<u>Scriptum 7</u>

ON SUMS OF SYSTEMS

J.G. van der Corput.

In this scriptum a system means always a set formed by a finite positive number of numbers $\geqq 0$. If A is a system, then $A(m)$ denotes the number of the positive elements $< m$ of A. If $A_1, A_2, \ldots, A_n$ are systems, then $A_1 + A_2 + \ldots + A_n$ denotes the sumset, formed by the numbers which can be written in at least one way in the form $a_1 + a_2 + \ldots + a_n$, where a_ν $(1 \leqq \nu \leqq n)$ belongs to A_ν. The system consisting only of the integer zero will be denoted by O, so that for each system A

$$A + O = A.$$

If $W, A_1, \ldots, A_n$ are systems and if ϱ is a positive integer $\leqq n$, then I call the sum

$$\sum_{1 \leqq \nu_1 < \nu_2 < \ldots < \nu_\varrho \leqq n} (W + A_{\nu_1} + \ldots + A_{\nu_\varrho})(m),$$

which consists of $\binom{n}{\varrho}$ terms, an elementary symmetric function of $A_1, A_2, \ldots, A_n$.

For instance

$(W+A_1)\ (m)+(W+A_2)\ (m)+\ldots+\ (W+A_n)\ (m)$

and

$(W+A_1\ +\ A_2\ +\ldots+\ A_n)\ (m)$

and the sum

$(W+A_1\ +\ A_2)\ (m)+\ (W+A_1+A_3)\ (m)+\ldots+(W+A_{n-1}+A_n)\ (m),$

consisting of $\frac{1}{2}n(n-1)$ terms; denote elementary symmetric functions of $A_1, A_2, \ldots, A_n$.

A symmetric function $\sigma(m; A_1, \ldots, A_n)$ of $A_1, \ldots, A_n$ is a function which can be written in the form

$$\sigma(m; A_1, \ldots, A_n) = \sum_{\lambda=1}^{l} \mu_\lambda \sigma_\lambda(m; A_1, \ldots, A_n),$$

where the coefficients μ_λ are $\geqq 0$ and where $\sigma_\lambda(m; A_1, \ldots, A_n)$ denotes an arbitrary elementary symmetric function of $A_1, \ldots, A_n$. For instance

$$\mu(W+A_1)(m)+\ldots+\mu(W+A_n)(m)+\mu'(U+A_1+\ldots+A_n)(m)$$

where μ and μ' are $\geqq 0$, is a symmetric function of $A_1, \ldots, A_n$.

A slowly changing function is a function $\varphi(m)$, defined for each positive m, such that for any choice of the positive numbers m and m'

$$\varphi(m+m') \leqq \varphi(m) + \varphi(m').$$

The principal theorem

<u>If $n \geqq 2$, if each of the systems $A_1, \ldots, A_n$ contains the number zero and if A_n contains at least one positive number, then it is possible to construct n-1 systems $B_1, \ldots, B_{n-1}$ with the following properties: A_ν $(1 \leqq \nu \leqq n-1)$ is a subset of B_ν, but $B_1+\ldots+B_{n-1}$ is a subset of $A_1+\ldots+A_n$.</u>

<u>Each symmetric function $\sigma(m; A_1, \ldots, A_n)$ satisfies for each positive m the inequality</u>

$\sigma(m; B_1,\ldots,B_{n-1},0) \leqq \sigma\, m; A_1,\ldots,A_n).$

<u>If k is positive and if a slowly changing function $\varphi(m)$ satisfies for each positive $m \leqq k$ the inequality</u>

(1) $A_1(m)+\ldots+A_n(m) \geqq \varphi(m)+1-n$

<u>then the inequality</u>

(2) $B_1(m)+\ldots+B_{n-1}(m) \geqq \varphi(m)+1-n$

<u>holds for each positive $m \leqq k$.</u>

<u>Remark</u>. Let the slowly changing function $\varphi(m)$ be monotonically not-decreasing. If the inequalities (1) hold for m=k and also for each positive number m<k which belongs to at least one of the systems $A_1,\ldots, A_n$, then the inequalities (1) hold for each positive $m \leqq k$.
For let m' be the smallest number $\leqq k$ and $\geqq m$, which belongs to at least one of the systems $A_1,\ldots, A_n$, if such a number exists; otherwise I choose m' = k. Inequality (1) holds with m' instead of m, so that we get

$$\sum_{\nu=1}^{n} A_\nu(m') \geqq \varphi(m')+1-n.$$

Since A_ν does not contain a number $\geqq m$ and $<m'$, the left hand side does not change its value if m' is replaced by m. The right hand side is then replaced by an equal or a smaller number, since $\varphi(m)$ is

monotonically not-decreasing. In this way we see that the inequalities (1) are valid for each positive $m \leq k$.

Let us first give some applications of this theorem. In these applications $\varphi(m)$ denotes always a slowly changing function and k denotes in these applications always a positive number.

