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SCRIPTUM 6

Regular systems of equations and supernumerary coordinates.

A course of twelve lectures delivered in 1947 at the Mathematical Centre of Amsterdam

by

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Preface.

In differential geometry many mathematical tools are used that are made or should have been made in the workshop of the analyst. But from the point of view of the analyst these tools are so simple and uninteresting that he prefers not to spend much time on them and to spare his time and energy for more difficult problems. The result is that in textbooks of analysis e.g. the theory of regular systems of equations is dealt with in a rather superficial way and that we look in vain for a general theory of supernumerary coordinates, so frequently used in all branches of mathematics. So differential geometers had to do what properly was not their job, and this is exactly what they have done or at least have tried to do.

These lectures given in 1947 at the Mathematical Centre of Amsterdam form an introduction to my courses on tensor calculus, theory of Pfaff's problem and its generalizations and other objects of differential geometry given or to be given.

Now many of the points considered in this preliminary course are dealt with more elaborately in "Pfaff's problem and its generalizations" by Mr. W.v.d.Kulk and me, Clarendon Press, Oxford 1949 (hereafter referred to as P.P.) and some also in my "Tensor calculus for physicists" that will be published by the Clarendon Press in 1951. Nevertheless the Clarendon Press has kindly agreed with the appearance of these lectures in the scripta of the Mathematical Centre of Amsterdam, and I wish to express here my most hearty thanks for this to the English editor.

My personal view is that most students, after having studied this scriptum will be eager to study in the more elaborate books the numerous applications of the theories developed in this short publication. Many references will be found in the text and at the end there is a list of the literature referred to.

Epe, October 1950. J.A.Schouten.

§ 1. The arithmetic n-dimensional manifold $\mathfrak{A}_n$ [1].

Every ordered set of n real or complex values of n variables $\overset{\kappa}{\xi}$; $\kappa = 1, \cdots, n$ is called an arithmetic point and the totality of all these points an arithmetic manifold or $\mathfrak{A}_n$. The $\overset{\kappa}{\xi}$ are called the components of the point and the point is shortly called "point $\overset{\kappa}{\xi}$".

A polycylinder in $\mathfrak{A}_n$ is the totality of all points satisfying inequalities of the form

$$1.1) \qquad |\overset{\kappa}{\xi} - \overset{\kappa}{\alpha}| < \overset{\kappa}{\beta} \quad {}^{2)}$$

where the $\overset{\kappa}{\alpha}$ are arbitrarily given real or complex numbers and the $\overset{\kappa}{\beta}$ arbitrarily given positive numbers.

A set of arithmetic points of $\mathfrak{A}_n$ is called a region of $\mathfrak{A}_n$ if:

1. the set is open i.e. every point of the set belongs to at least one polycylinder consisting only of points of the region;

2. for every choice of two points of the region there exists at least one finite chain of polycylinders, each consisting only of points of

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1) For § 1 and § 2 cf. Veblen and Whitehead 1932.1; Behnke and Thullen 1934.2; Schouten and v.d. Kulk 1949.1 (hereafter referred to as P.P.) Ch II § 1,2.

2) In this publication the five indices $\kappa, \lambda, \mu, \nu, \omega$ always take the values $1, \cdots, n$.

the region, such that the first point lies in the first and the second in the last polycylinder and consecutive polycylinders have at least one point in common.

Obviously every polycylinder is a region and the whole $\mathfrak{A}_n$ is a region. But not every region is a polycylinder. Every region is called a <u>neighbourhood</u> of every one of its points. For "neighbourhood" of $\overset{\kappa}{\underset{0}{\xi}}$ " we write shortly $\mathfrak{N}(\overset{\kappa}{\underset{0}{\xi}})$.

<u>§ 2. The geometric n-dimensional manifolds X_n and $\mathcal{E}_n$.</u>

We consider a set M of elements of some kind which are in one-to-one correspondence to the poin of a region $\underset{0}{\mathfrak{R}}$ of $\mathfrak{A}_n$. With respect to the element of M we only presume that they are no points of $\mathfrak{A}_n$. They may e.g. be homogeneous linear forms in n variables or polynomials of degree $n-1$ in one variable or points of an arithmetic manifold different from $\mathfrak{A}_n$. The one-to-one correspondence between M and $\underset{0}{\mathfrak{R}}$ is called a <u>coordinate system over</u> M. If the point $\overset{\kappa}{\xi}$ corresponds to an element of M we call the $\overset{\kappa}{\xi}$ the <u>coordinates of this element with respect to the coordinate system</u> (κ) and we write ξ^{κ} instead of $\overset{\kappa}{\xi}$ if they are considered as coordinates of an element and not as components of an arithmetic point.

Now we make use of the following theorem, proved in every reliable textbook on analysis:

Theorem I (Theorem of inversion)

If in the system of equations

2.1) $$\overset{\kappa'}{\xi} = \overset{\kappa'}{f}(\overset{\kappa}{\xi}) \quad ; \quad \kappa' = 1', \cdots, n'$$

the functions $\overset{\kappa'}{f}$ are analytic[1] in $\underset{\circ}{\overset{\kappa}{\xi}}$ (hence in an $\mathcal{N}(\underset{\circ}{\overset{\kappa}{\xi}})!$) and if the functional determinant of these functions

2.2) $$\Delta \overset{def}{=} Det\left(\partial\overset{\kappa'}{\xi} / \partial\overset{\kappa}{\xi}\right) \quad ; \quad \kappa' = 1', \cdots, n'$$

is $\neq 0$ in $\underset{\circ}{\overset{\kappa}{\xi}}$ (hence $\neq 0$ in an $\mathcal{N}(\underset{\circ}{\overset{\kappa}{\xi}})!$), the $\overset{\kappa}{\xi}$ can be solved from (2.1) and in this solutiom

2.3) $$\overset{\kappa}{\xi} = \overset{\kappa}{f}(\overset{\kappa'}{\xi}) \quad ; \quad \kappa' = 1', \cdots, n'$$

the functions $\overset{\kappa}{f}$ are analytic in an $\mathcal{N}(\underset{\circ}{\overset{\kappa'}{\xi}})$, $\underset{\circ}{\overset{\kappa'}{\xi}}$ being defined by

2.4) $$\underset{\circ}{\overset{\kappa'}{\xi}} = \overset{\kappa'}{f}(\underset{\circ}{\overset{\kappa}{\xi}})$$

The functional determinant of the $\overset{\kappa}{f}$ in $\underset{\circ}{\overset{\kappa'}{\xi}}$ is equal to Δ^{-1}.

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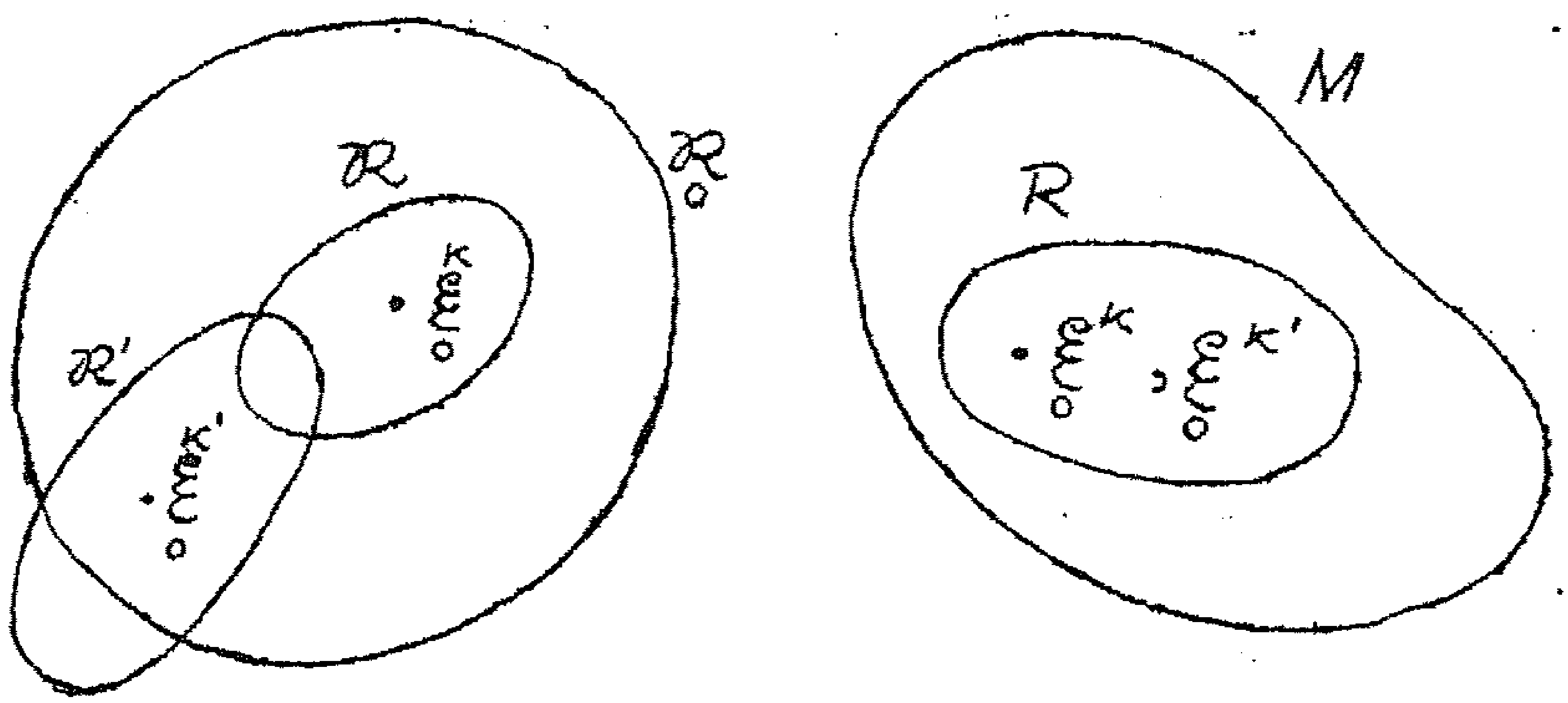
1) A function defined in an $\mathcal{N}(\underset{\circ}{\overset{\kappa}{\xi}})$ is said to be analytic in $\underset{\circ}{\overset{\kappa}{\xi}}$ if there exists an $\mathcal{N}(\underset{\circ}{\overset{\kappa}{\xi}})$ where it can be expanded into a power series in $\overset{\kappa}{\xi} - \underset{\circ}{\overset{\kappa}{\xi}}$, convergent in this latter $\mathcal{N}(\underset{\circ}{\overset{\kappa}{\xi}})$.

As a consequence of this theorem there exists in $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa})$ a neighbourhood $\mathfrak{R}$ of $\overset{\circ}{\xi}{}^{\kappa}$ and in $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa'})$ a neighbourhood $\mathfrak{R}'$ of $\overset{\circ}{\xi}{}^{\kappa'}$ for the points of which the equations (2.1) and also (2.3) establish a one-to-one correspondence.

Now let us presume that $\mathfrak{R}$ is contained in $\overset{\circ}{\mathfrak{R}}$. Then there exists in M a subset R of arithmetic points whose elements are in one-to-one correspondence to the points of $\mathfrak{R}$ and therefore also in one-to-one correspondence to the points of $\mathfrak{R}'$. This latter correspondence is, according to our definition, another coordinate system over R. This coordinate system we denote by (κ') and we write $\xi^{\kappa'}$ for the $\overset{\circ}{\xi}{}^{\kappa'}$ if they are consideredas coordinates of the elements of M. (fig.1.)

Fig.1



The equations (2.1,3) represent a <u>point transformation</u> in $\mathfrak{A}_n$ and its inverse. If the $\overset{\circ}{\xi}{}^{\kappa}, \overset{\circ}{\xi}{}^{\kappa'}$ are replaced by $\xi^{\kappa}, \xi^{\kappa'}$ we get the equations

2.5)
$$\xi^{\kappa'} = f^{\kappa'}(\xi^{\kappa}) \qquad ; \kappa' = 1', \ldots, n'$$
$$\xi^{\kappa} = f^{\kappa}(\xi^{\kappa'})$$

representing a coordinate transformation in $\mathcal{R}$ in $\mathcal{M}$. A transformation of elements in $\mathcal{M}$ has the form

2.6) $$\eta^{\kappa} = \mathcal{F}^{\kappa}(\xi^{\lambda})$$

or,
with respect to another coordinate system (κ')
$$\eta^{\kappa'} = \mathcal{F}^{\kappa'}(\xi^{\lambda'}) \quad ; \kappa', \lambda' = 1', \ldots, n'.$$

Collecting results we have:

A coordinate system over a set of elements $\mathcal{R}$ is a one-to-one correspondence between the elements of $\mathcal{R}$ and the points of a region of $\mathfrak{A}_n$, and a transformation of coordinates in $\mathcal{R}$ means passing to another one-to-one correspondence between these elements and the points of another region of $\mathfrak{A}_n$ [1].

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1) Cf. Veblen and Whitehead 1932.1, p.32. We consider here only ordinary coordinates. For supernumerary coordinates see § 9.

At the beginning we have to agree upon the coordinate transformations to be allowed. Often it is required that this set of transformations forms a group, i.e.

1. the result of two transformations of the set, applied after each other, is in the set;
2. if a transformation is in the set its inverse exists and is in the set;
3. the set contains the identical transformation.

E.g.:

1. the group of all permutations of coordinates;
2. the affine group $\mathcal{G}_a$ of all invertible linear transformations;
3. the special affine group $\mathcal{G}_{ho}$ of all invertible linear homogeneous transformations;
4. the orthogonal group $\mathcal{G}_{or}$ of all orthogonal transformations (One of the properties of this group is: $\Delta = \pm 1$ but this is not a sufficient condition for orthogonality).
5. the group $\mathcal{G}_{ro}$ of all rotations ($\mathcal{G}_{or}$ with $\Delta = + 1$).

But the group property is not always required. E.g. the set $\mathcal{K}$ of all invertible transformations analytic in some region is not a group. If a transformation T_1 transforms a region $\mathcal{R}'''$ into $\mathcal{R}'$ and some other transformation T_2 a region $\mathcal{R}''$ into $\mathcal{R}$, the transformation $T_2 T_1$ can be formed if and only $\mathcal{R}'$ and $\mathcal{R}''$ have a point (hence a region) in common. Such a set is called a <u>pseudo-group.</u> The result of two transformations of a pseudo-group, if existing, belongs to the pseudo-group[1].

The set $\mathcal{R}$ equipped with an original coordinate system and with all <u>allowable coordinate systems</u> i.e. all coordinate systems that can be derived from the original one by means of allowable coordinate transformations is called an <u>n-dimensional geometric manifold.</u> The elements are called <u>geometric points</u> or shortly <u>points.</u>

The choice of $\mathcal{R}_0$ and of the allowable coordinate systems is entirely free. These choices fix the geometric properties of the geometric manifold.

If we choose $\mathcal{R}_0$ arbitrarily, and $\mathcal{f}$, we get the X_n: the space of ordinary generalized differential geometry.

If we choose $\mathcal{R}_0 = \mathfrak{A}_n$ and $\mathcal{G}_a$ we get the $\mathcal{E}_n$, the space of ordinary n-dimensional affine geometry.

If we choose $\mathcal{R}_0 = \mathfrak{A}_n$ and $\mathcal{G}_{ko}$ we get the centred $\mathcal{E}_n$, the space of ordinary n-dimensional affine geometry with fixed origin.

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1) Cf. Veblen and Whitehead 1932.1, p.38.

If we choose $\overset{\circ}{\mathcal{R}} = \mathfrak{A}_n$ and g_{or} we get the R_n, the space of ordinary n-dimensional metric geometry.
R_3 = ordinary space.

In an X_n the notion of polycylinder can not be used because there is not a preferred coordinate system. Instead of polycylinders we use cells, a cell being defined as a set of points satisfying the inequalities

2.7) $$|\xi^\kappa| < 1$$

in some allowable coordinate system. A point set R in X_n is called a region if there exists an allowable coordinate system (κ) determining a one-to-one correspondence between the points of R and the arithmetic points of a region $\mathcal{R}$ of $\mathfrak{A}_n$. $\mathcal{R}$ is called the fundamental region of R with respect to (κ) [1]. Evidently every cell is a region. But not every region needs to be a cell. Also the X_n itself is a region and every region in X_n is itself an X_n. Every X_n that is a region of another X_n is said to be imbedded in the latter. Every region of an X_n is called a neighbourhood of every one of its points. For "neighbourhood of $\overset{\circ}{\xi}{}^\kappa$" we write shortly $\mathfrak{N}(\overset{\circ}{\xi}{}^\kappa)$.

p is said to be analytic in $\overset{\circ}{\xi}{}^\kappa$ if for any choice of the coordinate system (κ) in R, p is a

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1) A region in an X_n can also be defined in the same way as a region in $\mathfrak{A}_n$ by using cells instead of polycylinders.

function of the ξ^κ

$$2.8) \qquad p = f(\xi^\kappa)$$

and if f is analytic in $\overset{\circ}{\xi}{}^\kappa$. If this condition is satisfied and if (κ') is another allowable coordinate system and

$$2.9) \qquad \xi^{\kappa'} = f^{\kappa'}(\xi^\kappa) \quad ; \; \kappa' = 1', \cdots, n'$$

it is well-known from the theory of functions of several variables that

$$2.10) \qquad f(f^\kappa(\xi^{\kappa'}))$$

is analytic in $\overset{\circ}{\xi}{}^{\kappa'}$. Hence analyticity is invariant for all allowable coordinate transformations. If p is analytic in $\overset{\circ}{\xi}{}^\kappa$ there exists an $\mathfrak{N}(\overset{\circ}{\xi}{}^\kappa)$ where p is analytic in every point.[1]

§ 3. The null form of the equations of an X_m in X_n[2].

