another example is the square root transform, used for variance stabilization; with a suitable continuity correction it becomes

$$v = 2(k+1)^{\frac{1}{2}} - 2\lambda^{\frac{1}{2}}. \qquad (4)$$

Our advice for normal approximations to $F_\lambda(k)$ is summarized in Table 1. The problem of finding accurate "normal deviates" was studied in a more general context by Peizer and Pratt (1968). Their proposal, listed as v^{***} in Table 1, is somewhat laborious for hand calculation. Our evaluation of simplicity is based on a preference for approximations requiring no more than square roots and a cumulative normal table. The use of third roots, natural logarithms and other special functions is unattractive in "quick and dirty" work. With an electronic computer, direct summation of (1) for say $\lambda<30$, and calculation of $\Phi(v^{***})$ for larger λ, guarantees correctness to 5 or 6 decimal places when a good approximation to Φ is used.

The choice of approximating deviates in Table 1 is based on theoretical considerations, summarized in the Appendix, and on a numerical investigation of errors, which included values of λ ranging from .5 to 300, and, for each λ, all values of k satisfying $.001 \leq F_\lambda(k) \leq .999$. For us accurate approximation near the customary significance levels is the most important. It is hardly ever interesting to be quite sure that a probability is 3.10^{-4} and not 4.10^{-4}, so small tails were not considered.

A brief report on the numerical errors is not an easy task. The errors $\Phi(u) - F_\lambda(k)$ will be sketched as a function of k for fixed λ. Figures 1, 2, 3, 4 pertain to λ = 2, 10, 30 and 100 respectively, different curves belonging to different deviates u. The horizontal scale gives k, and in italics $F_\lambda(k)$; the vertical scale gives the error $\Phi(u) - F_\lambda(k)$. Though it is hardly possible to give information on, say, u and v^{***} in the same picture, the simple linear error scale is used in order to facilitate the interpretation of the pictures by the reader.

For example for λ = 10 (fig. 2) and k = 5, the italics give $F_{10}(5)$ = .067. The graph for v gives at k = 5 an error of +.010. One thus finds that $\Phi(v)$ = .067 + .010 = .077, an overestimation of the tail with a relative error of some 15 percent.

It is also interesting to give the *minimal* parameter value for which the *relative* error at the .05 and .95 point of the distribution is less than 1 percent, i.e. for which the approximation lies between .0495 and .0505 (between .9495 and .9505). This minimal λ exceeds 300 for v and w, lies between 7 and 10 for v^{*}, between 3 and 4 for v^{**} and below .5 for v^{***}. For the .025 and .975 point the approximation lies between .02475 and .02525 (between .97475 and .97525) if λ exceeds a value that is more than 300 for w, and is about 70 for v (for v^{*}, v^{**}, v^{***} as above). Whereas v is especially accurate near the .025 and .975 points, most other approximations have a relative error that is large for small tails and small for large tails. More details will be given in a subsequent publication.

2. THE CONFIDENCE PROBLEM

When k events have been observed, in a process where the number of events has a Poisson distribution with unknown expectation λ, the upper bound λ_2 with confidence level $1-\alpha$ is the solution of $F_{\lambda_2}(k) = \alpha$. The similar lower bound λ_1 satisfies $1-F_{\lambda_1}(k-1) = \alpha$. Note that the confidence level $1-\alpha$ holds for each bound separately. Thus the random interval (λ_1,λ_2) contains the fixed unknown value λ with probability at least $1-2\alpha$, and when a confidence interval with confidence level $1-\alpha$ is desired one should replace α by $\frac{1}{2}\alpha$ in all subsequent formulae. The exact value of the probability of incorrect statements depends on the value of λ, because of the discrete nature of the Poisson distribution.

Tables for λ_1 and λ_2 exist for small values of k and customary values of α. Moreover λ_2 equals the $1-\alpha$ fractile of a χ^2 distribution with 2k+2 degrees of freedom, and λ_1 equals the α fractile of χ^2 with 2k d.f. Nevertheless, when such tables are not available or do not contain the relevant values of k and/or α, approximations to confidence bounds become useful.

An advice for such approximations is given in Table 2. As before, there is a subjective element in the evaluation of accuracy and computational labour. Some mathematical background of the Table is given in the appendix. In a numerical investigation many expressions (some are listed in the appendix) were tried for α = .1, .05, .025, .01 and .005. No choice was uniformly superior for all α and k.

In our short report on errors we compare the approximated to the exact bounds, as a comparison of probabilities of correct statements becomes rather complicated. We shall not consider k=0 for upper bounds, k=0 or k=1 for lower bounds, as one easily finds the exact values to be $-\ln\alpha$, 0 and $-\ln(1-\alpha)$ respectively in these cases.

The "quick work" suggestion of Table 2 differed less than .25 from the exact bound, with a few exceptions having error .34 or less for upper bounds with α = .1.

The combined "more accurate" advice has an absolute difference between approximated and exact upper bound of at most .05 provided that: $k \geq 1$ when $\alpha = .1$; $k \geq 5$ when $\alpha = .05$; $k \geq 12$ when $\alpha = .025$; $k \geq 8$ when $\alpha = .01$; $k \geq 4$ when $\alpha = .005$. For lower bounds the same statement holds for: all k when $\alpha = .1$ or $\alpha = .05$, $k \geq 7$ when $\alpha = .025$, $k \geq 2$ when $\alpha = .01$ and $k \geq 5$ when $\alpha = .005$. As a rule this approximation is conservative, i.e.

the approximated upper [lower] bounds are higher [lower] than the exact values. The only exceptions observed to this conservatism were lower bounds for $\alpha = .005$ ($k < 70$) and $\alpha = .01$ ($k < 4$). Anderson and Burstein (1967, 1968) use $k - \xi k^{\frac{1}{2}} + R$ and the corresponding upper bound for all α, but we found that $\{(k+B)^{\frac{1}{2}} - \frac{1}{2}\xi\}^2$ was much better for $\alpha = .05$ and .1, and slightly better for $\alpha = .025$.