I. If both A and B contain the number zero and if

$$A(m) + B(m) \geq \varphi(m) - 1 \quad (0 < m \leq k),$$

then

$$(A+B)(m) \geq \varphi(m) - 1 \quad (0 < m \leq k).$$

Proof. According to the principal theorem, applied with n=2, we can construct a subset S of A+B such that for each positive $m \leq k$

$$S(m) \geq \varphi(m) - 1, \text{ hence } (A+B)(m) \geq \varphi(m) - 1.$$

Particular cases: 1 (Theorem of Khintchine) x):
If both A and B contain the number zero and
$A(m) \geq \alpha(m-1)$ and $B(m) \geq \beta(m-1)$ $(0 < m \leq k)$,
where $\alpha + \beta \leq 1$, then

$$(A+B)(m) \geq (\alpha+\beta)(m-1) \quad (0 < m \leq k).$$

That is clear, since

$$\varphi(m) = (\alpha+\beta)m + 1 - \alpha - \beta$$

----

x) A.Ya.Khintchine, Zur additiven Zahlentheorie, Matematiceski Sbornik 39, 27 - 34 (1932).

satisfies the relation

$$\varphi(m) + \varphi(m') - \varphi(m + m') = 1 - \alpha - \beta \geqq 0$$

and changes therefore slowly.

2 (Famous theorem of Mann) x):

If both A and B contain the number zero and

$$A(m) + B(m) \geqq \gamma(m-1) \quad (0 < m \leqq k),$$

where $\gamma \leqq 1$, then

$$(A+B)\,(m) \geqq \gamma\,(m-1) \quad (0 < m \leqq k).$$

That is obvious, since $\varphi(m) = \gamma m + 1 - \gamma$ changes slowly.

II. If A_ν ($\nu = 1, \ldots, n$) contains the number zero and if

$$A_1(m) + \ldots + A_n(m) \geqq \varphi(m) - 1 \quad (0 < m \leqq k),$$

then

$$(A_1 + \ldots + A_n)\,(m) \geqq \varphi(m) - 1 \quad (0 < m \leqq k).$$

Proof. The particular case n=1 is obvious and the case n=2 has already been treated in the first application, so that I may assume that $n \geqq 3$ and that the proof has already been given for n-1 instead of n.

The principal theorem, applied with the slowly changing function $\varphi(m)+n-2$ instead of $\varphi(m)$, gives n-1 systems $B_1, \ldots, B_{n-1}$ such that $B_1 + \ldots + B_{n-1}$ is a subset of $A_1 + \ldots + A_n$ and that

$$B_1(m) + \ldots + B_{n-1}(m) \geqq \varphi(m) - 1 \quad (0 < m \leqq k).$$

This implies according to our induction hypothesis

----

x) H.B.Mann, A proof of the fundamental theorem on the density of sums of sets of positive integers, Annals of Math., 43, 523 - 529 (1942).

$$(B_1+\ldots+B_{n-1})(m) \geqq \varphi(m)-1 \quad (0 < m \leqq k),$$

which yields immediately the required inequality.

Particular cases:

1. If k is a positive integer, if A_ν $(\nu=1,\ldots,n)$ is formed by integers $\geqq 0$ and contains the number zero and if

(3) $A_1(m)+A_2(m)+\ldots+A_n(m) \geqq m-1$ $(m=1,2,\ldots,k)$,

then each positive integer $<k$ can be written in the form $a_1+a_2+\ldots+a_n$, where a_ν $(1 \leqq \nu \leqq n)$ occurs in A_ν.

Proof. Inequality (3) holds (according to the second remark added to the principal theorem) for each positive $m \leqq k$, hence

$$(A_1+\ldots+A_n)(m) \geqq m-1 \quad (0 < m \leqq k),$$

so that each positive integer $<k$ belongs to $A_1+\ldots+A_n$.

2. (Theorem of Dyson) x):

If A_ν $(1 \leqq \nu \leqq n)$ contains the number zero and

$$A_1(m)+\ldots+A_n(m) \geqq \gamma(m-1) \quad (0 < m \leqq k),$$

then

$$(A_1+\ldots+A_n)(m) \geqq \gamma(m-1) \quad (0 < m \leqq k).$$

Proof. In this case we choose $\varphi(m) = \gamma m+1-\gamma$.

----

x) F.J Dyson, A theorem on the densities of sets of integers, Journal of the London Math.Society 20, 8 - 15 (1945).

3. <u>If</u> A_ν $(1 \leqq \nu \leqq n)$ <u>contains the number zero and</u>

$$A_\nu(m) \geqq \alpha m + \gamma - \frac{\nu}{n} \quad (\nu = 1,2,\ldots n;\ 0 < m \leqq k),$$

<u>where α and γ are $\geqq 0$, then</u>

$$(A_1 + A_2 + \ldots + A_n)(m) \geqq n\alpha m + n\gamma - 1 \quad (0 < m \leqq k)$$

<u>Proof.</u> We have for $0 < m \leqq k$

$$\sum_{\nu=1}^{n} A_\nu(m) \geqq \sum_{\nu=1}^{n} \left\{\alpha m + \gamma - \frac{\nu}{n}\right\},$$

where $\{u\}$ is the smallest integer $\geqq u$. The identity

$$\sum_{\nu=1}^{n} \left\{u - \frac{\nu}{n}\right\} = \{nu\} - 1$$

is obvious in the interval $0 \leqq u < \frac{1}{n}$, since in that case both sides are equal to zero and if u is replaced by $u + \frac{1}{n}$, the increase of both sides is equal to 1, so that the identity is valid for all real u. Consequently

$$\sum_{\nu=1}^{n} A_\nu(m) \geqq \{n\alpha m + n\gamma\} - 1 \geqq \varphi(m) - 1,$$

where $\varphi(m) = n\alpha m + n\gamma$ changes slowly. This gives the required inequality.