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1) We always consider analytic functions. But many of the theorems dealt with here can also be formulated and proved if only the existence and the continuity of the derivatives up to a certain order is presupposed.

2) Cf. Kähler 1934.1; P.P. Ch. II § 3.

We consider N functions $\mathcal{F}^{\alpha}(\xi^{\kappa})$; $\alpha = 1, \cdots, N$, analytic in $\overset{\circ}{\xi}{}^{\kappa}$. The matrix of the nN derivatives of the $\mathcal{F}^{\alpha}$ with respect to the ξ^{κ}

$$3.1) \qquad \overset{\longleftarrow N \longrightarrow}{n \updownarrow \left\| \partial_{\lambda} \mathcal{F}^{\alpha} \right\|} \quad ; \; \alpha = 1, \cdots, N$$

is called the <u>functional matrix of the system</u> $\mathcal{F}^{\alpha}$ and its rank r 1) in $\overset{\circ}{\xi}{}^{\kappa}$ the <u>rank of the system</u> in that point. Evidently $r \leq n$ and $r \leq N$. Hence r is the maximum number of linearly independent differentials among the $d\mathcal{F}^{\alpha}$ in $\overset{\circ}{\xi}{}^{\kappa}$.

If we form the matrix of the $n'N$ derivatives of the $\mathcal{F}^{\alpha}$ with respect to $n' < n$ of the variables ξ^{κ} the rank r' of this matrix in $\overset{\circ}{\xi}{}^{\kappa}$ is called the <u>rank of the system</u> $\mathcal{F}^{\alpha}$ <u>with respect to these variables in that point</u>. Evidently $r' \leqq n'$; $r' \leqq N$; $r' \leqq r$.

The functions $\mathcal{F}^{\alpha}$ are said to be (<u>functionally</u>) <u>independent</u> in $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa})$ if in $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa})$ none of them can be expressed as a function of the others and <u>(functionally)</u> <u>dependent</u> in the other case.

The proof of the following theorem will be found in every reliable textbook on analysis.

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1) A matrix has rank r if it contains at least one non-vanishing subdeterminant with r rows but none with $r+1$ rows.

Theorem II (Theorem of independency).

N functions $\mathcal{F}^{\alpha}(\xi^{\kappa})$; $\alpha = 1, \dots, N$, analytic in $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa})$ are independent in $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa})$ if and only if the rank r of the system $\mathcal{F}^{\alpha}$ is equal to N in at least one point of $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa})$.

According to this theorem the functions $\mathcal{F}^{\alpha}$ are always dependent if $N > n$. If the $\mathcal{F}^{\alpha}$ are independent and if $r = N$ the following theorem holds:

Theorem III (Theorem of adaption).

If $N \leq n$ functions $\mathcal{F}^{\alpha}(\xi^{\kappa})$; $\alpha = 1, \dots, N$, are analytic in an $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa})$ and if $r = N$ in every point of $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa})$, there exists an allowable coordinate system ξ^{h}; $h = 1, \dots, n$, in $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa})$, such that

3.2) $$\mathcal{F}^{\alpha} = \xi^{\alpha} \quad ; \quad \alpha = 1, \dots, N.$$

Proof.

We take $n - N$ functions

3.3) $$\mathcal{F}^{N+1}(\xi^{\kappa}); \dots ; \mathcal{F}^{n}(\xi^{\kappa}),$$

analytic in $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa})$, such that the rank of the system $\mathcal{F}^{h}$; $h = 1, \dots, n$ is n in each point of $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa})$. Then the transformation

$$3.4) \qquad \bar{\xi}^{h} = \mathcal{F}^{h}(\xi^{\kappa}) \; ; \; h = 1, \ldots, n$$

is an allowable coordinate transformation and consequently the $\bar{\xi}^{h}$ form an allowable coordinate system in $\mathfrak{N}(\xi_0^{\kappa})$.

Now we consider a system of N equations

$$3.5) \qquad \mathcal{F}^{\alpha}(\xi^{\kappa}) = 0 \quad ; \quad \alpha = 1, \cdots, N$$

with functions $\mathcal{F}^{\alpha}$, analytic in $\mathfrak{N}(\xi_0^{\kappa})$.

Every point of $\mathfrak{N}(\xi_0^{\kappa})$ satisfying (3.5) is called a nullpoint of (3.5) and the set M of all nullpoints the null manifold of (3.5).

The system (3.5) is called the null form of M. The rank of the system $\mathcal{F}^{\alpha}$ in a null point of (3.5) is called the rank of (3.5) in that point; the rank of the system $\mathcal{F}^{\alpha}$ with respect to n' of the ξ^{κ} in a null point of (3.5) is called the rank of (3.5) with respect to these variables in that point.

Two systems of equations, having the same null points in $\mathfrak{N}(\xi_0^{\kappa})$ are said to be equivalent in that region. It has to be remarked that two equivalent systems need not have the same rank in all null points. (E.g. $x=0$, $y=0$ and $x^2=0$, $y=0$ in the point $x=0$, $y=0$).

ξ^{κ} a null point of (3.5). If r is the rank and if $r \doteq N$ in $\overset{\circ}{\xi}{}^{\kappa}$ (and consequently also $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa})$), (3.5) is said to be <u>minimal re-</u>
n $\overset{\circ}{\xi}{}^{\kappa}$. The number $n-N$ is called the <u>di-</u>
<u>of the null manifold</u> in $\overset{\circ}{\xi}{}^{\kappa}$. Evidently
; $N \leqq n$. If a system is minimal regular
l point $\overset{\circ}{\xi}{}^{\kappa}$, there exists an $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa})$ where
tem is minimal regular in all points. From
nition it follows immediately <u>that if a</u>
<u>s minimal regular in</u> $\overset{\circ}{\xi}{}^{\kappa}$ <u>every one of its</u>
<u>ems is also minimal regular in</u> $\overset{\circ}{\xi}{}^{\kappa}$. The
"minimal regular" and "dimension" are in-
for all allowable coordinate transforma-

ystem (3.5) with the null point $\overset{\circ}{\xi}{}^{\kappa}$ being
here are four possible cases:
there does not exist an equivalent system
in an $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa})$, minimal regular in $\overset{\circ}{\xi}{}^{\kappa}$.
n the system is called <u>irregular</u> in $\overset{\circ}{\xi}{}^{\kappa}$.
l manifold has no dimension in $\overset{\circ}{\xi}{}^{\kappa}$;
there exists an equivalent system, mini-
ular of dimension m in $\overset{\circ}{\xi}{}^{\kappa}$ but among the N
ntials $d\mathfrak{F}^{\alpha}$ in $\overset{\circ}{\xi}{}^{\kappa}$ there exist no $n-m$
e linearly independent. Then the system
ed semi-regular of dimension m in $\overset{\circ}{\xi}{}^{\kappa}$;
as under 2 but among the N differentials
there exist $n-m < N$ linearly independent
hen the system is called (<u>supernumerary</u>)

regular of dimension m in $\overset{\circ}{\xi}{}^{\kappa}$ [1];

4. as under 3 but $n - m = N$. Then according to our definition the system is minimal regular in $\overset{\circ}{\xi}{}^{\kappa}$.

The notions irregular, semiregular, regular and minimal regular are invariant for all allowable coordinate transformations.

Here follow some examples in R_3:

3.6)
1. $xy = 0$; $xz = 0$: irregular in $x = 0, y = 0$.
2. $x^2 = 0$; $y = 0$: semiregular in $x = 0, y = 0; m = 1$.
3. $x^2 = 0; x = 0; y = 0$: regular in $x = 0, y = 0$; $m = 1$.
4. $x = 0$; $y = 0$: minimal regular in $x = 0, y = 0$; $m = 1$.

A subsystem of a system regular of dimension m in $\overset{\circ}{\xi}{}^{\kappa}$ need not be regular in $\overset{\circ}{\xi}{}^{\kappa}$. If a system is regular in $\overset{\circ}{\xi}{}^{\kappa}$ there exists always an $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa})$ where the system is regular of dimension m in every point.

If a system has in $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa})$ an equivalent subsystem, minimal regular in $\overset{\circ}{\xi}{}^{\kappa}$, it is evident that it is regular in $\overset{\circ}{\xi}{}^{\kappa}$. Conversely, a system regular in $\overset{\circ}{\xi}{}^{\kappa}$ always contains an equivalent subsystem, minimal regular in $\overset{\circ}{\xi}{}^{\kappa}$. The proof will be postponed till we can make use of the first basistheorem (theorem IV).

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1) We adopt here the definition of Kähler (1934.1, p. 12). Other authors, e.g. v.Weber 1900.1, S 48 call regular what we call minimal regular. Our exposition differs from that given by Kähler by the introduction of the notions "semiregular" and "minimal regular" and by its form, which is a little more adapted to geometrical applications

Accordingly a supernumerary regular system differs from a minimal system only by containing some superfluous equations. But it is not always convenient to drop these equations, because it may happen that the remaining system has not an invariant form.

A null form of a manifold, minimal regular of dimension m in the null point $\overset{\circ}{\xi}{}^{\kappa}$ being given, according to the theorem of adaption II it is always possible to choose the coordinates $\overset{\circ}{\xi}{}^{\kappa}$ in such a way that the system takes the form

$$3.7) \qquad \xi^{\wp} = 0 \quad ; \wp = m+1, \cdots, n .$$

The ξ^{α} ; $\alpha = 1, \cdots, m$ can be used as coordinates in the null manifold. Now the pseudo-group $\mathfrak{k}$ consists of all invertible analytic transformations of the ξ^{κ} and this pseudo-group contains the sub-pseudo-group $\mathfrak{k}'$ of all analytic transformations of the ξ^{α} ; $\alpha = 1, \cdots, m$ leaving the $\xi^{\wp}$; $\wp = m+1, \cdots, n$ invariant. Consequently $\mathfrak{k}$ induces into the null manifold the psuedo-group $\mathfrak{k}'$ and this means that this manifold is an X_m. The X_m is said to be <u>imbedded in the</u> X_n. Hence <u>a system regular or minimal regular in</u> $\overset{\circ}{\xi}{}^{\kappa}$ <u>represents an</u> X_m <u>imbedded in</u> X_n <u>in an</u> $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa})$[1].

1) According to our definition an X_m in X_n is always free of singularities. Hence an X_2 in ordinary space can never be a surface with singular points or curves, but it can be a part of such a surface free from singularities.

If an X_m in ... through $\underset{0}{\xi}^\kappa$ is represented by the $n-m$ equations

$$3.8) \qquad C^x(\xi^\kappa) = 0 \quad ; \quad x = m+1, \dots, n$$

the quantity

$$3.9) \qquad C^x_\lambda \overset{def}{=} \partial_\lambda C^x \quad ; \quad x = m+1, \dots, n$$

has the rank $n-m$ in an $\mathfrak{N}(\underset{0}{\xi}^\kappa)$. If another coordinate system (κ') is introduced, (3.8) passes into

$$3.10) \qquad C^x(\xi^{\kappa'}) \overset{def}{=} C^x(f^\kappa(\xi^{\kappa'})) = 0 \; ; \; \begin{matrix} \kappa' = 1', \dots, n' ; \\ x = m+1, \dots, n \end{matrix} \quad 1)$$

If then

$$3.11) \qquad C^x_{\lambda'} \overset{def}{=} \partial_{\lambda'} C^x(\xi^{\kappa'}) \; ; \; \kappa' = 1', \dots, n' ; \; x = m+1, \dots, n$$

we have

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1) We remark that the definition (3.10) is not according to the custom in the theory of functions because the C^x in (3.8) and the C^x in (3.10) stand for different functions. This discrepancy can be avoided by not using the C^x as function symbols and introducing extra function symbols, e.g. $C^x = \varphi^x(\xi^\kappa) = \psi^x(\xi^{\kappa'})$. In fact this must always be done in more complicated cases where ambiguity could arise. But in the simple case here we prefer the shorter notation (3.10).

.12) $C^{x}_{\lambda'} = A^{\lambda}_{\lambda'} C^{x}_{\lambda}$; $A^{\lambda}_{\lambda'} \overset{def}{=} \partial_{\lambda'} \xi^{\lambda}$; $\partial_{\lambda'} \overset{def}{=} \frac{\partial}{\partial \xi^{\lambda'}}$; [1)]

$\lambda' = 1', \cdots, n'$; $x = m+1, \cdots, n$

ne matrix of $A^{\lambda}_{\lambda'}$ having the rank n, it follows
nat $C^{x}_{\lambda'}$ has the rank $n-m$ and that consequent-
y (3.10) is a null form of the X_m, minimal re-
ular in $\xi^{\kappa'}$. Once more we see that the notion
dimension" is really invariant for all allow-
ble coordinate transformations.

The functions $C^{x}(\xi^{\kappa})$ in (3.8) are said
o form a basis of the X_m in ξ^{κ} [2)]. Hence to
very null form, minimal regular in ξ^{κ} there
xists a definite basis. The relation between
ifferent bases are dealt with in the follow-
ng two well-known theorems:

) We shall use the summation convention: if in one term the same index appears twice, once as an upper index and once as a lower index, summation over it has to be effected. (3.12) stands for $C^{x}_{\lambda'} = \sum_{\lambda} A^{\lambda}_{\lambda'} C^{x}_{\lambda}$.

) We define a basis in another way as Kähler, who allows also systems of more than $n-m$ functions. (1934.1, p.13).

Theorem IV (First basis theorem)

If a function $s(\xi^\kappa)$ is analytic in an $\mathfrak{N}(\overset{\circ}{\xi}{}^\kappa)$ and zero in all points of an X_m in $\mathfrak{N}(\overset{\circ}{\xi}{}^\kappa)$ through $\overset{\circ}{\xi}{}^\kappa$ with the null form (3.8), minimal regular in $\overset{\circ}{\xi}{}^\kappa$, there always exists an $\mathfrak{N}(\overset{\circ}{\xi}{}^\kappa)$ such that s satisfies in this $\mathfrak{N}(\overset{\circ}{\xi}{}^\kappa)$ an equation of the form

$$3.13) \qquad s = \varphi_x(\xi^\kappa)\, C^x \quad ; \quad x = m+1, \dots, n$$

where the functions φ_x are analytic in this latter $\mathfrak{N}(\overset{\circ}{\xi}{}^\kappa)$.

Proof.

According to the theorem of adaption III there exists a coordinate system ξ^h ; $h = 1, \dots, m$ such that the equations (3.8) take the form

$$3.14) \qquad \xi^x = 0 \quad ; \quad x = m+1, \dots, n$$

We may choose the ξ^α ; $\alpha = 1, \dots, m$ in such a way that $\overset{\circ}{\xi}{}^h = 0$; $h = 1, \dots, n$. Now s being analytic in $\mathfrak{N}(\overset{\circ}{\xi}{}^h)$ there exists an $\mathfrak{N}(\overset{\circ}{\xi}{}^h)$ where s can be expanded into a convergent power series in the ξ^h. s vanishes in every point of X_m, hence all terms of this series not containing at least one of the variables ξ^x ; $x = m+1, \dots, n$ as a factor must necessarily vanish. Consequently in this latter

$\mathfrak{N}(\overset{\circ}{\xi}{}^{k})$ can be written in the form

$$3.15) \qquad s = \psi_x(\xi^k)\,\xi^x = \varphi_x(\xi^\kappa)\,\mathcal{C}^x(\xi^\kappa)$$

where the φ_x are analytic in an $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa})$.

<u>Theorem V</u> <u>(Second basis theorem)</u>

<u>If both</u> $\mathcal{C}^x$; $x = m+1, \cdots, n$ <u>and</u> $\mathcal{C}^{x'}$; $x' = (m+1)', \cdots, n'$ <u>constitute each a basis in</u> $\overset{\circ}{\xi}{}^{\kappa}$ <u>of an</u> X_m <u>through</u> $\overset{\circ}{\xi}{}^{\kappa}$, <u>there exist</u> $(n-m)$ <u>functions</u> $\mathcal{C}_x^{x'}(\xi^\kappa)$, <u>analytic in an</u> $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa})$ <u>such</u> that in this $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa})$

$$3.16) \qquad \mathcal{C}^{x'} = \mathcal{C}_x^{x'}\,\mathcal{C}^x \;;\; x = m+1, \cdots, n \;;\; x' = (m+1)', \cdots, n'$$

and

$$3.17) \qquad Det(\mathcal{C}_x^{x'}) \neq 0$$

This theorem follows immediately from the first basis theorem. We call the transformations (3.16) <u>basis transformations.</u>

From the second basis theorem V we see that the index x is subject to linear homogeneous transformations with coefficients analytic in an $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa})$, and a non-vanishing determinant. As a consequence of (3.16) we have

3.18) $C_\lambda^{x'} = C_x^{x'} C_\lambda^x + C^x \partial_\lambda C_x^{x'}$; $x = m+1, \cdots, n$; $x' = (m+1)', \cdots, n'$,

hence in all points of X_m

3.19) $C_\lambda^{x'} = C_x^{x'} C_\lambda^x$.