The "still better" formula of Table 2 was indeed nearly always better, the only exceptions being for lower bounds $3 \leq k \leq 11$ when $\alpha = .01$ and $k \geq 7$ when $\alpha = .005$, for upper bounds $k \geq 10$ when $\alpha = .005$. It has an error which rarely exceeded .02 and never exceeded .041 in our investigation.

The first "very accurate" suggestion has an error not exceeding .051, and for $\alpha \geq .025$ not exceeding .021. The second one differs never more than .007 from the exact confidence bound: it is better than the first one for small k and small α, and this is the region where the largest errors occur.

In the preceding statements about errors, k=0 for upper bounds and k=0 or 1 for lower bounds were not considered; the simple exact solution for these cases has already been mentioned.

3. THE FRACTILE PROBLEM

When a hypothesis is tested about an unknown Poisson parameter, it is frequently desired to determine k, from given α and λ, such that $F_\lambda(k) \leq \alpha < F_\lambda(k+1)$, or $F_\lambda(k-1) < 1-\alpha \leq F_\lambda(k)$. An equality sign, i.e. $F_\lambda(k) = \alpha$ or $= 1-\alpha$, is only possible for the very special values of λ which are an upper or lower $(1-\alpha)$ level confidence bound corresponding to some integer k.

Table 3 gives advice on normal approximations to Poisson P fractiles for given values of P and λ. The preceding remark indicates how one uses these fractiles when determining a rejection region. In many cases the probability of exceedance for an observed value (section 1) will be more informative. Furthermore, the exact [approximated] fractile has a jump of +1 at the exact [approximated] corresponding confidence bound values for λ, so section 2 gives some information on the errors. Therefore we shall be brief about fractile approximation, and skip the technical complications arising from the discrete character of the distribution.

In our numerical investigation of approximations to the Poisson P fractile, we took P = .005, .01, .025, .05, .1, .9, .95, .975, .99 and .995, and λ had the special (confidence) values mentioned above. For these values, accuracy was measured by the absolute difference between approximated and exact k.

For the "quick work" suggestion of Table 3, this difference never exceeded .85 and hardly ever exceeded .4 for the ten values of P mentioned above. It was found that $\lambda + \xi\lambda^{\frac{1}{2}}$ was better than our "quick work" idea for P = .025 and all λ, P = .05 and $\lambda<5$, P = .99 and $\lambda<12$, P = .995 and $\lambda<5$. As the "quick work" idea is defeated by $\lambda + \xi\lambda^{\frac{1}{2}} - \frac{1}{2}$ for roughly $.07 \leq P \leq .93$, it is only retained in Table 1 because it is simple and reasonably accurate for all P simultaneously.

The "more accurate" suggestion was found to give an error never exceeding .11 provided that $\lambda>1$, never exceeding .03 provided that $\lambda>33$. Unless P = .995 it was found to be conservative (to underestimate quantiles for P near 0, to overestimate them for P near 1).

The error of the "very accurate" formula was never found to be more than .02 provided that $P \geq .025$ and $\lambda>1$, but for P = .01 or P = .005 it can be .08 even if $\lambda>1$.

APPENDIX

Monotonicity considerations guarantee that for given λ and k there exists a unique "exact normal deviate" $\xi = \xi(k,\lambda)$ satisfying $F_\lambda(k) = \Phi(\xi)$. Its explicit solution from this equation is impossible, but for $\lambda\to\infty$ and bounded w or v [defined in (3) and (4)] ξ has the asymptotic expansions

$$\xi = w + \frac{1-w^2}{6\sqrt{\lambda}} + \frac{5w^3-2w}{72\lambda} + \frac{128+79w^2-249w^4}{6480\lambda\sqrt{\lambda}} + O(\lambda^{-2}) , \qquad (A.1)$$

$$\xi = v + \frac{v^2-4}{12\sqrt{\lambda}} + \frac{10v-v^3}{72\lambda} + \frac{21v^4-371v^2-52}{6480\lambda\sqrt{\lambda}} + O(\lambda^{-2}) . \qquad (A.2)$$

The well-known expansion (A.1) can be derived by combining

$$F_\lambda(k) = \sum_{j=0}^{k} e^{-\lambda}\lambda^j/j! = \int_\lambda^\infty t^k e^{-t}/k!\ dt \qquad (A.3)$$

with the Cornish-Fisher (1937) expansion valid for the gamma distribution. An alternative proof uses Cornish-Fisher directly for the Poisson distribution, with correction terms for its lattice character (Esseen, 1945, p.61). The expansion (A.2) follows from (A.1) if w is written as a function of v and λ.