Khintchine has treated some special cases of this result.

III. <u>If</u> $A_\nu (1 \leqq \nu \leqq n)$ <u>contains the number zero and</u>

$$A_1(m) + A_2(m) + \ldots + A_n(m) \geqq \varphi(m) - 1 \quad (0 < m \leqq k),$$

<u>then we have for $\varrho = 1,2,\ldots n$.</u>

$$(4)\quad \sum_{1\leqq \nu_1<\nu_2<\ldots<\nu_\varrho\leqq n} (A_{\nu_1}+A_{\nu_2}+\ldots+A_{\nu_\varrho})(m)\geqq$$

$$\geqq \binom{n-1}{\varrho-1}(\varphi(m)-1)\quad (0<m\leqq k)$$

Proof. In the special case $n=1$ we have $\varrho=1$, the left hand side is equal to $A_1(m)$ and the right hand side is equal to $\varphi(m)-1$. We may therefore assume that $n\geqq 2$ and that the proof has already been given for $n-1$ instead of n. The principal theorem gives $n-1$ systems $B_1,\ldots, B_{n-1}$ such that

$$B_1(m)+B_2(m)+\ldots+B_{n-1}(m)\geqq\varphi(m)-1\quad (0<m\leqq k)$$

The left hand side of (4), which is a symmetric function of $A_1,\ldots, A_n$ remains the same or decreases if A_ν $(1\leqq\nu\leqq n-1)$ is replaced by B_ν and if B_n is replaced by 0. Consequently the left hand side of (4) is

$$\geqq \sum_{1\leqq\nu_1<\ldots<\nu_\varrho\leqq n-1}(B_{\nu_1}+\ldots+B_{\nu_\varrho})(m)+$$

$$+\sum_{1\leqq\nu_1<\ldots<\nu_{\varrho-1}\leqq n-1}(B_{\nu_1}+\ldots+B_{\nu_{\varrho-1}}).$$

According to our induction hypothesis the first term is at most equal to $\binom{n-2}{\varrho-1}(\varphi(m)-1)$ and the second term is at most equal to $\binom{n-2}{\varrho-2}(\varphi(m)-1)$, so that the left hand side of (4) is

$$\geqq\left\{\binom{n-2}{\varrho-1}+\binom{n-2}{\varrho-2}\right\}(\varphi(m)-1)=\binom{n-1}{\varrho-1}(\varphi(m)-1).$$

IV. If each of the systems A, B, C and D contain the number zero and if they satisfy the inequalities

$A(m)+B(m) \geqq \varphi(m)-1$ and $C(m)+D(m) \geqq \psi(m)-1$ $(0 < m \leqq k)$, where $\varphi(m)$, $\psi(m)$ and $\varphi(m)+\psi(m)-1$ change slowly, then

(5) $(A+C)(m)+(A+D)(m)+(B+C)(m)+(B+D)m \geqq 2\varphi(m)+$
$+2\psi(m)-4 \qquad (0 < m \leqq k)$

Proof. Applying the principal theorem on the two systems A and B we find a system E with

(6) $$E(m) \geqq \varphi(m)-1 \qquad (0 < m \leqq k).$$

Doing the same with C and D we obtain a system F with

(7) $$F(m) \geqq \psi(m)-1 \qquad (0 < m \leqq k).$$

Since the left hand side of (5) is a symmetric function of A and B, its value remains the same or decreases if A is replaced by E and B is replaced by 0, hence

$$(A+C)(m)+(A+D)(m)+(B+C)(m)+(B+D)(m) \geqq$$
$$\geqq (E+C)(m)+(E+D)(m)+C(m)+D(m).$$

The right hand side is a symmetric function of C and D, so that its value remains the same or decreases if C is replaced by F and D is replaced by 0, hence

$$(A+C)(m)+(A+D)(m)+(B+C)(m)+(B+D)(m) \geqq (E+F)(m)+$$
$$+E(m)+F(m).$$

From (6) and (7) it follows that
$E(m)+F(m) \geqq \varphi(m)+\psi(m)-2$, hence
$(E+F)(m) \geqq \varphi(m)+\psi(m)-2$,

which gives the required result.

This result is a special case of the following application.

V. <u>If each of the systems</u> $A_1, \ldots, A_n, A'_1, \ldots, A'_t$ <u>contain the number zero and if</u>

$$\sum_{\nu=1}^{n} A_\nu(m) \geqq \varphi(m)-1 \quad \text{and} \quad \sum_{\tau=1}^{t} A'_\tau(m) \geqq \psi(m)-1 \quad (0<m\leqq k),$$

<u>where</u> $\varphi(m), \psi(m)$ <u>and</u> $\varphi(m)+\psi(m)-1$ <u>change slowly, then we have for</u> $\varrho=1,2,\ldots,n$, <u>for</u> $\lambda=1,2,\ldots,t$ <u>and for each positive</u> $m \leqq k$

$$(8) \quad \sum_{\substack{1\leqq\nu_1<\ldots<\nu_\varrho\leqq n \\ 1\leqq\tau_1<\ldots<\tau_\lambda\leqq t}} (A_{\nu_1}+\ldots+A_{\nu_\varrho}+A'_{\tau_1}+\ldots+A'_{\tau_\lambda})(m) \geqq$$

$$\geqq \binom{n-1}{\varrho-1}\binom{t-1}{\lambda-1}(\varphi(m)+\psi(m)-2)$$

<u>Proof.</u> In the special case n=t=1 the left hand of (8) is equal to $(A_1+B_1)(m)$ and therefore $\geqq \varphi(m)+$ $+\psi(2)-2$, since $\varphi(m)+\psi(m)-1$ changes slowly.