If a coordinate transformation and a basis transformation are effected simultaneously, the transformation of C_λ^x in all points of X_m is

3.20) $C_{\lambda'}^{x'} = A_{\lambda'}^\lambda C_x^{x'} C_\lambda^x$; $x = m+1, \cdots, n$; $x' = (m+1)', \cdots, n'$; $\lambda' = 1', \cdots, n'$.

C_λ^x is called the <u>covariant connecting quantity</u> of the X_m in X_n .

Using the first basis theorem we can now give the proof postponed on p. 14. Be (3.5) a system regular of dimension m in $\overset{\circ}{\xi}{}^\kappa$. It is always possible to choose a subsystem of $n-m$ equations having in $\overset{\circ}{\xi}{}^\kappa$ the rank $n-m$. This subsystem is minimal regular in $\overset{\circ}{\xi}{}^\kappa$ and we will prove that it is equivalent to (3.5) in an $\mathfrak{N}(\overset{\circ}{\xi}{}^\kappa)$. By interchanging the indices α it can always be attained that

3.21) $\mathcal{F}^{\alpha'}(\overset{\circ}{\xi}{}^\kappa) = 0$; $\alpha' = 1, \cdots, n-m$

in the subsystem.

Now let $\mathcal{G}^{\alpha'}(\xi^\kappa)=0$ be a minimal regular system, equivalent to (3.5). Then, in consequence of the first basis theorem the $\mathcal{F}^{\alpha'}(\xi^\kappa)$ can be expressed linearly in the $\mathcal{G}^{\alpha'}$ in an $\mathcal{N}(\overset{\circ}{\xi}{}^\kappa)$

$$3.22) \qquad \mathcal{F}^{\alpha'}(\xi^\kappa)=\mathcal{P}^{\alpha'}_{\beta'}\mathcal{G}^{\beta'}(\xi^\kappa)\ ;\ \alpha',\beta'=1,\cdots,n-m\ .$$

and in this expression $Det(\mathcal{P}^{\alpha'}_{\beta'})\neq 0$ because both $\mathcal{F}^{\alpha'}$ and $\mathcal{G}^{\alpha'}$ have rank $n-m$. Hence the $\mathcal{F}^{\alpha'}$ form a basis for the X_m in an $\mathcal{N}(\overset{\circ}{\xi}{}^\kappa)$ and accordingly the $\mathcal{F}^{\alpha}$ can be expressed linearly in the $\mathcal{F}^{\alpha'}$ in an $\mathcal{N}(\overset{\circ}{\xi}{}^\kappa)$. That proves that the equations $\mathcal{F}^{\alpha'}=0$ form an equivalent subsystem of (3.5).

§ 4. The parametric form of the equations of an X_m in X_n [1].

We consider an X_m with the coordinates η^a, $a=1,\cdots,m$ and a system of n equations

$$4.1) \qquad \xi^\kappa=\mathcal{B}^\kappa(\eta^a)\ ;\ a=1,\cdots,m$$

with functions $\mathcal{B}^\kappa$ analytic in $\overset{\circ}{\eta}{}^a$. If the matrix of

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1) Cf P.P. Ch. II § 4.

$$4.2)\qquad B_b^{\kappa} \stackrel{\text{def}}{=} \partial_b B^{\kappa}\ ;\ b = 1,\cdots,m\ ;\ \partial_b \stackrel{\text{def}}{=} \frac{\partial}{\partial \eta^b}$$

has the rank m in $\overset{\circ}{\eta}{}^a$, the equations (4.1) establish a one-to-one correspondence between the points of X_m in an $\mathfrak{N}(\overset{\circ}{\eta}{}^a)$ and certain points of X_n in an $\mathfrak{N}(\overset{\circ}{\xi}{}^\kappa)$; $\overset{\circ}{\xi}{}^\kappa \stackrel{\text{def}}{=} B^\kappa(\overset{\circ}{\eta}{}^a)$.

Consequently every point of this $\mathfrak{N}(\overset{\circ}{\eta}{}^a)$ can be identified with its corresponding point in $\mathfrak{N}(\overset{\circ}{\xi}{}^\kappa)$. This process we call the imbedding of an X_m into X_n. We call (4.1) a parametric form (also: parametric representation) of the X_m in X_n, minimal regular of dimension m in $\overset{\circ}{\eta}{}^a$ and B_b^κ the contravariant connecting quantity of the X_m in X_n.

This definition of an X_m in X_n is in accordance with the definition of § 3 because (3.7) can be written in the form

$$4.3)\qquad \begin{aligned} \xi^\alpha &= \xi^\alpha \quad ;\ \alpha = 1,\cdots,m\ ;\ \varsigma = m+1,\cdots,n. \\ \xi^\varsigma &= 0 \end{aligned}$$

and the ξ^α can be looked upon as parameters. Then this system has the parametric form.

If in (4.1) the ξ^κ and η^a are transformed simultaneously we get another parametric form

$$4.4)\qquad \xi^{\kappa'} = B^{\kappa'}(\eta^{a'})\ ;\ a' = 1',\cdots,m'\ ;\ \kappa' = 1',\cdots,n'$$

minimal regular of dimension m in $\overset{\circ}{\eta}{}^{a'}$ and the B_b^κ transform in the following way

4.5) $\quad B_{b'}^{\kappa'} = B_{b'}^{b} A_\kappa^{\kappa'} B_b^\kappa \; ; \quad B_{b'}^{b} \overset{def}{=} \partial_b \eta^{b'}$

$b = 1, \dots, m \; ; \; b' = 1', \dots, m' \; ;$
$\kappa' = 1', \dots, n'.$

If (3.8) and (4.1) represent the same X_m in X_n we have

4.6) $\quad C^x(B^\kappa(\eta^a)) = 0$

identical in the η^α and from this, by differentiation, we get

4.7) $\quad B_b^\kappa C_\kappa^x = 0 \quad ; \quad b = 1, \dots, m \; ; \; x = m+1, \dots, n$

identical in the η^a.

As we have seen above it is possible to derive a minimal regular parametric form of an X_m from a minimal regular null form by means of a transformation of coordinates. The conversion is also true. B_b^κ having the rank m in $\overset{\circ}{\eta}{}^a$, by means of interchanging of the indices κ it can always be arranged that the determinant of the B_b^α ; $\alpha = 1, \dots, m$; $b = 1, \dots, m$ does not vanish. Then the transformation

4.8) $$\xi^{\kappa} = B^{\kappa}(\xi^{a}) + \delta^{\kappa}_{x}\xi^{x}\,;\quad a=1,\cdots,m\,;\ x=m+1,\cdots,n,$$ 1)

is an allowable coordinate transformation in an $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa})$. Effecting this transformation we get from (4.1)

4.9)
$$a)\quad B^{\alpha}(\xi^{a}) = B^{\alpha}(\eta^{a})\,;\quad a=1,\cdots,m\,;\ \alpha=1,\cdots,m$$
$$b)\quad B^{\xi}(\xi^{a}) + \delta^{\xi}_{x}\xi^{x} = B^{\xi}(\eta^{a})\,;\quad x=m+1,\cdots,n\,;\ \xi=m+1,\cdots,n\,.$$

The determinant of B^{α}_{b} is $\neq 0$ in $\overset{\circ}{\eta}{}^{a}$. Hence, according to the theorem of inversion I the η^{a} can be solved from (4.9a) as functions of the ξ^{a} in an $\mathfrak{N}(\overset{\circ}{\xi}{}^{a})$. $\eta^{a}=\xi^{a}$ being a solution, this is the only solution for which $\overset{\circ}{\eta}{}^{a}=\overset{\circ}{\xi}{}^{a}$. Hence (4.9b) is equivalent to

4.10) $$\xi^{x} = 0\quad ;\quad x=m+1,\cdots,n$$

---

1) δ^{κ}_{x} is the generalized Kronecker symbol. It stands for + 1 if κ and x have corresponding values e.g. $m+1$ and $m+1$, and for zero in all other cases.

and this system is minimal regular in $\overset{\circ}{\xi}{}^{k}$.

It is also possible to derive a minimal regular parametric form of an X_m in X_n from a minimal regular null form and vice versa without using coordinate transformations. In order to prove the first assertion we need the theorem

<u>Theorem VI</u> (<u>Existence theorem of implicite functions</u>)

<u>If the system</u>

$$4.11) \qquad \mathcal{F}^{\alpha}(\xi^{\kappa}) = 0 \quad ; \quad \alpha = 1, \cdots, N$$

<u>is regular of dimension</u> m <u>in</u> $\overset{\circ}{\xi}{}^{\kappa}$ <u>and if by interchanging the indices</u> α <u>it has been attained that</u>

$$4.12) \qquad \mathcal{F}^{\alpha'}(\xi^{\kappa}) = 0 \quad ; \quad \alpha' = 1, \cdots, n-m$$

<u>is an equivalent subsystem of (4.11) minimal regular in</u> $\overset{\circ}{\xi}{}^{\kappa}$ [1], <u>the indices</u> κ <u>can be interchanged in such a way that the rank of (4.12) with respect to the</u> ξ^{ς} ; $\varsigma = m+1, \cdots, n$ <u>is equal to</u> $n-m$. <u>Then an</u> $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa})$ <u>exists in which the</u> ξ^{ς} can be solved from (4.12)

---

1) That this is always possible was proved above.

4.13) $\quad \xi^{\varsigma} = f^{\varsigma}(\xi^{\alpha})\ ; \quad \alpha = 1, \cdots, m\ ; \quad \varsigma = m+1, \cdots, n$

and for the $\underset{\circ}{\xi}^{\kappa}$ the equations

$$\underset{\circ}{\xi}^{\varsigma} = f^{\varsigma}(\underset{\circ}{\xi}^{\alpha})\ ; \quad \alpha = 1, \cdots, m\ ; \quad \varsigma = m+1, \cdots, n$$

hold[1].

The proof of this theorem will be found in every reliable textbook on analysis.

Now if we complete the system (4.13) with the identities

4.15) $\quad \xi^{\alpha} = \xi^{\alpha}$

we have in fact a parametric form of the X_m, minimal regular in $\underset{\circ}{\xi}^{\alpha}$ representing the same X_m as (4.11).

Conversely, to derive a null form from a parametric form we need the theorem

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1) The theorem of inversion I is a special case of this theorem. In fact, the system (2.1) in the $2n$ variables ξ^{κ}, $\xi^{\kappa'}$ is minimal regular of dimension n in $\underset{\circ}{\xi}^{\kappa}$, $\underset{\circ}{\xi}^{\kappa'}$ and the rank with respect to the $\xi^{\kappa'}$ is $\overset{\circ}{n}$.

Theorem VII (Theorem of elimination[1,2]).

If a system of N equations is given, regular of dimension m in $\underset{\circ}{\xi}^{\kappa}$ and if the rank with respect to $\xi^1, \dots, \xi^M$ in an $\mathcal{N}(\underset{\circ}{\xi}^{\kappa})$ is $R < n-m$, from these equations a system of at most $n-m-R$ equations in $\xi^{M+1}, \dots, \xi^n$ can be derived, valid in an $\mathcal{N}(\underset{\circ}{\xi}^{\kappa})$ and minimal regular in $\underset{\circ}{\xi}^{\kappa}$. The converse is also true.

Proof. The given system be replaced by an equivalent subsystem of $n-m$ equations, minimal regular in $\underset{\circ}{\xi}^{\kappa}$. The differentials of the left hand sides of the other equations being linearly dependent on the differentials of the left hand sides of the chosen equations, the rank of the subsystem with respect to $\xi^1, \dots, \xi^M$ is also R. Consequently, after a suitable interchange of indices the subsystem can be written in the form

$$\text{a)}\quad \mathcal{F}^{\alpha}(\xi^{\kappa}) = 0\,; \quad \mathrm{Det}\left(\partial_{\beta}\mathcal{F}^{\alpha}\right) \neq 0 \text{ in } \mathcal{N}(\underset{\circ}{\xi}^{\kappa})$$

$$4.16)\qquad \text{b)}\quad \mathcal{F}^{\varrho}(\xi^{\kappa}) = 0\,; \quad \alpha, \beta = 1, \dots, R\,; \quad \varrho = R+1, \dots, n-m$$

---

1) Cf. e.g. v. Weber 1900.1, p. 50. Though very important, this theorem is not always stated explicitly in its most general form.

2) A geometric illustration of this theorem will be given in § 7.

According to the existence theorem of implicite functions VI the ξ^{α} can be solved from (4.16a) in an $\mathcal{N}(\underset{\circ}{\xi}{}^{\kappa})$ and consequently (4.16a) can be replaced by the system

$$4.17)\quad G^{\alpha}(\xi^{\kappa}) \overset{def}{=} \xi^{\alpha} - f^{\alpha}(\xi^{\varrho}) = 0 \;;\; \alpha = 1, \cdots, R \;;\; \varrho = R+1, \cdots, n,$$

minimal regular in $\underset{\circ}{\xi}{}^{\kappa}$. Hence (4.16) is equivalent to

$$4.18)\quad \begin{array}{ll} a) & G^{\alpha}(\xi^{\kappa}) = 0 \\ b) & F^{\lambda}(\xi^{\kappa}) = 0 \end{array} \;;\; \alpha = 1, \cdots, R \;;\; \lambda = R+1, \cdots, n-m$$

and this system is minimal regular in $\underset{\circ}{\xi}{}^{\kappa}$. If we replace ξ^{α} by $f^{\alpha}(\xi^{\varrho})$, (4.18b) passes into

$$4.19)\quad G^{\lambda}(\xi^{\varrho}) \overset{def}{=} F^{\lambda}(f^{\alpha}(\xi^{\varrho}), \xi^{\varrho}) = 0 \;;\; \begin{array}{l} \alpha = 1, \cdots, R \\ \lambda = R+1, \cdots, n-m \\ \varrho = R+1, \cdots, n \end{array}$$

and (4.18) into

$$4.20)\quad \begin{array}{lll} a) & G^{\alpha}(\xi^{\kappa}) = 0 & \alpha = 1, \cdots, R \;;\; \lambda = R+1, \cdots, n-m \\ b) & G^{\lambda}(\xi^{\varrho}) = 0 & \varrho = R+1, \cdots, n \end{array}$$

we will prove now that the system (4.20) is minimal regular in $\underset{\circ}{\xi}{}^{\kappa}$. N.a.s. condition is that the matrix

$$4.21) \quad \begin{array}{c|cc} & \xleftrightarrow{\;R\;} & \xleftrightarrow{\;n-R\;} \\ \hline R & \delta^\alpha_\beta & \partial_\varsigma \mathcal{G}^\alpha \\ n-m-R & 0 & \partial_\varsigma \mathcal{G}^{\mathscr{C}} \end{array} \qquad \alpha = 1,\dots,R;\ \mathscr{C} = R+1,\dots,n-m;\ \varsigma = R+1,\dots,n$$

has the rank $n-m$ in $\overset{\circ}{\xi}{}^k$. Hence it is n.a.s. that $\partial_\varsigma \mathcal{G}^{\mathscr{C}}$ has the rank $n-m-R$ in $\overset{\circ}{\xi}{}^k$, in other words, that no equations

$$4.22) \quad \mu_{\mathscr{C}} \partial_\varsigma \mathcal{G}^{\mathscr{C}} = 0 \ ; \quad \mathscr{C} = R+1,\dots,n-m;\ \varsigma = R+1,\dots,n$$

exist with coefficient $\mu_{\mathscr{C}}$ that do not all vanish in $\overset{\circ}{\xi}{}^k$. According to (4.17, 19) the equation (4.22) is equivalent to

$$4.23) \quad -\mu_{\mathscr{C}}(\partial_\alpha \mathcal{F}^{\mathscr{C}})\partial_\varsigma \mathcal{G}^\alpha + \mu_{\mathscr{C}}(\partial_\varsigma \mathcal{F}^{\mathscr{C}}) = 0 ; \quad \begin{array}{l}\alpha = 1,\dots,R;\ \varsigma = R+1,\dots,n \\ \mathscr{C} = R+1,\dots,n-m\end{array}$$

and from (4.20a) follows the identity

$$4.24) \quad -\mu_{\mathscr{C}}(\partial_\alpha \mathcal{F}^{\mathscr{C}})\partial_\beta \mathcal{G}^\alpha + \mu_{\mathscr{C}}(\partial_\beta \mathcal{F}^{\mathscr{C}}) = 0 ; \quad \begin{array}{l}\alpha,\beta = 1,\dots,R; \\ \mathscr{C} = R+1,\dots,n-m\end{array}$$

Now (4.24) and (4.23) together express that the matrix

$$
4.25)\qquad
\begin{array}{c|cc|}
 & \overleftrightarrow{\ R\ } & \overleftrightarrow{\ n-R\ } \\
R & \partial_\beta \mathcal{G}^\alpha & \partial_\varsigma \mathcal{G}^\alpha \\
n-m-R & \partial_\beta \mathcal{F}^\ell & \partial_\varsigma \mathcal{F}^\ell
\end{array}
\qquad
\begin{array}{l}
\alpha = 1, \cdots, R; \\
\ell = R+1, \cdots, n-m; \\
\varsigma = R+1, \cdots, n
\end{array}
$$

has a rank $< n-m$ in $\mathring{\xi}^\kappa$. But this is impossible because (4.18) is minimal regular in $\mathring{\xi}^\kappa$. Hence (4.20) is really minimal regular in $\mathring{\xi}^\kappa$. Consequently $\mathcal{F}^\alpha$, $\mathcal{F}^\ell$ and likewise $\mathcal{G}^\alpha$, $\mathcal{G}^\ell$ constitute a basis of the X_m represented by (4.16), and also by (4.20). According to the second basis theorem every basis can be transformed into every other basis by a linear homogeneous transformation and this implies that the rank R of $\mathcal{F}^\alpha$, $\mathcal{F}^\ell$ with respect to $\xi^1, \cdots, \xi^M$ is the same as the rank of $\mathcal{G}^\alpha$, $\mathcal{G}^\ell$ with respect to these same variables (cf. (3.16, 17)). The matrix of the derivatives of $\mathcal{G}^\alpha$, $\mathcal{G}^\ell$ with respect to $\xi^1, \cdots, \xi^M$ has the form

$$
4.26)\qquad
\begin{array}{c|cccccc|}
 & & \overleftrightarrow{\ R\ } & & & \overleftrightarrow{\ M-R\ } & \\
 & 1 & \cdots & 0 & -\partial_{R+1} f^1 & \cdots & -\partial_M f^1 \\
R & \vdots & & \vdots & \vdots & & \vdots \\
 & 0 & \cdots & 1 & -\partial_{R+1} f^R & \cdots & -\partial_M f^R \\
 & 0 & \cdots & 0 & \partial_{R+1} \mathcal{G}^{R+1} & \cdots & \partial_M \mathcal{G}^{R+1} \\
n-m-R & \vdots & & \vdots & \vdots & & \vdots \\
 & 0 & \cdots & 0 & \partial_{R+1} \mathcal{G}^{n-m} & \cdots & \partial_M \mathcal{G}^{n-m}
\end{array}
$$

Now if one of the derivatives of the $\mathcal{G}^{\varrho}$ with respect to $\xi^{R+1},\dots,\xi^{M}$ were $\neq 0$, this matrix would have a rank $> R$. Consequently all these derivatives have to vanish and that means that the $\mathcal{G}^{\varrho}$ contain only the variables $\xi^{M+1},\dots,\xi^{n}$. Hence the equations (4.20b) do not contain $\xi^{1},\dots,\xi^{M}$ and the rank of this system has to be $n-m-R$ because otherwise the rank (4.21) could not be $n-m$. That proves the first part of the theorem. The conversion is trivial.