When (A.3) is combined with asymptotic normality of the gamma distribution, one finds the deviate $y = (k+1-\lambda)(k+1)^{-\frac{1}{2}}$. Riordan (1949) gives the expansion ($k\to\infty$, y bounded)

$$\xi = y + \frac{y^2-1}{3\sqrt{k+1}} + \frac{7y^3-y}{36(k+1)} + \frac{219y^4-14y^2-13}{1620(k+1)\sqrt{k+1}} + O((k+1)^{-2}) , \qquad (A.4)$$

of which each partial sum is usually less accurate than the same number of terms from (A.1) or (A.2). If one uses (A.3) and the Wilson-Hilferty approximation to χ^2, which makes $\{(\chi^2/n)^{1/3} - 1 + \frac{2}{9n}\}\{\frac{2}{9n}\}^{-\frac{1}{2}}$ unit normal, one obtains a deviate

$$z = 3(k+1)^{\frac{1}{2}} - \frac{1}{3}(k+1)^{-\frac{1}{2}} - 3\{\lambda(k+1)^{\frac{1}{2}}\}^{1/3} , \qquad (A.5)$$

which is somewhat laborious (third root!), slightly superior to v^* for large λ, but hardly ever superior to v^{**}. The Fisher approximation to χ^2, which makes $(2\chi^2)^{\frac{1}{2}} - (2n-1)^{\frac{1}{2}}$ unit normal, gives the deviate $v_1 = 2(k+\frac{3}{4})^{\frac{1}{2}} - 2\lambda^{\frac{1}{2}}$, mentioned in Table 1 for quick work and probabilities between .06 and .94.

In Table 1 it was advised to use v and v_1, and never w, w_1 or y. For a motivation of this advice, observe that straightforward series expansions lead to

$$\begin{aligned}
y &= (k+1-\lambda)(k+1)^{-\frac{1}{2}} &&= \xi + (4-4\xi^2)/(12\sqrt{\lambda}) + O(\lambda^{-1}) ,\\
w &= (k+\tfrac{1}{2}-\lambda)\lambda^{-\frac{1}{2}} &&= \xi + (2\xi^2-2)/(12\sqrt{\lambda}) + O(\lambda^{-1}) ,\\
v_1 &= 2(k+\tfrac{3}{4})^{\frac{1}{2}}-2\lambda^{\frac{1}{2}} &&= \xi + (1-\xi^2)/(12\sqrt{\lambda}) + O(\lambda^{-1}) ,\\
w_1 &= (k-\lambda)\lambda^{-\frac{1}{2}} &&= \xi + (2\xi^2-8)/(12\sqrt{\lambda}) + O(\lambda^{-1}) ,\\
v &= 2(k+1)^{\frac{1}{2}}-2\lambda^{\frac{1}{2}} &&= \xi + (\xi^2-4)/(12\sqrt{\lambda}) + O(\lambda^{-1}) .
\end{aligned} \tag{A.6}$$

For large λ it follows that y, w and v_1 are accurate near $\xi = \pm 1$ (tails of .16) and w_1 and v near $\xi = \pm 2$ (tails of .023). Far from these special values the error of v is $(-\frac{1}{2})$ times the error of w_1, etc. As Taylor expansion gives e.g. $\Phi(v) - \Phi(\xi) = (v-\xi)\,\Phi'(\xi) + O(v-\xi)^2$, the error $\Phi(v) - F_\lambda(k)$ behaves like the error in the deviate and is about $(-\frac{1}{2})$ times $\Phi(w_1) - F_\lambda(k)$. When all numerators which have denominator $12\sqrt{\lambda}$ in (A.6) are similarly examined as functions of ξ, our suggestions in Table 1 become apparent. The picture does not essentially change when also the $O(\lambda^{-1})$ terms in (A.6) are considered, and numerical results confirm our findings with only a few exceptions. A more complete discussion will be published elsewhere.

Next we consider approximations with error $O(\lambda^{-1})$. Similar asymptotic considerations establish the superiority of v^* over the first two terms of (A.1), (A.2) or (A.3), and over the Wilson-Hilferty result (A.5). The well-known $\Phi(w) + (1-w^2)\,\Phi'(w)/(6\sqrt{\lambda})$ is usually more accurate than $\Phi(w + (1-w^2)/(6\sqrt{\lambda}))$, but also more laborious, and it is worse than $\Phi(v^*)$.

Finally v^{***}, v^{**} and the first three terms of the expansions for ξ all have error $O(\lambda^{-3/2})$, the advantage of v^{***} over v^{**} and of v^{**} over the others being a smaller coefficient of $\lambda^{-3/2}$ for most values of ξ.

The deviates v^{*} and v^{**} were found by expanding the general formula $2(k+1+\mathcal{B}+\mathcal{C}k^{-\frac{1}{2}})^{\frac{1}{2}} - 2(\lambda+\beta+\gamma\lambda^{-\frac{1}{2}})^{\frac{1}{2}}$, and choosing two or three of the expressions $\mathcal{B}$, $\mathcal{C}$, β, γ as zero, the other(s) as such functions of w that give agreement up to the highest possible order. The use of polynomials in v or v_1 instead of w turned out to be generally less accurate. Many other approximating deviates were tried, some new and some previously published. Though some of them will occasionally be good for special values of λ and k, the advice of Table 1 will be effective in most cases.

For an explanation of section 2, observe that the $1-\alpha$ confidence bounds λ_2 and λ_1 both satisfy the expansion (Campbell, 1923), for $h\to\infty$ and ξ fixed,

$$\lambda_i = h + \xi\sqrt{h} + \frac{\xi^2-1}{3} + \frac{\xi^3-7\xi}{36\sqrt{h}} + \frac{16-7\xi^2-3\xi^4}{810h} + O(h^{-3/2}) , \qquad \text{(A.7)}$$

if one takes $h = k+1$ and ξ such that $\Phi(\xi) = 1-\alpha$ for the upper bound, $h = k$ and ξ such that $\Phi(\xi) = \alpha$ for the lower bound. The expansion can be obtained from the inverse relation to (A.4) expressing y in terms of ξ and h. It also follows from the expansion for χ^2 fractiles (Goldberg and Levine, 1946). The first few terms suffice if h is large, convergence being slower for small α.