Consequently we may suppose that at least one of the numbers n and t, say n, is $\geqq 2$ and that the assertion has already been proved for n-1 instead of n.

The principal theorem gives systems $B_1, \ldots, B_n$ with

$$B_1(m)+\ldots+B_{n-1}(m) \geqq \varphi(m)-1 \quad (0<m\leqq k),$$

such that the left hand side of (8) remains the same or decreases if A_ν $(1 \leqq \nu \leqq n-1)$ is replaced by B_ν and B_n is replaced by 0. The left hand side of (8) is therefore

$$\geqq \sum_{\substack{1 \leqq \nu_1 < \ldots < \nu_\varrho \leqq n-1 \\ 1 \leqq \tau_1 < \ldots < \tau_\lambda \leqq t-1}} (B_{\nu_1} + \ldots + B_{\nu_\varrho} + A'_{\tau_1} + \ldots + A'_{\tau_\lambda})(m) +$$

$$+ \sum_{\substack{1 \leqq \nu_1 < \ldots < \nu_{\varrho-1} \leqq n-1 \\ 1 \leqq \tau_1 < \ldots < \tau_\lambda \leqq t-1}} (B_{\nu_1} + \ldots + B_{\nu_{\varrho-1}} + A'_{\tau_1} + \ldots + A'_{\tau_\lambda})(m)$$

According to our induction hypothesis these two terms are respectively

$$\geqq \binom{n-2}{\varrho-1}\binom{t-1}{\lambda-1} \left(\varphi(m) + \psi(m) - 2\right) \text{ and}$$

$$\geqq \binom{n-2}{\varrho-2} \binom{t-1}{\lambda-1}\left(\varphi(m) + \psi(m) - 2\right),$$

so that their sum is

$$\geqq \binom{n-1}{\varrho-1}\binom{t-1}{\lambda-1} \left(\varphi(m) + \psi(m) - 2\right).$$

This completes the proof.

It is clear that the results obtained in the applications III, IV and V can be generalised. For instance: <u>if each of the systems</u> $A_1, \ldots, A_n$, $A'_1, \ldots, A'_t$, $A''_1, \ldots, A''_s$ <u>contain the number zero and if</u>

$$\sum_{\nu=1}^{n} A_\nu(m) \geqq \varphi(m)-1, \quad \sum_{\tau=1}^{t} A'_\tau(m) \geqq \psi(m)-1,$$

$$\sum_{\sigma=1}^{s} A''_\sigma(m) \geqq \chi(m)-1 \qquad (0 < m \leqq k),$$

where $\varphi(m)$, $\psi(m)$, $\chi(m)$, $\varphi(m)+\psi(m)-1$, $\varphi(m)+\chi(m)-1$, $\psi(m)+\chi(m)-1$ and $\varphi(m)+\psi(m)+\chi(m)-2$ change slowly, then we have for

$1 \leqq \varrho \leqq n$, $1 \leqq \lambda \leqq t$ and $1 \leqq \mu \leqq s$

and for each positive $m \leqq k$ that

$$\sum_{\substack{1 \leqq \nu_1 < \dots < \nu_\varrho \leqq n \\ 1 \leqq \tau_1 < \dots < \tau_\lambda \leqq t \\ 1 \leqq \sigma_1 < \dots < \sigma_\mu \leqq s}} (A_{\nu_1}+\dots+A_{\nu_\varrho}+B'_{\tau_1}+\dots+B'_{\tau_\lambda}+A''_{\sigma_1}+\dots+A''_{\sigma_\mu})(m)$$

$$\geqq \binom{n-1}{\varrho-1}\binom{t-1}{\lambda-1}\binom{s-1}{\mu-1} \; (\varphi(m)+\psi(m)+\chi(m)-2)\,.$$

So we can go on.

VI. If A_ν $(1 \leqq \nu \leqq n)$ consists of integers $\geqq 0$ and contains the number zero and if $S = A_1+\dots+A_n$ does not contain the positive integer k, then there exists a positive integer $m \leqq k$ such that

$$S(k+1)-S(k+1-m) \geqq A_1(m+1)+A_2(m+1)+\dots+A_n(m+1)+1$$

Proof. Suppose for a moment that for each positive integer $m \leqq k$

$$(9) \quad S(k+1)-S(k+1-m) \leqq A_1(m+1)+A_2(m+1)+\dots+A_n(m+1)\,.$$

Let T be the system formed by the numbers $t \geqq 0$ and $\leqq k$ such that $k-t$ does not belong to S. This set T contains the number zero since k does not belong to S. The interval $k+1-m \leq x \leq k$ contains m integers. $S(k+1)-S(k+1-m)$ of these integers belong to S and $T(m)$ of these integers does not belong to S, since the integers $x = k-y$ belonging to the interval in consideration and not belonging to S are characterized by the fact that y is ≥ 0 and $< m$ with the property that $k-y$ does not belong to S. Consequently

$$m = S(k+1)-S(k+1-m)+T(m),$$

hence by (9)

$$A_1(m+1)+\ldots+A_n(m+1)+T(m) \geq m,$$

valid for $m = 1,2,\ldots,n$. In this way we find for $m = 2,3,\ldots,k+1$

$$A_1(m)+A_2(m)+\ldots+A_n(m)+T(m) \geq m-1.$$

This inequality is obvious for $m = 1$ and therefore valid for $m = 1,2,\ldots,k+1$. From the first special case of application II it follows that each positive integer $< k+1$, in particular the integer k, can be written as $a_1+a_2+\ldots+a_n+t$, where a_ν belongs to A_ν and where t belongs to T. Consequently $k-t=a_1+a_2+\ldots+a_n$ would belong to S, contrary to the definition of T. This completes the proof.