Now be

$$4.27)\qquad \xi^{\kappa} = \mathcal{B}^{\kappa}(\eta^{\alpha})$$

a parametric form of an X_m in X_n, minimal regular in $\overset{}{\underset{0}{\eta}}{}^{\alpha}$. The equations

$$4.28)\qquad \xi^{\kappa} - \mathcal{B}^{\kappa}(\eta^{\alpha}) = 0$$

constitute a system in the $n+m$ variables $\xi^{\kappa},\eta^{\alpha}$, minimal regular of dimension m in $\underset{0}{\xi}{}^{\kappa},\underset{0}{\eta}{}^{\alpha}$. The rank of this system with respect to the η^{α} is m. According to the theorem of elimination VII there exists a system of $n-m$ equations in the ξ^{κ} only, minimal regular in $\underset{0}{\xi}{}^{\kappa}$. This system is the null form of the X_m looked for.

If a system of n equations

4.29) $\quad \xi^\kappa = \bar{\mathcal{B}}^\kappa(\eta^\alpha)\,; \quad \alpha = 1,\dots,M$

is given with functions $\bar{\mathcal{B}}^\kappa$ analytic in $\overset{\circ}{\eta}{}^\alpha$ and if the matrix of

4.30) $\quad \bar{\mathcal{B}}^\kappa_\beta \overset{\text{def}}{=} \partial_\beta \bar{\mathcal{B}}^\kappa \quad ; \quad \beta = 1,\dots,M$

has a rank $m < M$ in an $\mathfrak{N}(\overset{\circ}{\eta}{}^\alpha)$ the rank of the system

4.31) $\quad \xi^\kappa - \bar{\mathcal{B}}^\kappa(\eta^\alpha) = 0 \quad ; \quad \alpha = 1,\dots,M$

in the $n+M$ variables ξ^κ, η^α is n , and m is the rank with respect to the variables η^α in an $\mathfrak{N}(\overset{\circ}{\xi}{}^\kappa, \overset{\circ}{\eta}{}^\alpha)$; $\overset{\circ}{\xi}{}^\kappa \overset{\text{def}}{=} \bar{\mathcal{B}}^\kappa(\overset{\circ}{\eta}{}^\alpha)$.

According to the theorem of elimination VII it is possible to derive from (4.31) a system of at most $n-m$ equations in the ξ^κ, minimal regular in $\overset{\circ}{\xi}{}^\kappa$. By a suitable interchange of the indices κ it can be arranged that these equations take the form

4.32) $\quad \xi^\varrho - f^\varrho(\xi^\alpha) = 0\,.$

This is a parametric form of an X_m in X_n , minimal regular in $\overset{\circ}{\xi}{}^\kappa$. Therefore we call a system of the form (4.29) with functions $\bar{\mathcal{B}}^\kappa$ analytic in an

$\mathfrak{R}(\eta^{\alpha})$ and with $\bar{B}_b^{\kappa}$ having the rank $m < M$ in that region a supernumerary regular parametric form of dimension m. If a supernumerary regular parametric form of an X_m is given, a minimal regular parametric form can always be obtained by replacing $M-m$ well-chosen parameters by constants.

§ 5. Irregular systems of equations[1].

In order to deal with irregular systems we need some results of the theory of functions of several variables[2]. If an X_m in an $\mathfrak{R}(\overset{0}{\xi}{}^{\kappa})$ is given by a system

$$5.1)\qquad \mathfrak{F}^{\alpha}(\xi^{\kappa}) = 0 \quad ; \quad \alpha = 1, \dots, n-m$$

we know that this system is minimal regular in all points in $\mathfrak{R}(\overset{0}{\xi}{}^{\kappa})$. If now $\mathfrak{R}(\overset{0}{\xi}{}^{\kappa})$ is enlarged but always within a region where the functions $\mathfrak{F}^{\alpha}$ are analytic, points may appear where the system (5.1) is no longer minimal regular. If $\overset{*}{\xi}{}^{\kappa}$ is such a point and if a system exists equivalent to (5.1) in an $\mathfrak{R}(\overset{*}{\xi}{}^{\kappa})$ and minimal regular

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1) Cf. P.P. Ch. II Exerc. 9 and 10. We give here a more elaborate treatment.

2) Cf. Behnke and Thullen 1934.2.

in $\underset{*}{\xi}^{\kappa}$, $\underset{*}{\xi}^{\kappa}$ may be looked upon as an ordinary point of the enlarged X_m. But if such a system does not exist, $\underset{*}{\xi}^{\kappa}$ is said to be a boundary point of the X_m. An X_m with its boundary points is called a <u>completed</u> X_m [1].

E.g. the system $x^2=0$, $y=0$ in X_2 is not minimal regular in the point $x=0$, $y=0$, but it can be replaced by the minimal regular system $x=0, y=0$. The system $x^2-y^3=0$ in X_2 is minimal regular in all null points except the point $x=0, y=0$ and this point is a boundary point. Each of the two brachnes of the curve forms with this boundary point a completed X_1.

In the theory of functions of several variables the following theorem is proved[2]

<u>Theorem VIII.</u>

<u>If $\underset{\circ}{\xi}^{\kappa}$ is a null point of the equation</u>

$$\mathcal{F}(\xi^{\kappa})=0 \tag{5.2}$$

<u>with a function $\mathcal{F}$ analytic in $\underset{\circ}{\xi}^{\kappa}$, the null points in a sufficiently small $\mathfrak{N}(\underset{\circ}{\xi}^{\kappa})$ coincide with the points of a finite number of completed X_{n-1}'s through $\underset{\circ}{\xi}^{\kappa}$ in $\mathfrak{N}(\underset{\circ}{\xi}^{\kappa})$. $\underset{\circ}{\xi}^{\kappa}$ is either an ordinary point or a boundary point of each of these X_{n-1}'s.</u>

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1) Cf. Behnke and Thullen 1934.2, p.25. Our boundary points are his "uneigentliche wesentliche Randpunkt

2) Cf. Behnke and Thullen 1934.2, p.59.

This theorem may be used to prove the theorem

Theorem IX

If $\overset{\circ}{\xi}{}^{\kappa}$ is either ordinary point or boundary point of a completed X_p and also of a completed X_q; $p \leqq q$ in $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa})$, the common points of X_p and X_q in a sufficiently small $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa})$ coincide with the points of a finite number of completed X_s's ; $s=p+q-n, \cdots, p$ for $p+q-n>0$ and $s=0,\cdots,p$ for $p+q-n \leqq 0$. $\overset{\circ}{\xi}{}^{\kappa}$ is either ordinary point or boundary point of each of these X_s's .

Proof.

Be

$$5.3) \qquad \xi^{\kappa} = f^{\kappa}(\eta^{a}) \quad ; \quad a = 1, \cdots, p$$

a parametric form of X_p with functions f^{κ} analytic in $\overset{\circ}{\eta}{}^{a}$, satisfying the equation

$$5.4) \qquad \overset{\circ}{\xi}{}^{\kappa} = f^{\kappa}(\overset{\circ}{\eta}{}^{a}) \quad ; \quad a = 1, \cdots, p$$

and minimal regular in some point $\underset{1}{\eta}{}^{a}$ of $\mathfrak{N}(\overset{\circ}{\eta}{}^{a})$.

Be

$$5.5) \qquad \mathfrak{F}^{\alpha}(\xi^{\kappa}) = 0 \quad ; \quad \alpha = q+1, \cdots, n$$

a null form of X_q with functions $\mathfrak{F}^{\alpha}$ analytic in $\overset{\circ}{\xi}{}^{\kappa}$ and minimal regular in some null point $\underset{2}{\xi}{}^{\kappa}$

of $\mathfrak{M}(\overset{\circ}{\xi}{}^{\kappa})$. Then the function

$$5.6)\qquad \mathcal{G}^{q+1}(\eta^{a}) \overset{\text{def}}{=} \mathcal{F}^{q+1}(f^{\kappa}(\eta^{a}))$$

is either identically zero in the η^{a} or analytic in $\overset{\circ}{\eta}{}^{a}$. In the latter case according to theorem VIII the null points of $\mathcal{G}^{q+1}$ coincide with the points of a finite number of completed X_{p-1}'s through $\overset{\circ}{\eta}{}^{a}$ in the X_p of the η^{a}. If $\mathcal{G}^{q+1}$ is identically zero we go on with $\mathcal{G}^{q+2}(\eta^{a})$. If $\mathcal{G}^{q+1}(\eta^{a})$ is not identically zero and if

$$5.7)\qquad \eta^{a} = \varphi^{a}(\xi^{a'}) \quad ; \quad a' = 1', \cdots, (p-1)'$$

with

$$5.8)\qquad \overset{\circ}{\eta}{}^{a} = \varphi^{a}(\overset{\circ}{\xi}{}^{a'})$$

is a parametric form of one of the X_{p-1}'s with functions φ^{a} analytic in $\xi^{a'}$ and minimal regular in some point $\overset{3}{\xi}{}^{a'}$ of $\mathfrak{M}(\overset{\circ}{\xi}{}^{a'})$, the function

$$5.9)\qquad \mathcal{H}^{q+2}(\xi^{a'}) \overset{\text{def}}{=} \mathcal{F}^{q+2}\{f^{\kappa}(\varphi^{a}(\xi^{a'}))\}$$

is either identically zero in $\xi^{a'}$ or analytic in $\overset{\circ}{\xi}{}^{a'}$. In the latter case all null points of $\mathcal{H}^{q+2}$ coincide with the points of a finite number of completed X_{p-2}'s through $\overset{\circ}{\xi}{}^{a'}$ in the X_{p-1} chosen. Going on in this way the proof is finished after a finite number of steps.

Now it is possible to prove the theorem

Theorem X

If a system

$$\mathcal{F}^{\alpha}(\xi^{\kappa}) = 0 \quad ; \quad \alpha = 1, \cdots, N \tag{5.10}$$

with functions $\mathcal{F}^{\alpha}$ analytic in the null point $\overset{\circ}{\xi}{}^{\kappa}$, is irregular in $\overset{\circ}{\xi}{}^{\kappa}$, the null points in a sufficiently small $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa})$ coincide with the points of a finite number of completed X_s's; $s = 0, 1, \cdots, n-1$. $\overset{\circ}{\xi}{}^{\kappa}$ is either an ordinary point or a boundary point of each of these X_s's .

Proof. According to theorem VIII every equation of (5.10) represents a finite number of completed X_{n-1}'s through $\overset{\circ}{\xi}{}^{\kappa}$. Hence we get all null points if we choose in all possible ways one of the completed X_{n-1}'s of each equation and determine for each choice the intersections of these N X_{n-1}'s . Repeated application of theorem IX leads then to a finite number of $\mathcal{E}_s$'s ; $s = 0, 1, \cdots, n-1$, each of them containing $\overset{\circ}{\xi}{}^{\kappa}$ either as an ordinary point or as a boundarypoint.

§ 6. The local $\mathcal{E}_n$ [1].

If

$$6.1)\qquad \xi^{\kappa'} = f^{\kappa'}(\xi^{\kappa})$$

is a transformation of $\mathfrak{F}$, by differentiation we get

$$6.2)\qquad d\xi^{\kappa'} = A_{\kappa}^{\kappa'} d\xi^{\kappa} ;\ A_{\kappa}^{\kappa'} \stackrel{def}{=} \partial_{\kappa} \xi^{\kappa'} ;\ Det(A_{\kappa}^{\kappa'}) \neq 0$$

and from this it follows that the pseudo-group $\mathfrak{F}$ induces in every point of X_n the special affine group $\mathcal{G}_{ho}$ (cf. § 2). The centred $\mathcal{E}_n$ of this group we call the <u>local</u> $\mathcal{E}_n$ in the point considered. To every point of X_n there belongs a local $\mathcal{E}_n$. It is usual to identify the centre of the local $\mathcal{E}_n$ with the point of the X_n to which the $\mathcal{E}_n$ belongs, and to call it the <u>contact point.</u> In differential geometry sometimes the $\mathcal{E}_n$ in ξ^{κ} is identified with an "infinitesimal neighbourhood" of ξ^{κ} in the X_n. This is not in any way correct but in a few cases it may have <u>some heuristic value</u> [2].

1) Cf. Schouten and Struik 1935.1, p.65; P.P. Ch.II § 5.

2) If an X_n is imbedded in an $\mathcal{E}_N$, $N > n$, the local $\mathcal{E}_n$ of a point of X_n may be identified with the tangent $\mathcal{E}_n$. But a difficulty arises here. Two tangent $\mathcal{E}_n$'s in different points may intersect and this has to be ignored because the local $\mathcal{E}_n$'s of two different points have nothing in common and are wholly independent.

In every local $\mathcal{E}_n$ quantities may be introduced. They are distinguished by the behaviour of their components under the transformations (6.1). This behaviour is called the manner of transformation. We only mention a few cases, useful hereafter[1].

1. Contravariant vector v^κ

$$v^{\kappa'} = A_\kappa^{\kappa'} v^\kappa \tag{6.3}$$

The geometric image is an arrow in $\mathcal{E}_n$.

2. Covariant vector w_λ

$$w_{\lambda'} = A_{\lambda'}^{\lambda} w_\lambda \tag{6.4}$$

The geometric image is a system of two parallel $\mathcal{E}_{n-1}$'s given in a definite order.

To every coordinate system (κ) in X_n there belong in every point n contravariant and n covariant measuring vectors with the components

$$\underset{\lambda}{e}^{\kappa} \stackrel{*}{=} \delta_\lambda^\kappa \; ; \quad \overset{\kappa}{e}_\lambda \stackrel{*}{=} \delta_\lambda^\kappa \tag{6.5}$$ [2)3)]

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1) Cf. Schouten and Struik 1935.1, p.6; Schouten 1938.2, p.2; Schouten and v.Dantzig 1940.1. Dorgelo and Schouten 1946.1; P.P.Ch.I. § 4, Ch. II § 5; Schouten 1951.1, Ch.II, IV.

2) We use the sign $\stackrel{*}{=}$ to express that the validity

(p.t.o.)

3. Contravariant p -vector $v^{\kappa_1\cdots\kappa_p}$

$$v^{\kappa'_1\cdots\kappa'_p} = A^{\kappa'_1\cdots\kappa'_p}_{\kappa_1\cdots\kappa_p}\, v^{\kappa_1\cdots\kappa_p}$$

This quantity is said to have the valence p and is alternating in all its indices, viz.

6.6) $$v^{\kappa_1\cdots\kappa_p} = v^{[\kappa_1\cdots\kappa_p]}$$ 1)

---

footnote 2) continued:

of an equation is only ascertained for the coordinate system or coordinate systems that are used in the equation. Hence for every equation with $\stackrel{*}{=}$ there may exist coordinate transformations that do not leave the equation invariant.

footnote 3) of preceding page:

δ^κ_λ is called the Kronecker symbol. It is defined as follows:

$$\delta^\kappa_\lambda = \begin{cases} 1 & \text{for } \kappa = \lambda \\ 0 & \text{for } \kappa \neq \lambda \end{cases}$$

δ^κ_λ is to be considered as a set of n^2 scalars, hence it is not transformed at all with coordinate transformations.