As both bounds λ satisfy $F_\lambda(h-1) = \Phi(-\xi)$, cf. section 2, one may put $-\xi \approx v = 2h^{\frac{1}{2}}-2\lambda^{\frac{1}{2}}$, which gives $\lambda\approx(h^{\frac{1}{2}}+\frac{1}{2}\xi)^2$. With a table of squares and square roots this may be even simpler to compute than $h+\xi h^{\frac{1}{2}}$, and it is theoretically more accurate when its absolute error $|4-\xi^2|/12$ is less than $|\xi^2-1|/3$, i.e. when $|\xi| > \sqrt{1.6}$. In practice this means $\alpha\leq .1$ for lower bounds but $\alpha<.1$ for upper bounds, as there the $O(h^{-\frac{1}{2}})$ term of the error more or less cancels the $O(1)$ term for $h+\xi h^{\frac{1}{2}}$, but has the same sign for $(h^{\frac{1}{2}}+\frac{1}{2}\xi)^2$.

Similar asymptotic considerations show the superiority of the first "more accurate" formula of Table 2 over $k - \xi k^{\frac{1}{2}} + R$ whenever $2 \leq |\xi| \leq 4$. The "still better" formula, following directly from the Wilson-Hilferty approximation to χ^2, has a smaller error (though also of order $h^{-\frac{1}{2}}$) unless $\sqrt{6} < |\xi| < 3$. We observe that

$$h + \frac{\xi^2-1}{3} + \xi(h + \frac{\xi^2-7}{18})^{\frac{1}{2}} = \{(h + \frac{\xi^2-7}{18})^{\frac{1}{2}} + \tfrac{1}{2}\xi\}^2 + \frac{\xi^2+2}{36} \qquad (A.8)$$

is usually slightly superior to Wilson-Hilferty for large k. Inversion of the very accurate Peizer-Pratt deviate v^{***} (Table 1) is too complicated.

The first "very accurate" formula, consisting of the first five terms of (A.7), is good for large α (small ξ) because the first neglected term has a contribution proportional to $\xi^5 h^{-3/2}$. The second "very accurate" suggestion was found empirically, by plotting the exact E satisfying $\lambda_1 = \{(k+B+CE^{-\frac{1}{2}})^{\frac{1}{2}}-\frac{1}{2}\xi\}^2$ as a function of k, and similarly for upper bounds. The value of E did not vary seriously with α. Though it is asymptotically equal to $k + O(k^{\frac{1}{2}})$, the formulae $\frac{7}{6}$ k + 3 and $\frac{3}{4}$ (k-1) give a better performance for small k, while for large k the whole term $CE^{-\frac{1}{2}}$ is so small that the wrong coefficient of k does no harm.

The Poisson P fractile treated in section 3 satisfies the expansion (Campbell, 1923), for $\lambda \to \infty$ and ξ fixed

$$k = \lambda + \xi\lambda^{\frac{1}{2}} + \frac{\xi^2-4}{6} - \frac{\xi^3+2\xi}{72\lambda^{\frac{1}{2}}} + \frac{3\xi^4+7\xi^2-16}{810\lambda} + O(\lambda^{-3/2}), \qquad (A.9)$$

where ξ is the unit normal P fractile, i.e. $\Phi(\xi) = P$. This expansion follows from (A.1) when $w = (k+\frac{1}{2}-\lambda)\lambda^{-\frac{1}{2}}$ is expressed in terms of ξ and λ. It is obvious that the "quick work" suggestion of Table 3 is the equivalent of v in Table 1. It has an error with leading term $(\xi^2-4)/12$, against $(4-\xi^2)/6$ for the more customary $\lambda+\xi\lambda^{\frac{1}{2}}$. As before the second "more accurate" suggestion has a smaller asymptotic error than the first one for $2 < |\xi| < 4$. The Wilson-Hilferty and Peizer-Pratt deviates lead to a very complicated expression for the quantile.

TABLE 1

Advice for the choice of a normal deviate u such that $\Phi(u)$ is an approximation to the probability $F_\lambda(k)$ of k or less events in a Poisson distribution with expectation λ. For Φ and F_λ cf. section 1, (1) and (2).

for quick work, use

$v = 2(k+1)^{\frac{1}{2}} - 2\lambda^{\frac{1}{2}}$, for reasonable accuracy near the customary significance levels (for probabilities between .06 and .94 it is better to use $v_1 = 2(k+\frac{3}{4})^{\frac{1}{2}} - 2\lambda^{\frac{1}{2}}$) ;

for more accurate work, use

$v^{*} = 2\{k + (w^2+8)\ /12\}^{\frac{1}{2}} - 2\lambda^{\frac{1}{2}}$, where $w = (k+\frac{1}{2}-\lambda)\ \lambda^{-\frac{1}{2}}$

or, still better,

$v^{**} = 2\{k + (w^2+5)\ /9\}^{\frac{1}{2}} - 2\{\lambda + (w^2-4)\ /36\}^{\frac{1}{2}}$.