Proof of the fundamental theorem.

Let e be the smallest number such that a positive integer $\tau \leq n-1$ exists with the property that e is an element of A_τ and that A_n contains at least one element a_n' such that $e+a_n'$ does not belong to A_τ; such a number exists since the largest element a_1 of A_1 and any positive element a_n of A_n have the property that a_1+a_n does not belong to A_1.

If the systems $A_1,\ldots,A_n$ are given, the number e is uniquely defined. That is not necessarily the case with τ, but if more than one value of τ enters into consideration, we can make a choice; for instance we can choose for τ the smallest possible value.

I cancel in A_n all elements a_n' such that $e+a_n'$ does not belong to A_τ; let C_n be the set formed by the remaining elements of A_n. I choose for C_τ the system A_τ to which the numbers $e+a_n'$ are added. Finally $C_\nu = A_\nu$ for $1 \leq \nu \leq n-1$, $\nu \neq \tau$.

Let us show that this new set $(C_1,\ldots,C_n)$ satisfies the following conditions:

A_ν $(\nu = 1,\ldots,n-1)$ is a subset of C_ν, but $C_1+\ldots+C_{n-1}$ is a subset of $A_1+\ldots+A_n$.

Each symmetric function $\sigma(m;A_1,\ldots,A_n)$ satisfies for each positive m the inequality

(10) $\sigma(m;C_1,\ldots,C_n) \leq \sigma(m;A_1,\ldots,A_n)$.

If k is positive and (1) holds for each positive $m \leq k$, then

(11) $C_1(m)+\ldots+C_n(m) \geq \varphi(m)-\psi(m)+1-n$

for each positive $m \leq k$.

We are ready with the whole proof as soon as we have found this result For if C_n consists only of the number zero, we have $C_n(m) = 0$, so that we can choose $B_\nu = C_\nu$ ($\nu = 1, \ldots, n-1$). If C_n contains at least one positive number we can repeat our argument with the set $(C_1, \ldots, C_n)$ instead of $(A_1, \ldots, A_n)$. In this way we construct a new set $(D_1, \ldots, D_n)$ such that the above mentioned conditions are satisfied with D_ν instead of C_ν. Continuing in this manner we obtain after a finite number of constructions a set $(E_1, \ldots, E_n)$ where E_n consists only of the number zero; this follows from the fact that C_n contains less elements than A_n, that D_n contains less elements than C_n, and so on. Then the sets $B_\nu = E_\nu$ ($\nu = 1, \ldots, n-1$) possess the required properties.

That A_ν ($\nu = 1, \ldots, n-1$) is a subset of C_ν follows immediately from the definition of C_ν. Let us now show that $C_1 + \ldots + C_{n-1}$ is a subset of $A_1 + \ldots + A_n$. Since $C_\nu = A_\nu$ ($1 \leq \nu \leq n-1$, $\nu \neq \tau$) it is sufficient to prove that $C_\tau + C_n$ is a subset of $A_\tau + A_n$. Each element of $C_\tau + C_n$ has the form $c_\tau + c_n$, where c_τ and c_n belong respectively to C_τ and C_n. If c_τ belongs to A_τ we have $c_\tau = a_\tau$ and $c_n = a_n$, where a_τ and a_n belong respectively to A_τ and A_n, so that $c_\tau + c_n$ belongs to $A_\tau + A_n$. If c_τ does not belong to A_τ, it is one of the numbers which have been added to A_τ, so that it has the form $e + a'_n$, where a'_n denotes one of the cancelled elements of A_n; since c_n is one of the elements of A_n, which have not been cancelled, the sum $e+c_n = a_\tau$ belongs to A_τ, so that $c_\tau + c_n = e+a'_n+c_n = a_\tau+a'_n$ belongs to $A_\tau + A_n$

In the proof of (10) we can suppose without loss of generality that $\sigma(m;A_1,A_2, ..,A_n)$ is an elementary symmetric function, so that it can be written as

$$\sigma(m;A_1,\dots,A_n) = \sum_{1\leq \nu_1<\nu_2<\dots<\nu_\ell\leq n} (W+A_{\nu_1}+A_{\nu_2}+\dots+A_{\nu_\ell})(m).$$

Let us decompose $\sigma(m;A_1, ..,A_n)$ into three parts $\sigma(m;A_1, ..,A_n) = \alpha(m)+\beta(m)+\gamma(m)$; $\alpha(m)$ is the contribution of the terms which involve neither A_τ nor A_n; furthermore $\beta(m)$ is the contribution of the terms which involve both A_τ and A_n and finally $\gamma(m)$ is the contribution of the terms which involve one and only one of the two sets A_τ and A_n.
If A_ν $(1\leq \nu \leq n)$ is replaced by C_ν, the functions $\alpha(m)$, $\beta(m)$ and $\gamma(m)$ become $\alpha^*(m)$, $\beta^*(m)$ and $\gamma^*(m)$, so that