1) $v^{[\kappa_1\cdots\kappa_p]}$ is the sum of all terms arising from permutation of the indices, the terms with even permutations of $\kappa_1\cdots\kappa_p$ being provided with a + sign, the terms with odd permutation of $\kappa_1\cdots\kappa_p$ being provided with a - sign, and this divided by the number of terms (viz. p!).

A p-vector is called simple if it is the alternating product of p vectors:

6.7) $$v^{\kappa_1 \dots \kappa_p} = \underset{1}{v}^{[\kappa_1} \dots \underset{p}{v}^{\kappa_p]}$$

N.a.s. conditions for a p-vector to be simple are, as can be proved,

6.8) $$v^{[\kappa_1 \dots \kappa_p}\, v^{\kappa] \dots \lambda_p} = 0$$

The image of a simple contravariant p-vector is a part of an $\mathcal{E}_p$ with a p-dimensional screw sense in it (inner orientation). If (6.8) is valid the $v^{\kappa_1 \dots \kappa_p}$ can be used as homogeneous contravariant Grassmann coordinates of the $\mathcal{E}_p$'s through O. A p-vector has $\binom{n}{p}$ independent components and an $\mathcal{E}_p$ through O can be fixed by $p(n-p)$ numbers. Hence a simple p-vector has $p(n-p)+1$ independent components and among the equations (6.8) there are just $\binom{n}{p}-p(n-p)-1$ independent ones. They will be determined hereafter.

4. Covariant q-vector $w_{\lambda_1 \dots \lambda_q}$

$$w_{\lambda_1' \dots \lambda_q'} = A_{\lambda_1' \dots \lambda_q'}^{\lambda_1 \dots \lambda_q}\, w_{\lambda_1 \dots \lambda_q}$$

All that has been said under (3) about valence, alternating property and simple contravariant p-vector holds mutatis mutandis for covariant

p-vectors. The image of a simple covariant q-vector is an n-dimensional cylinder with an $(n-1)$-dimensional hypersurface consisting of ∞^{q-1} parallel $\mathcal{E}_{n-q}$'s provided with a q-dimensional screw sense around it (outer orientation). The components of simple covariant q-vectors can be used as homogeneous covariant Grassmann coordinates of the $\mathcal{E}_{n-q}$'s through O.

5. Affinor of contravariant valence p and covariant valence q [1], e.g.

$$\mathcal{P}^{\kappa'\lambda'}_{\cdot\ \cdot\ \mu'} = A^{\kappa'\lambda'\mu}_{\kappa\lambda\mu'}\,\mathcal{P}^{\kappa\lambda}_{\cdot\ \cdot\ \mu}\ ;\quad p=2\ ;\ q=1$$

An affinor $\mathcal{P}^{\kappa}_{\cdot\lambda}$ represents a homogeneous linear transformation in the local $\mathcal{E}_n$

$$y^{\kappa} = \mathcal{P}^{\kappa}_{\cdot\lambda}\,x^{\lambda}$$

A special case is the unity affinor defined by

$$A^{\kappa}_{\lambda} \overset{*}{=} \delta^{\kappa}_{\lambda}$$

corresponding to the identical transformation.

If the components of a quantity are given as functions of the ξ^{κ} we get a field. If the quantity is only defined over an X_m in X_n, the X_m is called the field region and m the field dimension.

---

1) p is the number of upper indices;
q the number of lower indices.

A field is said to be analytic in $\overset{\circ}{\xi}{}^{\kappa}$ if its components with respect to any allowable coordinate system (κ) are functions of the ξ^{κ} analytic in $\overset{\circ}{\xi}{}^{\kappa}$. Evidently analyticity of a field is invariant for all allowable coordinate transformations.

If a set of quantities with arbitrary valences is given, each carrying an upper (lower) index κ, all other indices can be replaced by the values belonging to them (f.i. λ by $1, \cdots, n$; λ' by $1', \cdots, n'$ etc.). Then we get a set of N contra (co-) variant vectors and the number of linearly independent ones among them is called the κ-rank of the set.[1] This rank is invariant for all allowable coordinate transformations.

The rank of a quantity with valence 2 is the same with respect to both indices and equal to the rank of the matrix of the components.

Be r the κ-rank of a given set of analytic fields in an $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa})$. If this rank has its maximum value r_m in $\overset{*}{\xi}{}^{\kappa}$ all $(r+1)$-rowed subdeterminants of the matrix of the vectors used for the determination of r vanish in $\overset{*}{\xi}{}^{\kappa}$ and at least one r-rowed subdeterminant is $\neq 0$ in this point. Consequently there exists an $\mathfrak{N}(\overset{*}{\xi}{}^{\kappa})$ where the κ-rank has the value r_m in every point. Such a region is called a region of constant κ-rank. The subdeterminants being analytic, the κ-rank must have the value r_m in every region of constant

1) Cf. Schouten and Struik 1935.1, p. 19; P.P. Ch. I § 5.

κ -rank because an analytic function is identically zero if it is zero in some region however small this region may be. That implies that the points where $\tau < \tau_m$ never fill a region of X_n .

§ 7. Section and reduction[1].

The simplest set of imbedded X_m's in X_n is given by equations of the form

$$7.1) \qquad \xi^\varsigma - c^\varsigma = 0 \qquad ; \varsigma = m+1, \dots, n.$$

with $n-m$ arbitrary parameters c^ς with respect to an arbitrary allowable coordinate system. These X_m's are called coordinate X_m's of (κ). Every set of $\infty^{n-m} X_m$'s in an $\mathfrak{N}(\overset{o}{\xi}{}^\kappa)$ that can be written as coordinate X_m's of some allowable coordinate system is called a normal system of X_m's . Two different X_m's of a normal system never have a point in common and through each point of the region considered there passes just one X_m of the normal system.

The $\infty^{n-m} X_m$'s of a normal system can be considered as points of an $(n-m)$-dimensional manifold. If the system is written in the form (7.1) the $\overset{o}{\xi}{}^\varsigma$ can be used as coordinates in this manifold. The pseudo group $\mathfrak{K}$ induces in this manifold the pseudo group of all invertible analytic transformations of the ξ^ς leaving invariant the ξ^α; $\alpha = 1, \dots, m$.

1) Cf. P.P. Ch.I § 9, II § 4,7.

Hence the manifold is an X_{n-m}. We call the process that leads to this X_{n-m} the <u>reduction of X_n with respect to the given normal system of X_m's</u>.

Now be

7.2) $$C^x(\xi^\kappa)=0 \;;\; x = m+1,\cdots,n$$

a minimal regular null form of an X_m and

7.3) $$\xi^\kappa = B^\kappa(\eta^a) \;;\; a = 1,\cdots,m$$

its minimal regular parametric form. In every point of X_m there exist three local affine spaces

1. the local $\mathcal{E}_n$ of the X_n;
2. the a -space, viz. the local $\mathcal{E}_m$ of the X_m;
3. the x -space, viz. the $\mathcal{E}_{n-m}$ of the index x.

These local spaces are connected by the connecting quantities B_b^κ and C_λ^x, with the relation $B_b^\kappa C_\kappa^x = 0$ 1).

Every contravariant vector v^a of X_m corresponds to one and only one vector v^κ of the X_n:

7.4) $$v^\kappa = v^a B_a^\kappa.$$

---

1) Cf. Schouten 1938.1; Schouten and v.Dantzig 1940.1.

These corresponding vectors span an $\mathcal{E}_m$ in the local $\mathcal{E}_n$ and this $\mathcal{E}_m$ can be identified with the a-space. After this identification we call the $\mathcal{E}_m$ the tangent $\mathcal{E}_m$ of the X_m in the point considered and we look upon v^κ and v^a as two different kinds of components of one and the same vector that may be considered as a vector of X_n and as a vector of X_m as well. This justifies the use of the same kernel letter v in (7.4).

Every covariant vector w_λ of X_n corresponds to one and only one covariant vector $'w_b$ of the X_m:

$$'w_b = w_\lambda B_b^\lambda \tag{7.5}$$

We call $'w_b$ the section of w_λ with the X_m. The section vanishes if and only if the $(n-1)$-direction of w_λ contains the tangent $\mathcal{E}_m$.

Every covariant vector w_y of the x-space corresponds to one and only one vector w_λ of the X_n

$$w_\lambda = w_y C_\lambda^y \tag{7.6}$$

The $(n-1)$-directions of the corresponding vectors contain all the tangent $\mathcal{E}_m$. If the local $\mathcal{E}_n$ is reduced with respect to the normal system of all $\mathcal{E}_m$'s parallel to the tangent $\mathcal{E}_m$ we get an $\mathcal{E}_{n-m}$ and the corresponding vectors pass into covariant vectors of this $\mathcal{E}_{n-m}$. Hence the x-space can be identified with this $\mathcal{E}_{n-m}$. After this identification we call the $\mathcal{E}_{n-m}$ the by- $\mathcal{E}_{n-m}$ of the X_m in the point considered and we look upon w_λ and w_y as two different kinds of components of one and the same vector that may be considered as a vector of X_n and as a vector of the by- $\mathcal{E}_{n-m}$. This justifies the use of the same kernel letter w in (7.6).

Every contravariant vector v^κ of X_n corresponds to one and only one vector v^x of the by-$\mathcal{E}_{n-m}$:

7.7) $$'v^x = C_\kappa^x v^\kappa \quad ; \quad x = m+1, \dots, n$$

We call $'v^x$ the reduction of v^κ with respect to the X_m. The reduction vanishes if and only if v^κ lies in the tangent $\mathcal{E}_m$.

In the same way co- and contravariant p-vectors can be dealt with. We use hereafter the section of a covariant q-vector $w_{\lambda_1 \dots \lambda_q}$ of X_n with an X_m:

7.8) $$'w_{b_1 \dots b_q} = B_{b_1}^{\lambda_1} \dots B_{b_q}^{\lambda_q} w_{\lambda_1 \dots \lambda_q} .$$

The notion of reduction of an X_n can be used to give a geometric illustration of the theorem of elimination VII. If the X_n is reduced with respect to the normal system of X_M's

7.9) $$\xi^{M+1} = const. \; ; \; \dots \; ; \; \xi^n = const.$$

we get an X_{n-M} and every X_m in X_n, having just an X_s in common with each of the X_M's of the normal system, is reduced to an X_{m-s} in this X_{n-M}. Now the equations (4.16) represent an X_m and this X_m is also represented by (4.20). (4.20b) contains only the variables $\xi^{M+1}, \cdots, \xi^n$ and consequently this system represents an X_{m-M+R} in the X_{n+M} of these variables. Hence $s = M-R$ and this means that from the equations of an X_m in X_n just $n-m-R$ equations in $\xi^{M+1}, \cdots, \xi^n$ can be deduced if and only if the X_m has just an X_{M-R} in common with each of the X_M's (7.9) or, in other words, if and only if the X_m reduces to an X_{m-M+R} if the X_n is reduced with respect to the normal system (7.9).

§ 8. Decomposition of a regular system according to Kähler[1].

A subsystem of a regular system need not be regular. If a regular system contains a regular subsystem the following theorem holds:[2]

Theorem XI (Theorem of decomposition of regular systems)

Let a system of N equations

8.1) $$\mathfrak{F}^{\alpha}(\xi^{\kappa}) = 0 \quad ; \alpha = 1, \cdots, N$$

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1) Cf P.P. Ch. II § 8.

2) Kähler, 1934.1, p.30 uses a part of this theorem without proof.

with functions $\mathcal{F}^{\alpha}$ analytic in the null point $\overset{0}{\xi}{}^{\kappa}$, contain a subsystem of N' equations, regular of dimension $m' \geq n - N'$ in $\overset{0}{\xi}{}^{\kappa}$.
If, using the existence theorem of implicit functions VI, we solve $n-m'$ of the ξ^{κ} from these N' equations as functions of the other m' (here called ξ^{α}) and if these solutions are introduced into the remaining $N-N'$ equations of (8.1), the arising $N-N'$ equations in the ξ^{α} constitute a regular system of dimension m in $\overset{0}{\xi}{}^{\alpha}$ if and only if the system (8.1) is regular of dimension m in $\overset{0}{\xi}{}^{\kappa}$.

Proof. By interchanging the indices α it can always be arranged that the equations

$$8.2) \qquad \mathcal{F}^{\alpha'}(\xi^{\kappa}) = 0 \quad ; \; \alpha' = 1, \cdots, N'$$

constitute the regular subsystem of dimension m' and

$$8.3) \qquad \mathcal{F}^{\alpha''}(\xi^{\kappa}) = 0 \quad ; \; \alpha'' = 1, \cdots, n-m'$$

a system equivalent to (8.2) and minimal regular in $\overset{0}{\xi}{}^{\kappa}$. By interchanging the indices κ it can always be arranged that the ξ^{ς}; $\varsigma = m'+1, \cdots, n$ can be solved from (8.3) as functions of the ξ^{α}; $\alpha = 1, \cdots, m'$:

8.4) $$G^{\varsigma}(\xi^{\kappa}) \overset{def}{=} \xi^{\varsigma} - f^{\varsigma}(\xi^{\alpha}) = 0 \; ; \; \alpha = 1, \dots, m' \; ; \; \varsigma = m'+1, \dots, n$$

(8.3) and also (8.4) represents an $X_{m'}$ in X_n in which the ξ^α can be used as coordinates. Substituting the solutions (8.4) into (8.1) we get a system of the form

8.5) $$\mathcal{H}^{\mathfrak{a}}(\xi^{\alpha}) \overset{def}{=} \mathcal{F}^{\mathfrak{a}}(\xi^{\alpha}, f^{\varsigma}(\xi^{\alpha})) = 0 \; ; \; \alpha = 1, \dots, m' \; ; \; \varsigma = m'+1, \dots, n \; ; \; \mathfrak{a} = 1, \dots, N$$

that reduces itself to a system of $N - N'$ equations

8.6) $$\mathcal{H}^{\mathfrak{e}'}(\xi^{\alpha}) = 0 \; ; \; \mathfrak{e}' = N'+1, \dots, N \; ; \; \alpha = 1, \dots, m'$$

because the $\mathcal{H}^{\mathfrak{a}'}$; $\mathfrak{a}' = 1, \dots, N'$ vanish identically in the ξ^α.

By differentiation of (8.5) we get

8.7) $$\partial_\beta \mathcal{H}^{\mathfrak{a}} = \partial_\beta \mathcal{F}^{\mathfrak{a}} + (\partial_\varsigma \mathcal{F}^{\mathfrak{a}}) \partial_\beta f^{\varsigma} = (\partial_\lambda \mathcal{F}^{\mathfrak{a}})(A^{\lambda}_{\beta} + A^{\lambda}_{\varsigma} \partial_\beta f^{\varsigma})$$

Looking upon (8.4) as a parametric form of the $X_{m'}$ with the parameters ξ^α, the contravariant connecting quantity B^{κ}_{β} is

8.8) $$\begin{aligned} B^{\alpha}_{\beta} &\overset{*}{=} A^{\alpha}_{\beta} \\ B^{\varsigma}_{\beta} &\overset{*}{=} \partial_\beta f^{\varsigma} \end{aligned} \; ; \; \alpha, \beta = 1, \dots, m' \; ; \; \varsigma = m'+1, \dots, n$$

Hence (8.7) can be written

$$8.9) \quad \partial_\beta \mathcal{H}^{\alpha} = B_\beta^\lambda \, \partial_\lambda \mathcal{F}^{\alpha} \; ; \beta = 1, \cdots m' ; \; \alpha = 1, \cdots, N$$

and this equation expresses <u>that for every value of α the covariant vector $\partial_\beta \mathcal{H}^{\alpha}$ is the section of the covariant vector $\partial_\lambda \mathcal{F}^{\alpha}$ with the $X_{m'}$.</u>

Be $n-s$ the rank of $\partial_\lambda \mathcal{F}^{\alpha}$ in $\overset{\circ}{\xi}{}^{\kappa}$. Then in this point the N covariant vectors $\partial_\lambda \mathcal{F}^{\alpha}$; $\alpha = 1, \cdots, N$ span an s-direction and this s-direction has to be contained in the tangent $\mathcal{E}_{m'}$ of the $X_{m'}$ represented by (8.4) because (8.2) is a subsystem of (8.1). From this special position of the s-direction it follows that it also must be spanned by the N sections $\partial_\beta \mathcal{H}^{\alpha}$ of the $\partial_\lambda \mathcal{F}^{\alpha}$ with the $X_{m'}$. Hence the rank of $\partial_\beta \mathcal{H}^{\alpha}$ in $\overset{\circ}{\xi}{}^{\alpha}$ has to be $m'-s$. But the $\partial_\beta \mathcal{H}^{\alpha'}$ being zero the rank of $\partial_\beta \mathcal{H}^{\kappa'}$ in $\overset{\circ}{\xi}{}^{\alpha}$ is also $m'-s$.

Now suppose that (8.1) be regular of dimension m in $\overset{\circ}{\xi}{}^{\kappa}$. Then (8.1) represents an X_m and $s = m$. This X_m is represented in the $X_{m'}$ by (8.6). The rank of $\partial_\beta \mathcal{H}^{\alpha}$ in $\overset{\circ}{\xi}{}^{\alpha}$ is $m'-m$ and consequently (8.6) is regular of dimension m in $\overset{\circ}{\xi}{}^{\alpha}$.