When the approximation is desired to be accurate near the P and 1-P fractile, but may be rather rough elsewhere, it saves time to replace w in the preceding two formulae by the unit normal P fractile ξ, defined by $\Phi(\xi) = P$. For some customary values of P, Table 4 gives $B = (\xi^2-4)\ /12$, and one has $(\xi^2+8)/12 = B+1$, $(\xi^2-4)/36 = \frac{1}{3}B$, $(\xi^2+5)/9 = \frac{4}{3}B+1$.

for very accurate work, use

$v^{***} = (k + \frac{2}{3} - \lambda + \frac{.02}{k+1})\ (1 + A)^{\frac{1}{2}}\ \lambda^{-\frac{1}{2}}$, where

$A = (1 - f^2 + 2f \ln f)\ (1 - f)^{-2}$ and $f = (k + \frac{1}{2})\lambda^{-1}$; Peizer and Pratt (1968) give a table to determine A from f;

never use $w = (k+\frac{1}{2}-\lambda)\ \lambda^{-\frac{1}{2}}$, $w_1 = (k-\lambda)\ \lambda^{-\frac{1}{2}}$ or $y = (k+1-\lambda)(k+1)^{-\frac{1}{2}}$, because our "quick work" suggestion is more accurate and not more laborious. However, w is useful as a first step in the calculation of v^{*} or v^{**} mentioned above.

TABLE 2

Advice for the approximation to confidence bounds for the Poisson parameter λ*, when k events have been observed.* In this table ξ denotes the unit normal $1-\alpha$ fractile, e.g. 1.9600 for $\alpha = .025$. The formulae give the lower bound with confidence coefficient $1-\alpha$; for the analogous upper bound replace k by k+1 and ξ by $-\xi$ throughout. The lower bound is zero for k=0 and $-\ln(1-\alpha)$ for k=1, the upper bound is $-\ln \alpha$ for k=0.

for quick work, use

$(k^{\frac{1}{2}} - \frac{1}{2}\xi)^2$ (but use $k - \xi k^{\frac{1}{2}}$ for $\alpha > .1$ and its upper bound analogue for $\alpha \geq .1$)

for more accurate work, use

$\{(k+B)^{\frac{1}{2}} - \frac{1}{2}\xi\}^2$ for $\alpha \geq .025$, $k - \xi k^{\frac{1}{2}} + R$ otherwise,
where $B = (\xi^2-4)/12$ and $R = (\xi^2-1)/3$, cf. Table 4;

or, still better,

$k\{1 - \frac{1}{3}\xi k^{-\frac{1}{2}} - \frac{1}{9}k^{-1}\}^3$;

for very accurate work, use

$k - \xi k^{\frac{1}{2}} + R - Sk^{-\frac{1}{2}} - Tk^{-1}$, where $R = (\xi^2-1)/3$, $S = (\xi^3-7\xi)/36$ and
$T = (3\xi^4+7\xi^2-16)/810$, cf. Table 4;

the alternative formula

$\{(k + B + CE^{-\frac{1}{2}})^{\frac{1}{2}} - \frac{1}{2}\xi\}^2$, where $B = (\xi^2-4)/12$ and $C = (\xi^3+2\xi)/72$, cf. Table 4,
is more accurate for lower bounds, with $E = \frac{3}{4}(k-1)$,
if $\alpha \leq .05$ and roughly $k \leq 15$, for upper bounds, with
$E = \frac{7}{6}k + 3$, if $.01 \leq \alpha \leq .05$ and roughly $k \leq 25$.

TABLE 3

Advice for the approximation to the P fractile of the Poisson distribution with expectation λ. This fractile k satisfies $F_\lambda(k) = P$, and ξ denotes the unit normal fractile, i.e. $\Phi(\xi) = P$. As the approximation (but also the exact fractile) is usually not an integer, the result will be rounded off in most applications.

for quick work, use

$$(\lambda^{\frac{1}{2}} + \tfrac{1}{2}\xi)^2 - 1 \;;$$

for more accurate work, use

$$\lambda + \xi\lambda^{\frac{1}{2}} + 2B \text{ if } .025 \leq P \leq .975 \;,\; \{(\lambda - B)^{\frac{1}{2}} + \tfrac{1}{2}\xi\}^2 - 1 \text{ otherwise,}$$

where $B = (\xi^2-4)/12$, cf. Table 4;

for very accurate work, use

$$\lambda + \xi\lambda^{\frac{1}{2}} + 2B - C\lambda^{-\frac{1}{2}} + T\lambda^{-1} \;,$$

where $B = (\xi^2-4)/12$, $C = (\xi^3+2\xi)/72$ and $T = (3\xi^4+7\xi^2-16)/810$, cf. Table 4.

TABLE 4

Some functions of the unit normal P fractile ξ used in the preceding tables. Note that the odd polynomials C and S change sign with ξ, but the others are even.

$\alpha = 1 - P$	.1	.05	.025	.01	.005
$P = 1 - \alpha$	.9	.95	.975	.99	.995
ξ	1.281552	1.644854	1.959964	2.326348	2.575829
$B = (\xi^2-4)/12$	-.1965	-.1079	-.0132	.1177	.2196
$C = (\xi^3+2\xi)/72$	.0648	.1075	.1590	.2395	.3089
$R = (\xi^2-1)/3$	.2141	.5685	.9472	1.4706	1.8783
$S = (\xi^3-7\xi)/36$	-.1907	-.1962	-.1720	-.1026	-.0261
$T = (3\xi^4+7\xi^2-16)/810$	.0044	.0307	.0681	.1355	.2006