$\sigma(m,C_1,\dots,C_n) = \alpha^*(m)+\beta^*(m)+\gamma^*(m)$

Since $\alpha(m)$ depends only on the choice of the sets $A_\nu = C_\nu$ ($\nu\neq\tau$ and $\neq n$), we have $\alpha(m) = \alpha^*(m)$.
The function $\beta(m)$ is a sum of terms of the form $(U+A_\tau+A_n)(m)$ and $\beta^*(m)$ is the corresponding sum of the terms $(U+C_\tau+C_n)(m)$. As we have seen above, $C_\tau+C_n$ is a subset of $A_\tau+A_n$, so that $U+C_\tau+C_n$ is a subset of $U+A_\tau+A_n$, therefore $\beta^*(m)\leq\beta(m)$. It is therefore sufficient to show that $\gamma^*(m)\leq\gamma(m)$, for each positive m. The function $\gamma(m)$ can be

written as a sum of terms of the form

$$(V+A_\tau)(m) + (V+A_n)(m),$$

whereas $\gamma^*(m)$ is the corresponding sum of the terms

$$(V+C_\tau)(m) + (V+C_n)(m)$$

In this way we see that it is sufficient to prove that

$$(V+C_\tau)(m) - (V+A_\tau)(m) \leqq (V+A_n)(m) - (V+C_n)(m)$$

Consequently it is sufficient to show that each element $h \leqq m$ of $V+C_\tau$ which does not occur in $V+A_\tau$ has the property that $h-e$ is an element of $V+A_n$ which does not occur in $U+C_n$. This number h has the form $v+c_\tau$, where v belongs to V and where c_τ belongs to C_τ but not to A_τ. Consequently c_τ is one of the numbers added to A_τ, so that is has the form $e+a_n'$, where a_n' is one of the numbers cancelled in A_n, so that a_n' occurs in A_n but not in C_n. The number $h-e = v+c_\tau-e = v+a_n'$ occurs in $V+A_n$. If $h-e$ would belong to $V+C_n$, it would have the form $v^*+c_n^*$, where v^* and c_n^* belong respectively to V and C_n. Then

$$e+c_n^* = h-v^* \leqq h \leqq m < g$$

Since c_n^* is an element of A_n which is not cancelled in A_n, $e+c_n^*$ is an element a_τ of A_τ; consequently $h = v^*+e+c_n^* = v^*+a_\tau$ would belong to $V+A_\tau$, which is not the case.

Finally we must show: if the set $(A_1,\ldots,A_n)$ satisfies inequality (1) for each positive $m \leqq k$, then the set

$(C_1, \ldots, C_n)$ satisfies (11) for each positive $m \leq k$. We have transformed the set $(A_1, \ldots, A_n)$ by a certain transformation into the set $(C_1, \ldots, C_n)$. I decompose this transformation into elementary transformations as follows. Let p be one of the elements of A_n such that e+p does not belong to A_τ. Let F_n be the system A_n without this element, and let F_τ be the system A_τ to which e+p has been added. Choose $F_\nu = A_\nu$ for $1 \leq \nu \leq n-1$, $\nu \neq \tau$. I shall show: if the set $(A_1, \ldots, A_n)$ satisfies inequality (1) for each positive $m \leq k$, then the set $(F_1, \ldots, F_n)$ satisfies for each positive $m \leq k$ the inequality

(12) $F_1(m)+\ldots+F_n(m) \geq \varphi(m)-\psi(m)+1-n$.

The proof of the principal theorem is established as soon as we have obtained this result. That is clear if for each element f_n of F_n the sum $e+f_n$ is an element of F_τ, since in that case the systems C_ν ($\nu = 1, \ldots, n$) is according to its definition identical with F_ν. Let us therefore consider the case that F_n contains an element f_n' such that $e+f_n'$ does not belong to F_τ. From the minimum property of e and from the fact that A_ν is a subset of C_ν ($\nu = 1, \ldots, n-1$) and that C_n is a subset of A_n it follows that e is the smallest number such that a positive integer $\lambda \leq n-1$ exists with the property that F_n contains at least one element f_n' such that $e+f_n'$ does not belong to F_λ. We can therefore transform the set $(F_1, \ldots, F_n)$ by an elementary transformation with the same numbers e and τ into a new set $(G_1, \ldots, G_n)$ and so on

Continuing in this way we obtain after a finite number of elementary transformations the set $(C_1, \ldots, C_n)$, mentioned above. Inequality (1) remains true if $(A_1, \ldots, A_n)$ is replaced by $(F_1, \ldots, F_n)$, also if $(A_1, \ldots, A_n)$ is replaced by $(G_1, \ldots, G_n)$, and so on, so that the inequality holds also, if $(A_1, \ldots, A_n)$ is replaced by $(C_1, \ldots, C_n)$.