Conversely, if (8.6) is regular of dimension m in $\overset{\circ}{\xi}{}^{\alpha}$, $\partial_\beta \mathcal{H}^{\kappa'}$ has the rank $m'-m$ in $\overset{\circ}{\xi}{}^{\alpha}$. Hence $s = m$. (8.6) represents an X_m in $X_{m'}$ and this same X_m is represented in X_n by (8.2,6) or by

(8.1), equivalent to (8.2,6). The rank of $\partial_\lambda \mathcal{F}^{\alpha}$ is $n-m$ and consequently (8.1) is regular in $\overset{\circ}{\xi}{}^\kappa$.

§ 9. Supernumerary coordinates[1].

The ξ^κ can be used as supernumerary coordinates in an X_m in X_n [2]. Then there exist $n-m$ independent relations between the ξ^κ, viz. the equations of a minimal regular null form of the X_m and the points of the X_m are in one-to-one correspondence to the sets of values ξ^κ satisfying these relations.

The homogeneous coordinates in ordinary projective geometry represent another kind of supernumerary coordinates. There exist no relations between these coordinates. Every set of values corresponds to one point but every point corresponds to ∞^1 set of values.

An $\mathcal{E}_m$ through the origin in a centred $\mathcal{E}_n$ can be fixed by its contravariant Grassmann coordinates $v^{\kappa_1 \cdots \kappa_m}$ and also by its covariant Grassmann coordinates $w_{\lambda_1 \cdots \lambda_{n-m}}$.(cf. § 6). Between these coordinates the relations

(9.1) $\quad v^{[\kappa_1 \cdots \kappa_m} v^{\lambda_1] \cdots \lambda_m} = 0 \;;\quad w_{[\lambda_1 \cdots \lambda_m} w_{\kappa_1] \cdots \kappa_m} = 0$

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1) Cf. Schouten and v.Dantzig 1935.2, p.33; P.P. Ch. II § 9.

2) In Schouten 1924.1 this method is frequently used.

exist and to every $\mathcal{E}_m$ there correspond ∞' sets of values.

From these examples we see that there exist different kinds of supernumerary coordinates. Though supernumerary coordinates are very frequently used, a general theory of them does not seem to exist. We need such a theory to answer the important question whether the regularity of a system of equations remains invariant if supernumerary coordinates are introduced. (§ 10)

The most general supernumerary coordinates in an $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa})$ of an X_n are defined by the equations

$$9.2)\quad \begin{array}{ll} a) & \xi^{\kappa} = \varphi^{\kappa}(\eta^{a}) \quad ; \; a = 1, \cdots, n+\varepsilon_1+\varepsilon_2 \; ; \; \varepsilon_1 \geqq 0 \; ; \; \varepsilon_2 \geqq 0 \; ; \\ & \qquad\qquad\qquad\qquad \varepsilon_1 + \varepsilon_2 \geqq 1 \; ; \; \mathfrak{r} = 1, \cdots, N' \\ b) & \psi^{\mathfrak{r}}(\eta^{a}) = 0 \end{array}$$

subject to the conditions

1. There exists a set of solutions $\overset{\circ}{\eta}{}^{a}$ of the equations

$$9.3)\quad \begin{array}{ll} \overset{\circ}{\xi}{}^{\kappa} = \varphi^{\kappa}(\overset{\circ}{\eta}{}^{a}) & ; \; a = 1, \cdots, n+\varepsilon_1+\varepsilon_2 \; ; \\ \psi^{\mathfrak{r}}(\overset{\circ}{\eta}{}^{a}) = 0 & \quad \mathfrak{r} = 1, \cdots, N' \end{array}$$

   such that the φ^{κ} are analytic in $\mathfrak{N}(\overset{\circ}{\eta}{}^{a})$ and the rank of $\partial_b \varphi^{\kappa}$ is n in that region;

2. The system (9.2b) is regular of dimension $n+\varepsilon_2$ in $\overset{\circ}{\eta}{}^{a}$ (hence it represents an $X_{n+\varepsilon_2}$, in the $X_{n+\varepsilon_1+\varepsilon_2}$ of the η^{a});

3. Among the $n+N'$ differentials $d\varphi^k$, $d\psi^{\mathscr{C}}$ there exist exactly $n+\varepsilon_1$ linearly independent ones in every point of an $\mathfrak{N}(\overset{\circ}{\eta}{}^a)$.

If $\varepsilon_2 = 0$, the $\overset{\circ}{\eta}{}^a$ are uniquely determined by (9.3). This is not true if $\varepsilon_2 \neq 0$. But if in this latter case $\overset{*}{\eta}{}^a$ is another solution of (9.3) in a sufficiently small $\mathfrak{N}(\overset{\circ}{\eta}{}^a)$, it follows from the form of our conditions that they hold for $\overset{*}{\eta}{}^a$ as well. Hence $\overset{\circ}{\eta}{}^a$ is in no way preferred.

Every set of values η^a satisfying (9.2b) corresponds to one and only one point of X_n but conversely every point of X_n corresponds to ∞^{ε_2} ($\infty^0 \overset{def}{=} 1$) sets of values η^a satisfying (9.2b).

The ξ^k, looked upon as coordinates in an X_m in X_n are supernumerary coordinates with $\varepsilon_1 = n-m$; $\varepsilon_2 = 0$. The $n+1$ projective coordinates in ordinary n-dimensional projective geometry are supernumerary coordinates with $\varepsilon_1 = 0$; $\varepsilon_2 = 1$. For the $v^{\kappa_1 \cdots \kappa_m}$ as coordinates of the $\mathcal{E}_m$'s through the origin in a centred $\mathcal{E}_n$ we have $\varepsilon_1 = \binom{n}{m} - m(n-m) - 1$; $\varepsilon_2 = 1$.

In order to prove that these latter coordinates satisfy the conditions 1-3 we introduce non supernumerary coordinates in the $X_{m(n-m)}$ of all $\mathcal{E}_m$'s not having a direction in common with the coordinate-$\mathcal{E}_{n-m}$ of the (affine) coordinates $\xi^{m+1}, \cdots, \xi^n$

in $\mathcal{E}_n$. For all these $\mathcal{E}_m$'s $v^{1\cdots m} \neq 0$ and $\xi^1, \cdots, \xi^m$ can be used as coordinates in every one of them. If this has been done the contravariant connecting quantity

$$9.4) \qquad B^{\kappa}_{\beta} \overset{*}{=} \partial \xi^{\kappa} / \partial \xi^{\beta} \quad ; \quad \beta = 1, \cdots, m$$

satisfies the equations

$$9.5) \qquad B^{\alpha}_{\beta} \overset{*}{=} \delta^{\alpha}_{\beta} \qquad ; \quad \alpha, \beta = 1, \cdots, m$$

and the B^{ς}_{β} ; $\beta = 1, \cdots, m$; $\varsigma = m+1, \cdots, n$ are the non supernumerary coordinates looked for. The $\mathcal{E}_m$ being spanned by the m contravariant vectors $B^{\kappa}_1, \cdots, B^{\kappa}_m$ we have

$$9.6) \qquad m!\, B^{[\kappa_1}_1 \cdots B^{\kappa_m]}_m \overset{*}{=} \frac{v^{\kappa_1 \cdots \kappa_m}}{v^{1 \cdots n}}$$

and consequently

$$9.7) \qquad B^{\varsigma}_{\beta} \overset{*}{=} \frac{v^{1 \cdots \beta-1\, \varsigma\, \beta+1 \cdots m}}{v^{1 \cdots m}} \qquad ; \quad \beta = 1, \cdots, m \, ; \quad \varsigma = m+1, \cdots, n$$

and

$$9.8) \qquad v^{\varsigma_1 \cdots \varsigma_s \alpha_{s+1} \cdots \alpha_m} \overset{*}{=} s!\, B^{[\varsigma_1}_{\alpha_1} \cdots B^{\varsigma_s]}_{\alpha_s} v^{1 \cdots m}$$

$$\varsigma_1, \cdots, \varsigma_s = m+1, \cdots, n \, ;$$

$\alpha_1, \dots, \alpha_m =$ even permutation of $1, \dots, m$; $s = 1, \dots, m$

by means of which all components of $v^{k_1 \dots k_p}$ not occuring in (9.7) can be expressed in those occuring in (9.7). To this end (9.7) has to be substituted into (9.8). Then all equations whose left hand sides contain a component occuring in (9.7) are identically satisfied and the other $\binom{n}{m} - m(n-m) - 1$ equations form a system minimal regular in all its null points because every one contains a variable not occuring in the other equations. The system obtained in this way is equivalent to

$$9.9) \qquad v^{[k_1 \dots k_m} v^{\lambda_1]\dots\lambda_n} = 0$$

and represents an $X_{m(n-m)+1}$ in the $X_{\binom{n}{m}}$ of all $v^{k_1 \dots k_m}$ with $v^{1 \dots m} \neq 0$. That implies that (9.9) is either regular or semiregular. In order to prove that (9.9) is regular in all its null points it has to be proved that the rank of (9.9) is $\binom{n}{m} - m(n-m) - 1$ in all null points. Among the equations (9.9) the following equations occur:

$$v^{[1\cdots m}v^{\xi_1]\alpha_2\cdots\alpha_m}=0$$

$$v^{[1\cdots m}v^{\xi_1]\xi_2\alpha_3\cdots\alpha_m}=0$$

9.10) $\vdots$

$$\begin{cases} v^{[1\cdots m}v^{\xi_1]\xi_2\cdots\xi_{n-m}\alpha_{n-m+1}\cdots\alpha_m}=0 & \text{for } n-m<m, \\ v^{[1\cdots m}v^{\xi_1]\xi_2\cdots\xi_m}=0 & \text{for } n-m\geqq m \end{cases}$$

The first set of these equations is identically satisfied, e.g.:

$$(m+1)v^{[1\cdots m}v^{\xi_1]2\cdots m}=v^{1\cdots m}v^{\xi_1 2\cdots m}-v^{\xi_1 2\cdots m}v^{1\cdots m}=0$$

The second set contains just $\binom{n-m}{2}\binom{m}{m-2}$ equations and each of them contains just one of the components $v^{\xi_1\xi_2\alpha_3\cdots\alpha_m}$ not occuring in one of the others and no components with more than two indices ξ. Accordingly the differentials of the left hand sides of the equations of this set are linearly independent. The third set contains just $\binom{n-m}{3}\binom{m}{m-3}$ equations and each of them contains just one of the components $v^{\xi_1\xi_2\xi_3\alpha_4\cdots\alpha_m}$ not occuring in one of the others and no components with more than three indices ξ. Hence the differentials of the left hand sides of the equations of the second and third set are linearly independent. Proceeding in this way we get at last $\binom{n}{m}-m(n-m)-1$ equations with left hand sides whose differentials are linearly independent. That implies that the rank of the system (9.9) is $\binom{n}{m} - m(n-m) - 1$ and that consequently (9.9) is regular of dimension $m(n-m)+1$ in all its null points. The condition $v^{1\cdots m} \neq 0$ drops out because (9.9) is invariant for all affine transformations of coordinates.

If now the $v^{\kappa_1\cdots\kappa_m}$ are looked upon as supernumerary coordinates in the $X_{m(n-m)}$ of all $\mathcal{E}_m$'s through the origin and the B_β^ξ as ordinary coordinates in the same $X_{m(n-m)}$, (9.7), (9.9) play the role of (9.2a), (9.2b), further $m(n-m)$ the role of n and $\binom{n}{m}$ the role of $n+\varepsilon_1+\varepsilon_2$, From (9.7) we see that the first condition is satisfied for $v^{1\cdots m}\neq 0$. (9.9) being regular of dimension $m(n-m)+1$ in all its null points, the second condition is satisfied with $\varepsilon_2 = 1$. The third condition is satisfied with $n+\varepsilon_1 = \binom{n}{m} - 1$ because, the right hand sides of (9.7) not containing any components with more than one index ξ, their differentials have to be independent of the differentials of the left hand sides of (9.10).

Not only the $v^{\kappa_1\cdots\kappa_p}$ but also the $B_\beta^\kappa;\ \beta=1,\ldots,m$ are supernumerary coordinates of the same $X_{m(n-m)}$ with $\varepsilon_1 = m^2$; $\varepsilon_2 = 0$ and the same holds for the mn components of m arbitrary vectors $\underset{1}{v}^\kappa,\ldots,\underset{m}{v}^\kappa$ spanning the $\mathcal{E}_m$ with $\varepsilon_1 = 0$; $\varepsilon_2 = m^2$. The condition $v^{1\cdots m}\neq 0$ drops out here.

Here is still another example. Be a normal system of X_m's in X_n given by the equations

$$\mathcal{F}^{\alpha}(\xi^{\kappa}) - c^{\alpha} = 0 \quad ; \quad \alpha = 1, \cdots, N$$

with N parameters c^{α} and functions $\mathcal{F}^{\alpha}$ analytic in $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa})$, chosen in such a way that there are just $n-m$ functionally independent ones among them. If the X_n is reduced with respect to this normal system, an X_{n-m} arises and in this X_{n-m} the c^{α} are supernumerary coordinates with $\varepsilon_1 = N-n+m$; $\varepsilon_2 = 0$. In the same X_{n-m} the ξ^{κ} can be used as supernumerary coordinates with $\varepsilon_1 = 0$; $\varepsilon_2 = m$.

§ 10. Invariance of regularity if supernumerary coordinates are introduced[1].

We will prove the theorem

Theorem XII

A system of N equations

10.1) $$\mathcal{F}^{\alpha}(\xi^{\kappa}) = 0 \quad ; \quad \alpha = 1, \cdots, N$$

with functions $\mathcal{F}^{\alpha}$ analytic in the null point $\overset{\circ}{\xi}{}^{\kappa}$ be given.

By means of the equations

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1) P.P. Ch. II § 9.

$$\xi^k = \varphi^k(\eta^a)\ ;\ \underset{\circ}{\xi}^k = \varphi^k(\underset{\circ}{\eta}^a)\ ;\ a = 1,\cdots,n+\varepsilon_1+\varepsilon_2\ ;$$

10.2)
$$\varepsilon_1 \geqq 0\ ;\ \varepsilon_2 \geqq 0\ ;\ \varepsilon_1+\varepsilon_2 \geqq 1\ ;$$
$$\psi^{\mathcal{C}}(\eta^a) = 0\ ;\ \psi^{\mathcal{C}}(\underset{\circ}{\eta}^a) = 0\ ;\ \mathcal{C} = 1,\cdots,N'$$

satisfying the conditions of § 9 a system of supernumerary coordinates η^a is introduced. Then the system (10.1) is regular of dimension m in $\underset{\circ}{\xi}^k$ if amd omly if the system

a) $$\mathcal{G}^{\alpha}(\eta^a) \overset{def}{=} \mathcal{F}^{\alpha}(\varphi^k(\eta^a)) = 0\ ;\ a = 1,\cdots,n+\varepsilon_1+\varepsilon_2\ ;$$

10.3)
$$\alpha = 1,\cdots,N\ ;$$

b) $$\psi^{\mathcal{C}}(\eta^a) = 0 \qquad \mathcal{C} = 1,\cdots,N'$$

is regular of dimension $m+\varepsilon_2$, in $\underset{\circ}{\eta}^a$ and this is the case if and only if the equations (10.3a) constite by themselves a system regular of dimension $m+\varepsilon_1+\varepsilon_2$ in $\underset{\circ}{\eta}^a$.

Proof. First we prove the second part of the theorem. Be (10.3) regular of dimension $m+\varepsilon_2$ in $\underset{\circ}{\eta}^a$. According to our conditions (10.3b) is regular of dimension $n+\varepsilon_2$ in $\underset{\circ}{\eta}^a$. Hence, by interchanging the indices $\mathcal{C}$ it can always be arranged that the equations

10.4) $$\psi^{\mathcal{C}'}(\eta^a) = 0\ ;\ \mathcal{C}' = 1,\cdots,\varepsilon_1\ ;\ a = 1,\cdots,n+\varepsilon_1+\varepsilon_2$$

form an equivalent subsystem of (10.3b), minimal regular in $\overset{}{\underset{0}{\eta}}{}^{a}$. (10.3) being regular in $\underset{0}{\eta}{}^{a}$ from (10.3a) a subsystem can be chosen constituting together with (10.4) an equivalent subsystem of (10.3), minimal regular in $\underset{0}{\eta}{}^{a}$. By interchanging the indices α it can always be arranged that this subsystem of (10.3a) is

10.5) $$\mathcal{G}^{\alpha'}(\eta^a)=0\ ;\ \alpha'=1,\cdots,n-m\ ;\ a=1,\cdots,n+\varepsilon_1+\varepsilon_2$$

Every subsystem of a minimal regular system being minimal regular, the system (10.5) is minimal regular in $\underset{0}{\eta}{}^{a}$. (10.4) and (10.5) being together equivalent to (10.3), they are also equivalent to the combination of (10.3b) and (10.5). Hence, if the remaining equations of (10.3a) are

10.6) $$\mathcal{G}^{\alpha''}(\eta^a)=0\ ;\ \alpha''=n-m+1,\cdots,N\ ;\ a=1,\cdots,n+\varepsilon_1+\varepsilon_2$$

from the first basis theorem it follows that there has to exist in $\mathcal{R}(\underset{0}{\eta}{}^{a})$ a relation of the form

10.7) $$\mathcal{G}^{\alpha''}=\alpha^{\alpha''}_{\alpha'}\mathcal{G}^{\alpha'}+\beta^{\alpha''}_{\varkappa'}\psi^{\varkappa'}\ ;\quad \begin{array}{l}\alpha'=1,\cdots,n-m\ ;\\ \alpha''=n-m+1,\cdots,N\ ;\\ \varkappa'=1,\cdots,\varepsilon_1\end{array}$$

Now, according to theorem VI, $n+\varepsilon_1$ of the η^a can be solved from (9.2) as functions of the ξ^k and the remaining η^a. Hence by interchanging the indices α it can be arranged that in some $\mathfrak{N}(\overset{o}{\xi}{}^\kappa; \overset{o}{\eta}{}^\alpha)$

$$10.8)\quad \eta^\alpha = \mathcal{H}^\alpha(\xi^\kappa, \eta^\theta)\ ;\ \alpha = 1, \cdots, n+\varepsilon_1\ ;\ \theta = n+\varepsilon_1+1, \cdots, n+\varepsilon_1+\varepsilon_2$$

If (10.8) is substituted into (10.7) we get equations of the form

$$10.9)\quad \mathcal{F}^{\alpha''}(\xi^\kappa) = \bar{\alpha}^{\alpha''}_{\alpha'}(\xi^\kappa, \eta^\theta)\, \mathcal{F}^{\alpha'}(\xi^\kappa)\ ;$$

$$\alpha' = 1, \cdots, n-m\ ;\ \alpha'' = n-m+1, \cdots, N\ ;\ \theta = n+\varepsilon_1+1, \cdots, n+\varepsilon_1+\varepsilon_2$$

and if in these equations the ξ^κ are replaced by $\varphi^\kappa(\eta^a)$ we get equations of the form (10.7) with vanishing $\beta^{\alpha''}_{\alpha'}$. Hence the system (10.3a) contains an equivalent subsystem minimal regular of dimension $m+\varepsilon_1+\varepsilon_2$ in $\overset{o}{\eta}{}^\alpha$, viz. (10.5), and this implies that (10.3a) is minimal regular of dimension $m+\varepsilon_1+\varepsilon_2$ in $\overset{o}{\eta}{}^{\alpha}$.