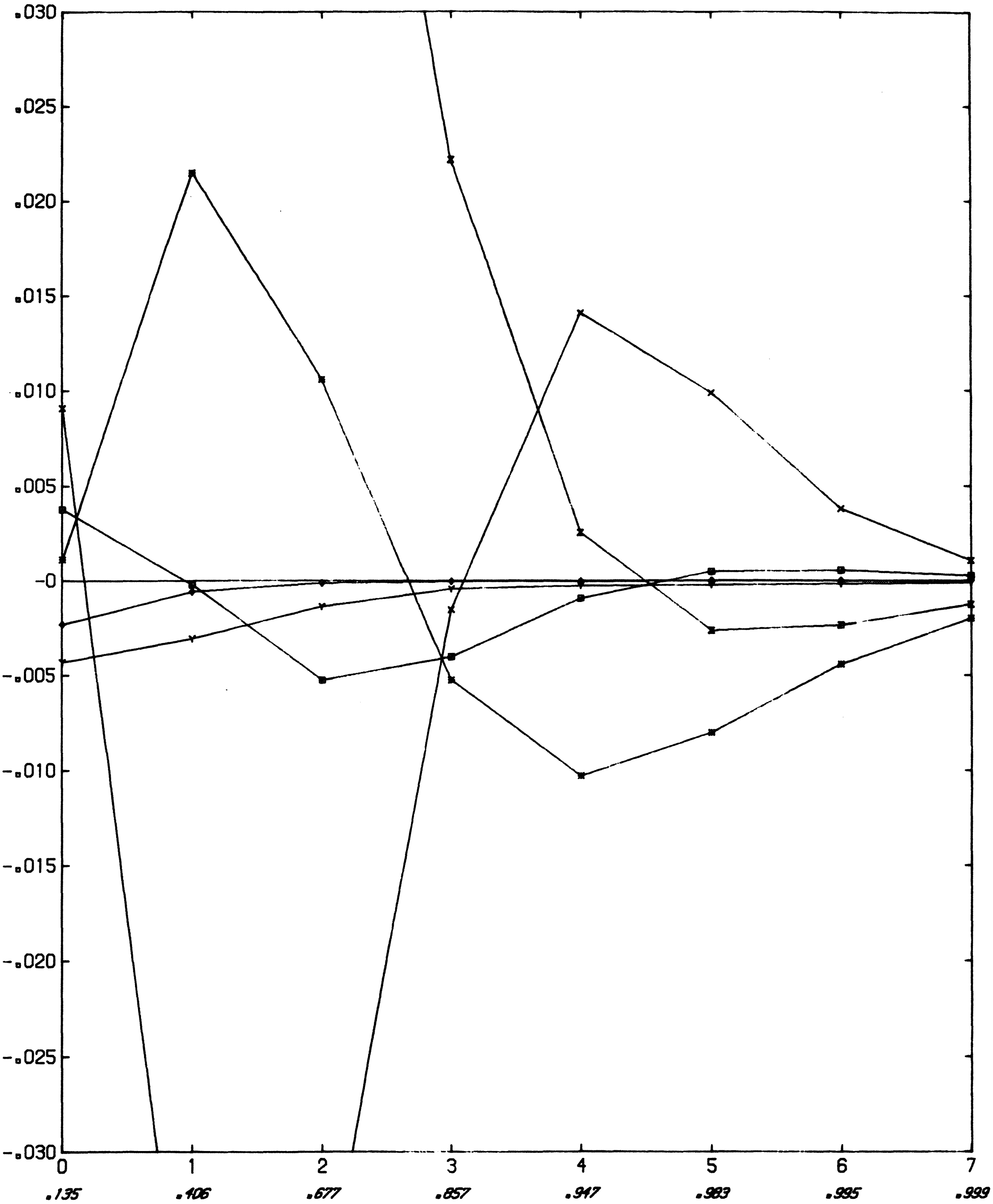

Fig. 1. Errors $\Phi(u) - F_{\lambda}(k)$ for $\lambda = 2$.

The horizontal scale gives k, and in italics the Poisson distribution function $F_{\lambda}(k)$. The vertical scale gives the errors. For the normal deviate u six functions of k and λ are considered. In the notation of Table 1 they are marked by × for w, ⧗ for v, ∗ for v_1, ◻ for v^{*}, Y for v^{**}, ◇ for v^{***}.