In this way we come to the last part of the proof, namely: if (1) holds for each positive $m \leq k$, then (12) holds also for each positive $m \leq k$. If $k \geq e+p$, I may suppose that this result has already been proved in the case that k is replaced by e. I divide this last part into three steps

I. If $p < m \leq e+p$, then

$$(13) \quad A_\nu(m) \geq A_\nu(m-p) + A_\nu(p)+1 \quad (\nu = 1, \ldots, n-1)$$

<u>Proof</u>. From the minimum property of e it follows that each element $a_\nu < e$ in A_ν $(1 \leq \nu \leq n-1)$ satisfies the condition that $a_\nu+p$ belongs to A_ν. The number of all elements $< m-p$ of A_ν is therefore at most equal to the number of elements $\geq p$ and $< m$ of A_ν, so that

$$A_\nu(m-p) \leq A_\nu(m) - A_\nu(p).$$

II. If $k \geq e+p$, then we have for each positive $m \leq e$

$$\sum_{\nu=1}^{n-1} A_\nu(m) \geqq \varphi(m)+1-n.$$

Proof. We know that there exists a subset T of A_n which contains the number zero and which satisfies for each positive $m \leqq e$ the inequality

$$\sum_{\nu=1}^{n-1} A_\nu(m) + T(m) \geqq \varphi(m)+1-n;$$

for instance the system A_n itself possesses these properties. Let T be a smallest subset of A_n with these properties. It is sufficient to show that T does not contain a positive number, for in that case $T(m) = 0$. Let us suppose for a moment that T contains at least one positive number, so that it is possible to transform the set $(A_1, \dots, A_{n-1}, T)$ into a set $(J_1, \dots, J_n)$ by an elementary transformation. We have assumed in the case $k \geqq e+p$ that the required proof has already been given with e instead of k. That means that the inequalities (1), valid for each positive $m \leqq e$ imply

$$\sum_{\nu=1}^{m} J_\nu(m) \geqq \varphi(m)+1-n,$$

for each positive $m \leqq e$.

The elementary transformation applied on the set $(A_1, \dots, A_{n-1}, T)$ has cancelled in T a certain positive element t and added to one of the systems A_ν $(\nu = 1, \dots, n-1)$ a number of the form e'+t. From the minimum property of e and from the fact that T is a subset of A_n it follows that $e \leq e'$, so that the added element e'+t is $> e' \geq e$. Consequently

$$J_\nu(m) = A_\nu(m)$$

for $\nu = 1, \ldots, n-1$ and for each positive $m \leq e$. In this manner we obtain

$$\sum_{\nu=1}^{n-1} A_\nu(m) + J_n(m) \geq \varphi(m)+1-n$$

for each positive $m \leq e$. This is impossible, since J_n is a proper subset of a smallest system T with this property.

<u>End of the proof</u>. We must show that the inequalities (1), valid for each positive $m \leq k$ imply the inequalities (12) for each positive $m \leq k$. This assertion is clear for the positive numbers $m \leq p$, since below p the sets A_ν and F_ν $(1 \leq \nu \leq n)$ are identical. The assertion is also evident for the numbers $m > e+p$, for in that case we have lost in A_n one term, namely p but we have gained in A_τ also one term, namely e+p. It is therefore sufficient to consider the case $p < m \leq e+p$. Then, according to I

$$\sum_{\nu=1}^{n-1} A_\nu(m) \geq \sum_{\nu=1}^{n-1} A_\nu(m-p) + \sum_{\nu=1}^{n-1} A_\nu(p)+n-1.$$

We have

$$F_\nu(m) \geq A_\nu(m) \ (\nu = 1, \ldots, n-1) \text{ and } F_n(m) \geq F_n(p)=A_n(p),$$

since F_n and A_n are identical below p. Consequently

$$\sum_{\nu=1}^{n} F_\nu(m) \geqq \sum_{\nu=1}^{n-1} A_\nu(m-p) + \sum_{\nu=1}^{n} A_\nu(p)+n-1.$$

The first term on the right hand side is according to II (applied with m-p instead of m) at least equal to $\varphi(m-p)+1-n$ and the second term is by hypothesis $\geqq \varphi(p)+1-n$. Consequently

$$\sum_{\nu=1}^{n} F_\nu(m) \geqq \varphi(m-p) + \varphi(p)+1-n \geqq \varphi(m)+1-n,$$

since $\varphi(m)$ changes slowly.

This completes the proof of the principal theorem.

In the preceding part of this scriptum we have restricted ourselves to numbers with weight 1, but we may attribute to each positive number m a positive weight $\psi(m)$. Let us denote by $A(m,u)$ the sum

$$A(m,u) = \sum_{0<a<m} \psi(a+u)$$

extended over the positive elements $a < m$ of the system A.

<u>Theorem.</u>

<u>Let $A_1,\ldots,A_n$, H be systems such that the number zero belongs to each of the systems $A_1,\ldots,A_n$, that A_n contains at least one positive number and that H is not empty.</u>

<u>Assume that $\psi(m)$ is positive and monotonically not-decreasing for positive m. Let k be positive.</u>

Assume

$$A_1(m,u) + A_2(m,u) + \dots + A_n(m,u) \geq \varphi(m+u) - \varphi(u) + \omega(u)$$

for each positive $m < k$ and for each u which is equal to an element of H augmented by different elements $< k$ of A_n; here $\varphi(m)$ is supposed to be real for positive m and

$$\omega(u) \geq -(n-1)\psi(u).$$

Then the systems $B_1, \dots, B_{n-1}$, constructed in the principal theorem, satisfy the inequality

$$B_1(m,h) + \dots + B_{n-1}(m,h) \geq \varphi(m+h) - \varphi(h) + \omega(h)$$

for each positive $m < k$ and for each element h of H.