Conversely, suppose that (10.3a) be regular of dimension $m+\varepsilon_1+\varepsilon_2$ in $\overset{o}{\eta}{}^\alpha$. Then by interchanging the indices α it can always be arranged that (10.5) is an equivalent subsystem of (10.3a) minimal regular of dimension $m+\varepsilon_1+\varepsilon_2$ in $\overset{o}{\eta}{}^\alpha$. The system (10.3) is then equivalent to its subsystem consisting of (10.4) and (10.5), both systems being minimal regular in $\overset{o}{\eta}{}^a$. So we have only to show that the whole system (10.4,5) is minimal regular in $\overset{o}{\eta}{}^a$, i.e.

[illegible]

that its rank in $\underset{0}{\eta}^a$ is $n-m+\varepsilon_1$. This rank can not be $> n-m+\varepsilon_1$ because the total number of equations is $n-m+\varepsilon_1$. If the rank were $< n-m+\varepsilon_1$ in $\underset{0}{\eta}^a$ there had to exist a relation of the form

$$10.10) \quad \alpha_{\alpha'} \partial_{\mathfrak{b}} \mathcal{G}^{\alpha'} + \beta_{\mathcal{C}'} \partial_{\mathfrak{b}} \psi^{\mathcal{C}'} = 0 \quad ; \quad \begin{array}{l} \alpha' = 1, \dots, n-m; \\ \mathcal{C}' = 1, \dots, \varepsilon_1; \\ \mathfrak{b} = 1, \dots, n+\varepsilon_1+\varepsilon_2 \end{array}$$

with coefficients of which neither all α's nor all β's could vanish simultaneously. From (10.10) it would follow in $\underset{0}{\eta}^a$

$$10.11) \quad \alpha_{\alpha'} (\partial_\kappa \mathcal{F}^{\alpha'}) \partial_{\mathfrak{b}} \varphi^\kappa + \beta_{\mathcal{C}'} \partial_{\mathfrak{b}} \psi^{\mathcal{C}'} = 0 \, ; \quad \begin{array}{l} \alpha' = 1, \dots, n-m; \\ \mathcal{C}' = 1, \dots, \varepsilon_1; \\ \mathfrak{b} = 1, \dots, n+\varepsilon_1+\varepsilon_2 \end{array}$$

but a relation of this form can not exist because the differentials $d\varphi^\kappa$ and $d\psi^{\mathcal{C}'}$ are linearly independent in $\underset{0}{\eta}^a$. Hence (10.4,5) is minimal regular in $\underset{0}{\eta}^a$ and this implies that (10.3) is regular of dimension $m+\varepsilon_2$ in $\underset{0}{\eta}^a$.

To prove the first part of the theorem we suppose first that (10.1) be regular of dimension m in $\underset{0}{\xi}^\kappa$. Then by interchanging the indices α it can always be arranged that

$$10.12 \quad \mathcal{F}^{\alpha'}(\underset{0}{\xi}^\kappa) = 0 \quad ; \quad \alpha' = 1, \dots, n-m$$

is an equivalent subsystem of (10.1) minimal regular of dimension m in $\overset{o}{\xi}{}^{\kappa}$. From (10.3a) we consider the equations corresponding to (10.12):

$$10.13) \qquad \mathcal{G}^{\alpha'}(\eta^{a}) = 0 \quad ; \quad \begin{array}{l} \alpha' = 1, \cdots, n-m \quad ; \\ a = 1, \cdots, n+\varepsilon_1+\varepsilon_2 \end{array}$$

Now we have

$$10.14) \qquad \partial_b \mathcal{G}^{\alpha'} = (\partial_\kappa \mathcal{F}^{\alpha'}) \partial_b \varphi^\kappa \quad ; \quad \begin{array}{l} \alpha' = 1, \cdots, n-m \, ; \\ b = 1, \cdots, n+\varepsilon_1+\varepsilon_2 \, . \end{array}$$

Besides $\partial_\kappa \mathcal{F}^{\alpha'}$ have the rank $n-m$ in all points of an $\mathfrak{U}(\overset{o}{\xi}{}^{\kappa})$ and $\partial_b \varphi^\kappa$ has the rank n in all points of an $\mathfrak{U}(\overset{o}{\eta}{}^{a})$. Hence $\partial_b \mathcal{G}^{\alpha'}$ must have the rank $n-m$ in all points of an $\mathfrak{U}(\overset{o}{\eta}{}^{a})$. That implies that (10.13) is minimal regular in $\overset{o}{\eta}{}^{a}$. The remaining equations of (10.1)

$$10.15) \qquad \mathcal{F}^{\alpha''}(\xi^{\kappa}) = 0 \quad ; \quad \alpha'' = n-m+1, \cdots, N$$

are dependent on (10.12). Therefore, if the ξ^κ in (10.15) are replaced by $\varphi^\kappa(\eta^a)$ the resulting equations

$$10.16) \qquad \mathcal{G}^{\alpha''}(\eta^{a}) = 0 \quad ; \quad \begin{array}{l} \alpha'' = n-m+1, \cdots, N \\ a = 1, \cdots, n+\varepsilon_1+\varepsilon_2 \end{array}$$

depend on (10.13). Hence (10.3a) is regular of dimension $m+\varepsilon_1+\varepsilon_2$ in $\overset{o}{\eta}{}^{a}$ and this implies, as we have proved already, that (10.3) is regular of dimension $m+\varepsilon_2$ in $\overset{}{\underset{0}{\eta}}{}^a$.

Finally let us assume again that (10.3) be regular of dimension $m+\varepsilon_2$ in $\underset{0}{\eta}{}^\alpha$. We have proved already that the indices α can be interchanged in such a way that

$$10.17)\quad \mathcal{G}^{\alpha'}(\eta^a)=0 \quad ; \alpha'=1,\cdots,n-m \, ; \quad a=1,\cdots,n+\varepsilon_1+\varepsilon_2$$

is an equivalent subsystem of (10.3a), minimal regular in $\underset{0}{\eta}{}^a$. Now we consider the equations

$$10.18)\quad \mathcal{F}^{\alpha'}(\xi^\kappa)=0 \quad ; \alpha'=1,\cdots,n-m$$

corresponding to (10.17). Then (10.14) holds and we know now that $\partial_b\mathcal{G}^{\alpha'}$ has the rank $n-m$ in all points of an $\mathfrak{N}(\underset{0}{\eta}{}^a)$ and $\partial_b\varphi^\kappa$ the rank n in all points of an $\mathfrak{N}(\underset{0}{\eta}{}^a)$. Hence $\partial_\kappa\mathcal{F}^{\alpha'}$ has the rank $n-m$ in all points of an $\mathfrak{N}(\underset{0}{\xi}{}^\kappa)$ and consequently (10.18) is minimal regular in $\underset{0}{\xi}{}^\kappa$. Now we have only to prove that (10.18) is equivalent to (10.1). If this were not true there would exist at least one point of $\mathfrak{N}(\underset{0}{\xi}{}^\kappa)$ where the number of linearly independent differentials among the $d\mathcal{F}^{\alpha}$ would be $> n-m$. Consequently, since $\partial_b\varphi^\kappa$ has the rank n in all points of $\mathfrak{N}(\underset{0}{\eta}{}^a)$, there had to exist at least one point of $\mathfrak{N}(\underset{0}{\eta}{}^\lambda)$ where the rank of

$$10.19) \quad \partial_b \mathcal{G}^{\alpha} = (\partial_\kappa \mathcal{F}^{\alpha}) \partial_b \varphi^\kappa ; \quad \begin{array}{l} \alpha = 1, \dots, N ; \\ b = 1, \dots, n+\varepsilon_1+\varepsilon_2 \end{array}$$

would be $> n-m$. But this is impossible because (10.3a) is regular of dimension $m+\varepsilon_1+\varepsilon_2$ in $\overset{\circ}{\eta}{}^{a}$. Hence (10.18) is equivalent to (10.1) and this implies that (10.1) is regular of dimension m in $\overset{\circ}{\xi}{}^{\kappa}$.

§ 11. Applications I. Integration of Goursat systems[1].

In this and the next section it will be shown that the theory of regular systems and of supernumerary coordinates plays an important role in the theory of integration of systems of partial differential equations.

A Goursat system is an arbitrary system of scalars, covariant vectors, bivectors,..., $n-1$-vectors:

$$11.1) \quad \begin{array}{lll} \overset{\chi_0}{u} & ; & \chi_0 = 1, \dots, N_0 \\ \overset{\chi_1}{u}_{\lambda} & ; & \chi_1 = 1, \dots, N_1 \\ \vdots & & \\ \overset{\chi_{n-1}}{u}_{\lambda_1 \dots \lambda_{n-1}} & ; & \chi_{n-1} = 1, \dots, N_{n-1} \end{array}$$

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1) Cf. Kähler 1934.1; P.P. Ch. VIII.

We suppose that the system of equations

$$11.2) \qquad \overset{\chi_0}{u} = 0 \quad ; \chi_0 = 1, \dots, N_0$$

be regular of dimension r_0 in the null point $\overset{0}{\xi}{}^\kappa$. Goursat's problem consists of the determination of all X_m's in an $\mathfrak{N}(\overset{0}{\xi}{}^\kappa)$ lying in the X_{r_0} represented by (11.2) and whose sections with the quantities $\overset{\chi_1}{u}_\lambda, \dots, \overset{\chi_{n-1}}{u}_{\lambda_1 \dots \lambda_{n-1}}$ all vanish. Such an X_m is called an <u>integral</u>-X_m of the Goursat system. If m has a given value, all quantities (11.1) with a valence $> m$ drop out automatically because in an X_m no q-vectors with $q > m$ exist.

<u>The natural derivative</u> (or <u>gradient</u>) <u>of a scalar</u> p is the covariant vector

$$11.3) \qquad \partial_\lambda p$$

and the <u>natural derivative</u> (or <u>rotation</u>) <u>of a covariant</u> q <u>-vector</u> $w_{\lambda_1 \dots \lambda_q}$ is the covariant $(q+1)$-vector

$$11.4) \qquad (q+1)\, \partial_{[\mu} w_{\lambda_1 \dots \lambda_q]}$$

The natural derivative of a natural derivative vanishes identically[1].

1) Cf. Schouten 1949.1 Ch. IV § 3.

It is easily proved that the section of all natural derivatives of the quantities (11.1) with every integral-X_m vanish. Therefore it is convenient to take these natural derivatives to the quantities of the system. If this has been done the Goursat system is said to be closed. A Goursat system consisting only of scalars, vectors and some of the rotations of these vectors or all of them is called a Cartan system. It becomes closed if the gradients of the scalars and the rotations of the vectors are added. In the following we always suppose that a Goursat or Cartan system is closed.

Every tangent-$\mathcal{E}_m$ of an integral-X_m with the (supernumerary) coordinates $\xi^\kappa, v^{\kappa_1 \cdots \kappa_m}$ satisfies the equations

$$\begin{array}{ll} & \overset{\chi_0}{u} = 0 \\ 11.5) & \overset{\chi_1}{u}_{\lambda_1} v^{\lambda_1 \cdots \lambda_m} = 0 \\ & \vdots \\ & \overset{\chi_m}{u}_{\lambda_1 \cdots \lambda_m} v^{\lambda_1 \cdots \lambda_m} = 0 \end{array}$$

in ξ^κ. Every $\mathcal{E}_m$ satisfying (11.5) is called an integral-$\mathcal{E}_m$. Hence every tangent $\mathcal{E}_m$ of an integral-X_m is an integral-$\mathcal{E}_m$. But conversely an integral-$\mathcal{E}_m$ need not be tangent-$\mathcal{E}_m$ of some integral X_m.

First all integral-$\mathcal{E}_m$'s have to be determined. An integral-$\mathcal{E}_0$ or <u>integral point</u> is a point of the X_{r_0} (11.2). If an integral-$\mathcal{E}_0$ is given, the integral-$\mathcal{E}_1$'s through this $\mathcal{E}_0$ fill a flat space in the local $\mathcal{E}_n$ denoted by $\mathcal{H}(\mathcal{E}_0)$. If the dimension of $\mathcal{H}(\mathcal{E}_0)$ is $1+r_1$, there exist just ∞^{r_1} integral-$\mathcal{E}_1$'s through $\mathcal{E}_0$ .In the same way the integral-$\mathcal{E}_{s+1}$'s through a given $\mathcal{E}_s$ fill a flat space $\mathcal{H}(\mathcal{E}_s)$. If the dimension of $\mathcal{H}(\mathcal{E}_s)$ is $s+1+r_{s+1}$ there exist just $\infty^{r_{s+1}}$ integral-$\mathcal{E}_{s+1}$'s through $\mathcal{E}_s$.

It may occur that for all integral-$\mathcal{E}_s$'s in the neighbourhood of some given $\mathcal{E}_s$ the number r_{s+1} has the same value. In that case this given $\mathcal{E}_s$ is said to be <u>regular</u>. A chain of integral elements

$$11.6) \quad \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_m \qquad {}^{1)}$$

is called a <u>regular chain</u> if $\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_{m-1}$ are all regular. <u>The last element of the chain is not necessarily regular.</u> An integral-X_m having at least one tangent-$\mathcal{E}_m$ that is last element of a regular chain is called a <u>regular integral</u>-X_m .

By means of the theorem [2] .

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1) $\subset$ means: is contained in.

2) P.P. Ch. VIII p.358.

Theorem XIII (**theorem of integrability of Cartan - Kähler**)

If $\mathcal{E}_m$ is last element of a regular chain there exists always at least one regular integral-X_m tangent to $\mathcal{E}_m$;

the construction of all regular integral-X_m's is brought back to the construction of all regular chains of $m+1$ elements and certain integration processes.

If regular chains exist with a last element $\mathcal{E}_g$ but not with a last element $\mathcal{E}_{g+1}$, g is called the genus of the Goursat system. Hence a Goursat system with genus g possesses always regular integral-X_g's but no regular integral manifolds with a dimension $> g$.

The theorem of integrability of Goursat systems is proved by means of some auxiliary theorems. First we need the theorem[1].

Theorem XIV (first theorem of uniqueness)

If in X_n be given an integral-X_m of the Goursat system (11.5), a tangent $\mathcal{E}_m$ with the coordinates $\underset{0}{\xi}^k, \underset{0}{v}^{k_1 \cdots k_m}$ and an $X_{n-r_{m+1}}$, satisfying the following conditions:

1. the integral-$\mathcal{E}_m$ is regular,
2. the system

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1) P.P. Ch. VIII p. 359.

$$\overset{\chi_0}{u} = 0 \qquad ; \ \chi_0 = 1, \dots, N_0$$

$$\overset{\chi_1}{u}_{\lambda_1} v^{\lambda_1 \dots \lambda_m} \qquad ; \ \chi_1 = 1, \dots, N_1$$

$$\vdots$$

11.7)
$$\overset{\chi_m}{u}_{\lambda_1 \dots \lambda_m} v^{\lambda_1 \dots \lambda_m} \qquad ; \ \chi_m = 1, \dots, N_m$$

$$v^{[\kappa_1 \dots \kappa_m} v^{\lambda_1] \dots \lambda_m} = 0$$

is regular in $\overset{\circ}{\xi}{}^{\kappa}, \overset{\circ}{v}{}^{\kappa_1 \dots \kappa_m}$,

3. the $X_{n-r_{m+1}}$ contains X_m and its tangent-$\mathcal{E}_{n-r_{m+1}}$ in $\overset{\circ}{\xi}{}^{\kappa}$ has just one $\mathcal{E}_{m+1}$ in common with $\mathcal{H}(\mathcal{E}_m)$;

there exists one and only one integral-X_{m+1} , containing X_m and being contained in $X_{n-r_{m+1}}$. The tangent-$\mathcal{E}_{m+1}$ in $\overset{\circ}{\xi}{}^{\kappa}$ of this X_{m+1} coincides with the section of $\mathcal{E}_{n-r_{m+1}}$ and $\mathcal{H}(\mathcal{E}_m)$.