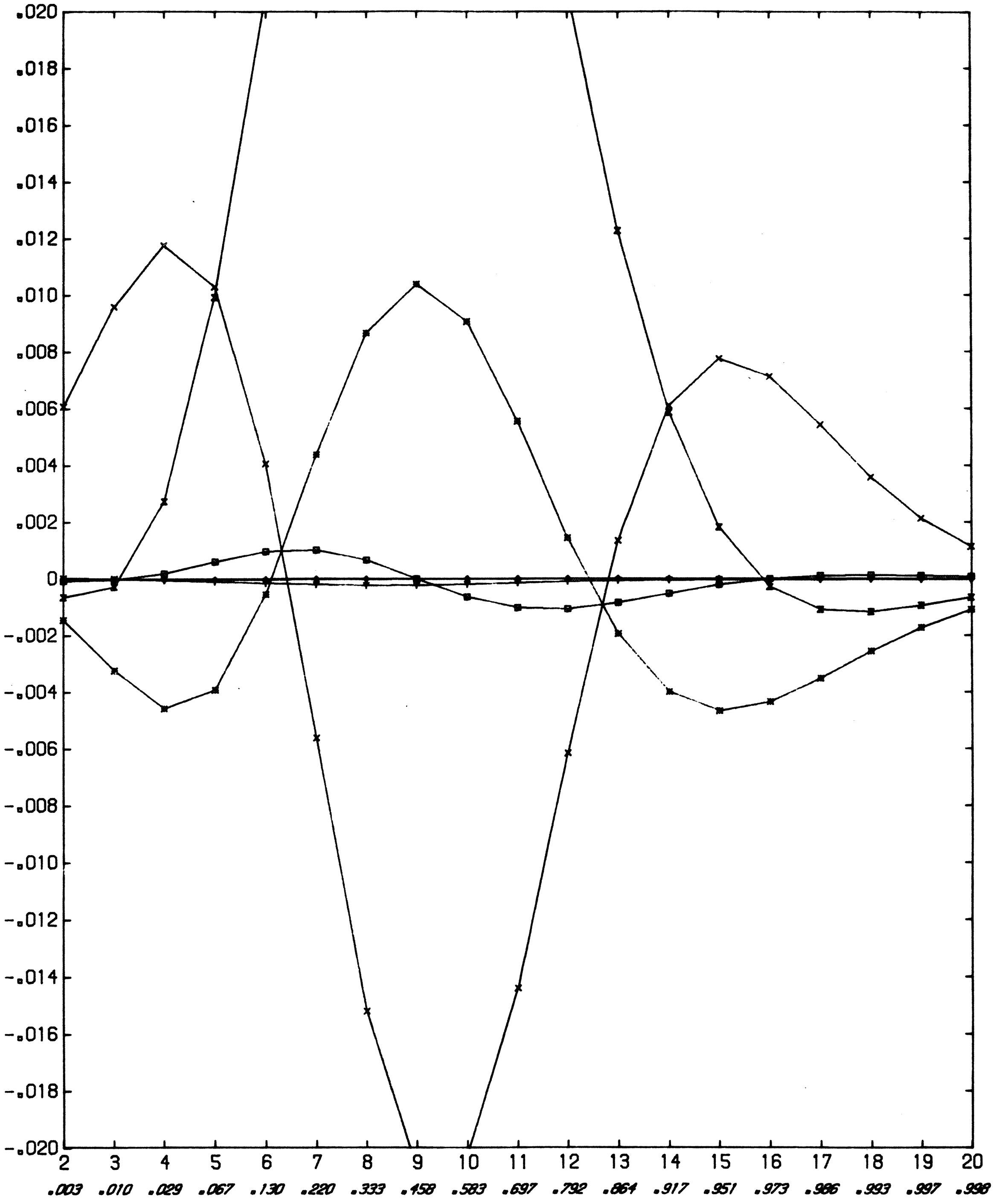

Fig. 2. Errors $\Phi(u) - F_{\lambda}(k)$ for $\lambda = 10$.

The horizontal scale gives k, and in italics the Poisson distribution function $F_{\lambda}(k)$. The vertical scale gives the errors. For the normal deviate u six functions of k and λ are considered. In the notation of Table 1 they are marked by × for w, ⧗ for v, ✱ for v_1, ⊡ for v^{*}, Y for v^{**}, ◇ for v^{***}.