The proof of this theorem is rather long and needs introduction of a suitable coordinate system and application of the existence theorem of Cauchy-Kowalewsky.

By means of algebraic operations it is always possible to determine a regular chain $\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_g$. Then by means of repeated application of the theorem XIV it is possible to construct a regular integral-X_0, -X_1, $\cdots$, -X_g each tangent to an element of the chain and in this way a proof of the theorem XIII could be constructed. But here a difficulty arises because at every step it has to be proved first that the system (11.7) is regular. For that proof we need the theorem

Theorem XV

Every element $\mathcal{E}_p$; $\overset{\circ}{\xi}^k, v^{k_1\cdots k_p}$ of a regular chain satisfies the condition that the system

11.8)
$$\begin{aligned} &\mathcal{U}^{\chi_s}_{\lambda_1\cdots\lambda_s} v^{\lambda_1\cdots\lambda_p} = 0 \qquad ; s = 0,1,\cdots,p \,; \quad \chi_s = 1,\cdots,N_s \\ &v^{[k_1\cdots k_p} v^{\lambda_1]\cdots\lambda_p} = 0 \end{aligned}$$

is regular of dimension

11.9)
$$M_p + 1 \overset{def}{=} \sum_{i=0}^{i=p} r_i - \tfrac{1}{2} p(p-1) + 1$$

in $\overset{\circ}{\xi}^k$, $\overset{\circ}{v}^{k_1\cdots k_p}$.

In the proof of this theorem the theory of supernumerary coordinates is used. The $\xi^k, v^{k_1\cdots k_p}$ are supernumerary coordinates with $\varepsilon_1 = \binom{n}{p} - p(n-p) - 1$; $\varepsilon_2 = 1$ in the $X_{n+p(n-p)}$ of all $\mathcal{E}_p$'s in the local spaces of X_n in $\mathfrak{M}(\overset{\circ}{\xi}^k)$. Instead of them the non-supernumerary coordinates $\xi^\kappa, \mathcal{B}_\beta^\varsigma$; $\beta = 1, \ldots, p$; $\varsigma = p+1, \ldots, n$ can be introduced. Then from the theorem XII it follows that we have to prove that the system

$$11.10) \quad \mathfrak{U}^{\chi_s}_{\lambda_1 \ldots \lambda_s} \mathcal{B}^{\lambda_1}_{\beta_1} \cdots \mathcal{B}^{\lambda_s}_{\beta_s} = 0 \; ; \; s = 0, 1, \ldots, q \; ; \; \beta_1, \ldots, \beta_s = 1, \ldots, p \; ; \; \chi_s = 1, \ldots, N_s$$

in the variables $\xi^\kappa, \mathcal{B}_\beta^\varsigma$ with $\mathcal{B}_\beta^\alpha = \delta_\beta^\alpha$; $\alpha, \beta = 1, \ldots, p$ is regular of dimension M_p in $\overset{0}{\xi}{}^\kappa, \overset{0}{\mathcal{B}}{}_\beta^\varsigma$. If now the restriction $\mathcal{B}_\beta^\alpha = \delta_\beta^\alpha$ is dropped, the $\xi^\kappa, \mathcal{B}_\beta^\kappa$ are supernumerary coordinates with $\varepsilon_1 = p^2$; $\varepsilon_2 = 0$ in the same $X_{n+p(n-p)}$ and from theorem XII it follows that we have to prove that the system (11.10) without this restriction is regular of dimension $M_p + p^2$ in $\overset{0}{\xi}{}^\kappa, \overset{0}{\mathcal{B}}{}_\beta^\kappa$. Finally instead of the $\mathcal{B}_\beta^\kappa$ the pn components of p arbitrary vectors $\mathcal{B}_1^\kappa, \ldots, \mathcal{B}_p^\kappa$ in the $\mathcal{E}_p$ may be introduced. With the ξ^κ these form a system of supernumerary coordinates with $\varepsilon_1 = 0$; $\varepsilon_2 = p^2$ and consequently we have to prove that the system

$$11.11) \quad \mathfrak{U}^{\chi_s}_{\lambda_1 \ldots \lambda_s} \mathcal{B}^{\lambda_1}_{1} \cdots \mathcal{B}^{\lambda_s}_{p} \; ; \; s = 0, 1, \ldots, p$$

is regular of dimension $M_p + p^2$ in $\overset{0}{\xi}{}^\kappa, \overset{0}{\mathcal{B}}{}_b^\kappa$; $b = 1, \ldots, p$.

The $\overset{0}{\mathcal{B}}{}_b^\kappa$ can be chosen in such a way that for every value of s the vectors $\overset{0}{\mathcal{B}}{}_1^\kappa, \ldots, \overset{0}{\mathcal{B}}{}_s^\kappa$ span the $\mathcal{E}_s$ of the chain. Then a proof by induction can be constructed.

§ 12. Applications II. Solution of a general system of partial differential equations by determining the integral manifolds of a Cartan system[1].

It is well-known that every system of partial differential equations can be brought back to a system of the first order. A system of the first order can be written in the form

$$12.1) \quad \mathfrak{F}^{\alpha}(\xi^{\kappa}, \partial_{\beta}\xi^{\varsigma}) = 0 \;;\; \alpha = 1, \cdots, N \;;\; \beta = 1, \cdots, m \;;\; \varsigma = m+1, \cdots, n$$

with the m independent variables ξ^{α}; $\alpha = 1, \cdots, m$ and the $n-m$ unknowns ξ^{ς}; $\varsigma = m+1, \cdots, n$. Now we consider the equations

$$12.2) \quad \mathfrak{F}^{\alpha}(\xi^{\kappa}, p_{\beta}^{\varsigma}) = 0 \;;\; \alpha = 1, \cdots, N \;;\; \beta = 1, \cdots, m \;;\; \varsigma = m+1, \cdots, n$$

and look upon the p_{β}^{ς} as the non-supernumerary coordinates of an $\mathcal{E}_m$ in the local $\mathcal{E}_n$ of ξ^{κ}. Then the $\xi^{\kappa}, p_{\beta}^{\varsigma}$ are non-supernumerary coordinates in the $X_{n+m(n-m)}$ of all $\mathcal{E}_m$'s in the local $\mathcal{E}_n$'s of X_n for which $v^{1 \cdots m} \neq 0$.

In this $X_{n+m(n-m)}$ we consider the $n-m$ so-called Pfaffians

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1) Cf. Cartan 1901.1; 1904.1; 1945.1; Goursat 1922.1 P.P. Ch. X.

.3) $d\xi^\varsigma - p^\varsigma_\beta d\xi^\beta \;;\; \beta = 1,\cdots,m \;;\; \varsigma = m+1,\cdots,n$

the $n+m(n-m)$ variables $\xi^\kappa, p^\varsigma_\beta$. The
nctions $\mathcal{F}^\alpha(\xi^\kappa, p^\varsigma_\beta)$ and $n-m$ covariant vectors
$X_{n+m(n-m)}$ belonging to the Pfaffians
2.3) and having the components

.4) $p^\varsigma_\beta \,,\, \delta^\varsigma_\eta \,,\, 0 \;;\; \beta = 1,\cdots,m \;;\; \varsigma,\eta = m+1,\cdots,n$

longing to the coordinates

.5) $\xi^\beta \,,\, \xi^\eta \,,\, p^\varsigma_\beta \;;\; \beta = 1,\cdots,m \;;\; \varsigma,\eta = m+1,\cdots,n$

nstitute a Cartan system and this system becomes
osed if we add the gradients of the $\mathcal{F}^\alpha$ and
e rotations of the vectors (12.4).

Each Pfaffian represents an $\mathcal{E}_{n+m(n-m)-1}$ in
ch local $\mathcal{E}_{n+m(n-m)}$ and consequently

.6) $d\xi^\varsigma - p^\varsigma_\beta d\xi^\beta = 0 \;;\; \beta = 1,\cdots,m \;;\; \varsigma = m+1,\cdots,n$

presents an $\mathcal{E}_{n+(m-1)(n-m)}$-field in the
$_{+m(n-m)}$. Besides in the $X_{n+m(n-m)}$ there
ists a normal system of ∞^m $X_{n+m(n-m-1)}$'s
th the equations

12.7) $\xi^{\alpha} = \text{constant} \quad ; \alpha = 1, \cdots, m$.

According to our definition in § 11 an integral-X_m of the Cartan system is an X_m lying in the null manifold of (12.2) whose sections with all the covariant vectors and bivectors of the Cartan system vanish, in other words, it is tangent to all $\mathcal{E}_{n+(m-1)(n-m)}$'s of the field (12.6). If such an X_m has nowhere a direction in common with an $X_{n+m(n-m-1)}$ of the normal system (12.7), the ξ^{α} can be used as coordinates in the X_m and the X_m can be represented by a parametric form

$$12.7) \quad \begin{aligned} &\text{a)} \quad \xi^{\varsigma} = f^{\varsigma}(\xi^{\alpha}) \quad ; \alpha, \beta = 1, \cdots, m \; ; \varsigma = m+1, \cdots, n \\ &\text{b)} \quad p_{\beta}^{\varsigma} = f_{\beta}^{\varsigma}(\xi^{\alpha}) \end{aligned}$$

But according to (12.2) and (12.6) these equations represent a solution of (12.1). Conversely every solution of (12.1) corresponds to an integral-X_m of the Cartan system having nowhere a direction in common with an $X_{n+m(n-m-1)}$ of the normal system (12.7). Hence the solution of the system of partial differential equations (12.1) is brought back to the determination of the most general integral-X_m of a Cartan system in $n+m(n-m)$ variables

In order to construct the most general integral-X_m of the Cartan system in an $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa}, \overset{\circ}{p}{}^{\xi}_{\beta})$ we have first to consider the equations (12.2). If this system is regular or semiregular in $\overset{\circ}{\xi}{}^{\kappa}$, $\overset{\circ}{p}{}^{\xi}_{\beta}$ it can be replaced by an equivalent system minimal regular in $\overset{\circ}{\xi}{}^{\kappa}$, $\overset{\circ}{p}{}^{\xi}_{\beta}$. If the system is irregular, by using theorem X, the null manifold can be decomposed into a finite number of X_s's ; $s = 0, 1, \cdots, n+m(n-m)-1$. For every X_s a system of equations can be found, minimal regular in $\overset{\circ}{\xi}{}^{\kappa}$, $\overset{\circ}{p}{}^{\xi}_{\beta}$, or in some other point in $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa}, \overset{\circ}{p}{}^{\xi}_{\beta})$. Consequently the Cartan system is decomposed into a finite number of Cartan systems each having a system of scalar equations minimal regular in a point of $\mathfrak{N}(\overset{\circ}{\xi}{}^{\kappa}, \overset{\circ}{p}{}^{\xi}_{\beta})$. If from these scalar equations the maximum number of variables $\xi^{\kappa}, p^{\xi}_{\beta}$ is solved and if these solutions are substituted into the equations (12.3) we get a scalar-free Cartan system. This latter reduction may be useful sometimes but in many cases it is more profitable to keep the scalar equations.

We now proceed with one of the Cartan systems with a minimal regular system of scalar equations. If g is the genus of the system and if $m \leq g$, regular integral-X_m's always exist (cf. § 11) and these X_m's can be determined by methods developed by Cartan[1]. But if $m > g$ there

1) Cartan 1901.1; 1904.1.

exist no regular integral-X_m's and we don't yet know if there are integral-X_m's at all.

In order to decide whether there exist for a given Cartan system integral-X_m's with $m>g$, Cartan's <u>method of prolongation</u> has to be used.

We consider a closed Cartan system

$$
\text{12.8)}\qquad
\begin{aligned}
&\mathfrak{F}^{\omega}(\xi^{\kappa}) && ; \omega = r_0+1,\dots,n\\
&\overset{\mathfrak{c}}{w}_{\lambda} && ; \mathfrak{c} = r_1+1,\dots,n\\
&\partial_{[\mu}\overset{\mathfrak{c}}{w}_{\lambda]} &&
\end{aligned}
$$

satisfying the conditions that the system $\mathfrak{F}^{\omega}=0$ is minimal regular in $\underset{\circ}{\xi}^{\kappa}$, that the $\partial_\lambda \mathfrak{F}^{\omega}$ depend linearly on the $\overset{\mathfrak{c}}{w}_{\lambda}$ and that the $\overset{\mathfrak{c}}{w}_{\lambda}$ are linearly independent. Then if $\mathcal{B}_{\beta}^{\kappa}$; $\beta = 1,\dots,m$ is the contravariant connecting quantity of an integral-X_m whose tangent $\mathcal{E}_m$ in every point of an $\mathfrak{X}(\underset{\circ}{\xi}^{\kappa})$ satisfies the inequality $v^{1\dots m}\neq 0$ the n.a.s. conditions for an X_m being integral-X_m are

$$
\text{12.9)}\qquad
\begin{aligned}
&\mathfrak{F}^{\omega}(\xi^{\kappa})=0 && ; \omega = r_0+1,\dots,n\\
&\mathcal{B}_{\beta}^{\lambda}\overset{\mathfrak{c}}{w}_{\lambda}=0 \quad ; \mathcal{B}_{\beta}^{\alpha}=\delta_{\beta}^{\alpha} && ; \mathfrak{c} = r_1+1,\dots,n\\
&\mathcal{B}_{\gamma}^{\mu}\mathcal{B}_{\beta}^{\lambda}\partial_{[\mu}\overset{\mathfrak{c}}{w}_{\lambda]}=0 && ; \alpha,\beta = 1,\dots,m
\end{aligned}
$$

Now we consider the $X_{n+m(n-m)}$ of the ξ^{κ}, B^{ξ}_{β} and the system of Pfaffian equations

12.10) $d\xi^{\xi} - B^{\xi}_{\beta}\, d\xi^{\beta} = 0 \ ; \beta = 1, \dots, m \ ; \xi = m+1, \dots, n$

in this manifold. Then the equations (12.9, 10) constitute a Cartan system in this $X_{n+m(n-m)}$ with the scalar equations (12.9). This Cartan system is called the <u>first (total) prolongation</u> of the given Cartan system. If an integral-X_m of the prolongation is given, having nowhere a direction in common with an $X_{n+m(n-m-1)}$ of the normal system

12.11) $\xi^{\alpha} = \text{constant} \ ; \alpha = 1, \dots, m$

from the $n+m(n-m-1)$ equations of this X_m the ξ^{ξ}, B^{ξ}_{β} can be solved as functions of the ξ^{α} and this solution represents an integral-X_m of the given Cartan system. Conversely from the parametric form (12.7a) of an integral-X_m of the given system, by differentiation, we get equations of the form (12.7b) and these equations together with (12.7a) represent an integral-X_m of the prolongation . Hence there exists a one-to-one correspondence between the integral-X_m's of a Cartan system and those integral-X_m's of the prolongation having nowhere a direction in common with an $X_{n+m(n-m-1)}$ of the normal system (12.11).

This way of prolongation is not the only one. In many cases it is more profitable to form a partial prolongation by using not all equations (12.10) but only a number $< n-m$ of linear combinations of them.

Now the principle of the method of prolongation can be described as follows. If the genus of the prolongation is $\geqq m$, the integral-X_m's of the prolongation can be formed and by them the integral-X_m's of the original system are determined. If the genus of the prolongation is $< m$ this system has to be prolongated once more, etc. Cartan has proved in 1904[1] that if at each step the prolongation is performed in a suitable way and if the manner of prolongation is always chosen most practically, the process gives after a finite number of steps either the integral-X_m's looked for <u>or the certainty</u> that no integral-X_m's exist[2].

1) Cf. 1904.1; 1945.1.

2) There only remains one difficulty. During the process of prolongation extra ordinary points may arise. There may exist integral-X_m's containing only extra ordinary points. This is not so bad because the extra ordinary points may be treated in the same way by prolongation and this does not disturb the finiteness of the whole process. But during this process new extra ordinary points may arise and they have to be treated in the same way, etc. Though it is highly probable that the <u>whole</u> process will remain finite, the proof is not yet given.

But this is only the principle of the method. For the performance of the prolongations and the choice of the manner of prolongation rather complicated calculations are necessary and there arises a lot of difficulties all overcome by Cartan in a very ingenious way. But the study of these investigations of Cartan is very difficult[1].

Here we are interested in the following difficulty. After some prolongation it may occur that the scalar equations of the new system no longer form a minimal regular system. Then according to theorem X the null manifold consists of a finite number of X_s's and to each X_s there belongs a system of equations minimal regular in some point of a neighbourhood of the point considered. The Cartan system considered splite up into a finite number of Cartan systems. If any one of these systems has a genus $\geqq m$ this system immediately furnishes a contribution to the integral-X_m's looked for. If the genus is $< m$ it has to be prolongated. At last we find that the integral-X_m's desired are all the integral-X_m's of a finite number of Cartan systems each of which has a genus m.

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1) In P.P. Ch. X an elaborate treatment of Cartan's beautiful methods is given, which is intended to be more intelligible.

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