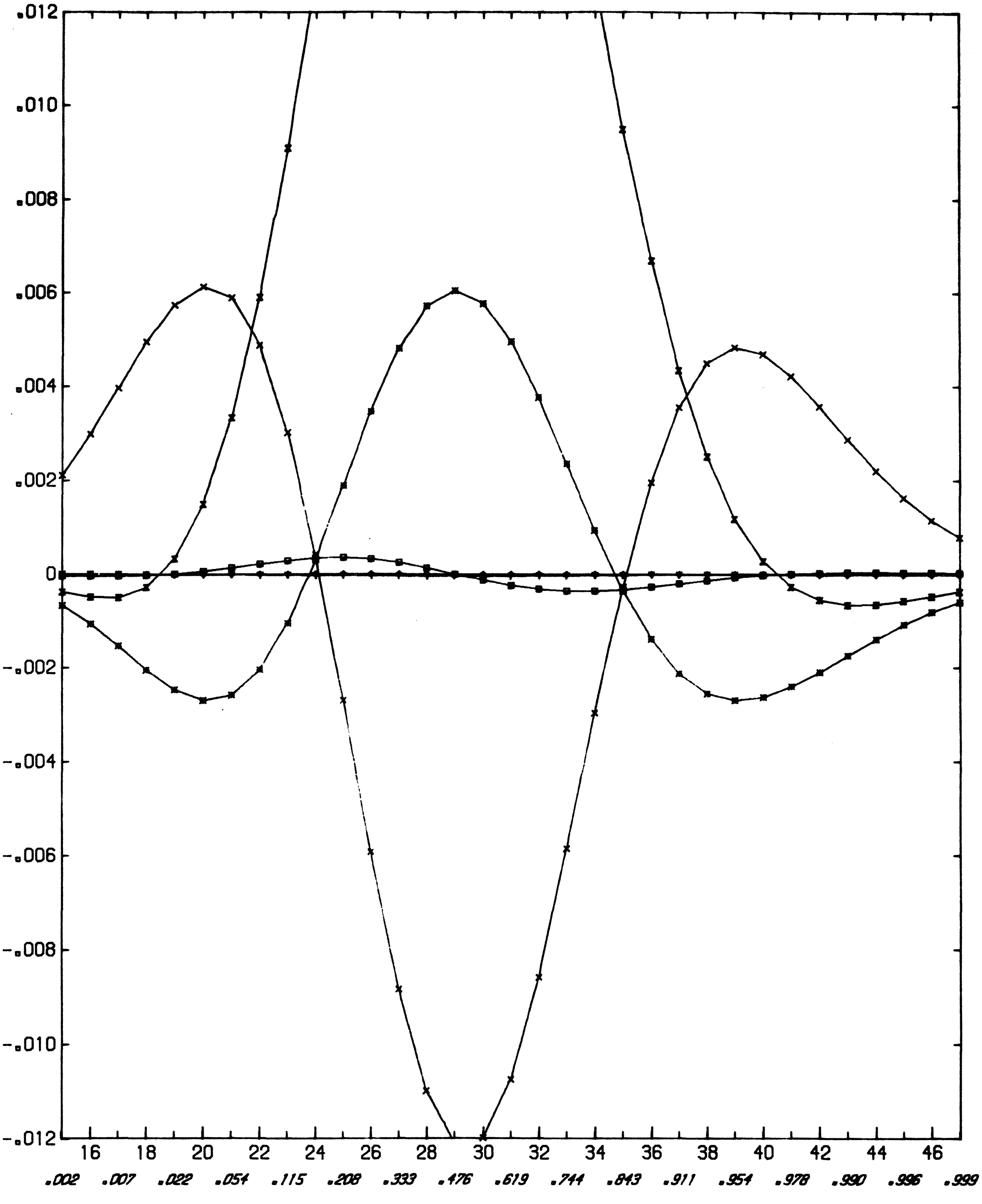

Fig. 3. Errors $\Phi(u) - F_{\lambda}(k)$ for $\lambda = 30$.

The horizontal scale gives k, and in italics the Poisson distribution function $F_{\lambda}(k)$. The vertical scale gives the errors. For the normal deviate u six functions of k and λ are considered. In the notation of Table 1 they are marked by × for w, ⧗ for v, ✱ for v_1, ◻ for v^{*}, Y for v^{**}, ◇ for v^{***}.

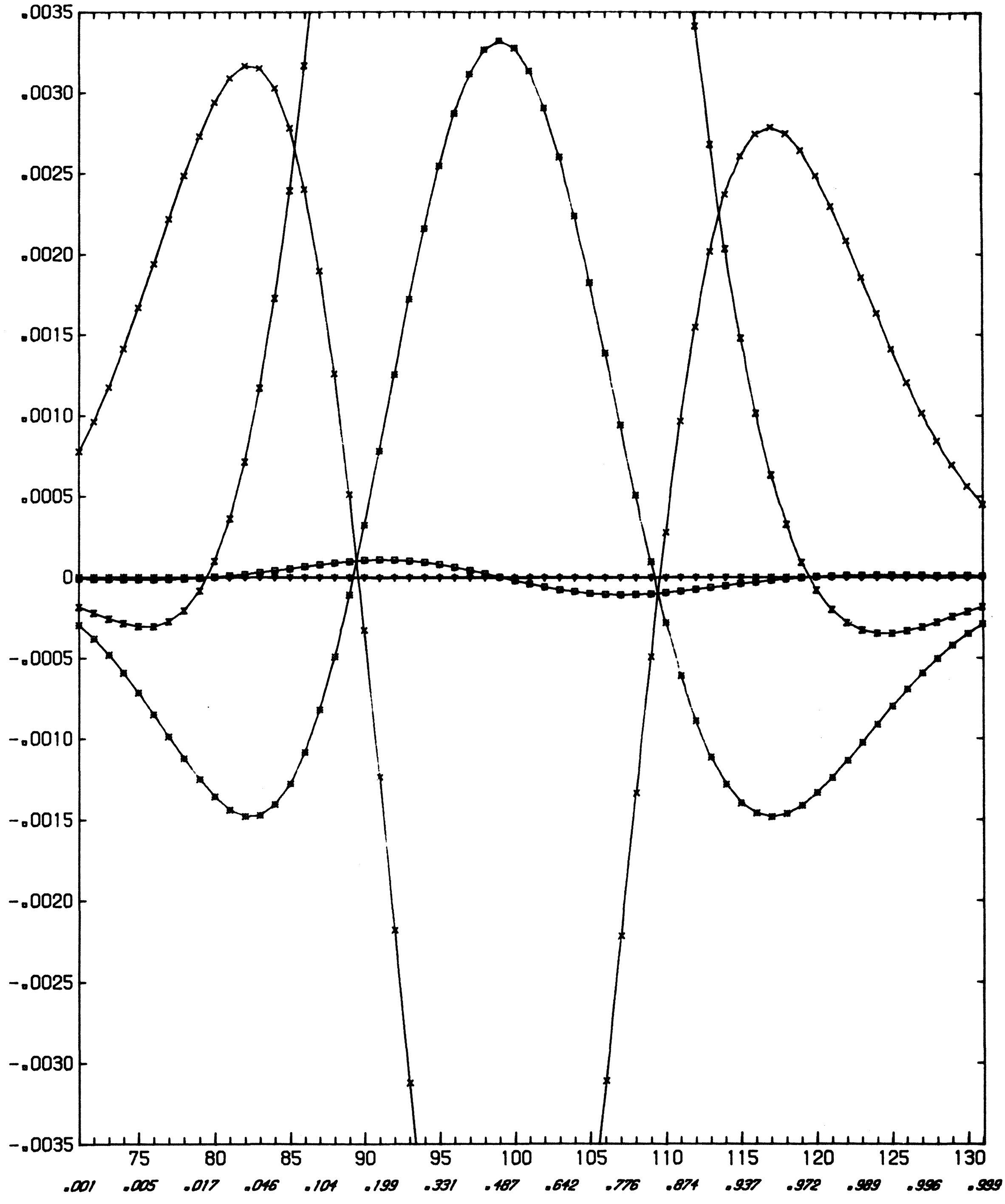

Fig. 4. Errors $\Phi(u) - F_{\lambda}(k)$ for $\lambda = 100$.
The horizontal scale gives k, and in italics the Poisson distribution function $F\lambda(k)$. The vertical scale gives the errors. For the normal deviate u six functions of k and λ are considered. In the notation of Table 1 they are marked by × for w, ⧗ for v, ✱ for v_1, ◘ for v^*, Y for v^{**}, ◇ for v^{***}.

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