Notice that the condition that $\varphi(m)$ changes slowly is not required in this theorem and that the theorem does not contain a result on symmetric functions The proof, which is practically the same as the end of the proof of the principal theorem, is also divided into three parts.

I. If $p < m \leq e+p$, then we have for each $u \geq 0$

$$A_\nu(m,u) \geq A_\nu(m-p,p+u) + A_\nu(p,u) + \psi(u+p).$$

Proof. We know that each element $a_\nu < e$ in A_ν $(1 \leq \nu \leq n-1)$ has the property that $a_\nu + p$ belongs to A_ν. Consequently

$$\sum_{0 \leq a_\nu < m-p} \psi(a_\nu+p+u) \leq \sum_{p \leq a_\nu < m} \psi(a_\nu+u) =$$

$$= \sum_{a_\nu < m} \psi(a_\nu+u) - \sum_{a_\nu < p} \psi(a_\nu+u),$$

hence

$$\psi(u+p) + A_\nu(m-p, p+u) = \sum_{0 \leq a_\nu < m-p} \psi(a_\nu+p+u)$$

$$\leq A_\nu(m,u) - A_\nu(p,u).$$

II. If $k \geq e+p$, then we have for each positive $m \leq e$ and for each u which is equal to an element of H augmented by different elements of A_n

$$\sum_{\nu=1}^{n-1} A_\nu(m,u) \geq \varphi(m+u) - \varphi(u)+\omega(u).$$

The proof is the same as in the principal theorem.

III. End of the proof. It is sufficient to show

$$(14) \quad \sum_{\nu=1}^{n} F_\nu(m,u) \geq \varphi(m+u) - \varphi(u)+\omega(u)$$

for each positive $u \leq k$ and for each u which is equal to an element of H augmented by different elements of F_n. Then u and also u+p is equal to an element of H augmented by different elements of A_n, so that

$$(15) \quad \sum_{\nu=1}^{n} A_\nu(m,u) \geq \varphi(m+u) - \varphi(u)+\omega(u)$$

and

$$(16) \quad \sum_{\nu=1}^{n} A_\nu(m,u+p) \geq \varphi(m+u+p) - \varphi(u+p)+\omega(u+p).$$

It is clear that (14) follows from (15) for each positive $m \leqq p$, since below p the systems A_ν and F_ν $(\nu = 1,\ldots,n)$ are identical. That is also the case for the numbers $m > e+p$, for then we loose in $A_n(m,u)$ one term, namely $\psi(p+u)$ and we gain in $A_\tau(m,u)$ the term $\psi(e+p+u) \geqq \psi(p+u)$. It is therefore sufficient to consider the case $p < m \leqq e+p$. We find by means of I

$$\sum_{\nu=1}^{n-1} A_\nu(m,u) \geqq \sum_{\nu=1}^{n-1} A_\nu(m-p,p+u) + $$

$$+ \sum_{\nu=1}^{n-1} A_\nu(p,u) + (n-1)\,\psi(p+u)$$

and we have

$$F_\nu(m,u) \geqq A_\nu(m,u) \quad (1 \leqq \nu \leqq n-1)$$

and

$$F_n(m,u) \geqq F_n(p,u) = A_n(p,u),$$

since F_n and A_n are identical below p. Consequently

$$(17) \quad \sum_{\nu=1}^{n} F_\nu(m,u) \geqq \sum_{\nu=1}^{n-1} A_\nu(m-p,p+u) + $$

$$+ \sum_{\nu=1}^{n} A_\nu(p,u) + (n-1)\,\psi(p+u).$$

The first term on the right hand side is according to (16) (applied with m-p instead of m) at least equal to

$$\varphi(m+u)-\varphi(p+u)+\omega(p+u) \geqq \varphi(m+u)-\varphi(p+u)+(n-1)\psi(p+u).$$

Combining this result with (17) and (15) we find

$$\sum_{\nu=1}^{n} F_\nu(m,u) \geqq \varphi(m+u)-\varphi(u)+\omega(u).$$

This establishes the proof.

In order to find a suitable application I deduce the following lemma.

Lemma. Let k be an integer $\geqq 2$; suppose

(18) $\psi(m) > 0$; $\psi^2(m+1) \geqq \psi(m)\psi(m+2)$ and $\chi(m) \geqq 0$

for each positive integer m. If the systems $A_1, \ldots, A_n$ consisting of integers $\geqq 0$ satisfy for $m = 2, 3, \ldots, k$ the inequality

$$(19) \quad \sum_{0<a_1<m} \psi(a_1) + \sum_{0<a_2<m} \psi(a_2) + \ldots + \sum_{0<a_n<m} \psi(a_n) \geqq$$

$$\geqq \sum_{h=1}^{m-1} \chi(h)\,\psi(h),$$

then we have for the same values of m and each integer $u \geqq 0$

$$(20) \quad \sum_{0<a_1<m} \psi(a_1+u) + \ldots + \sum_{0<a_n<m} \psi(a_n+u) \geqq$$

$$\geqq \sum_{h=1}^{m-1} \chi(h)\,\psi(h+u).$$

Proof. Put $t_1 = 1$. Let $t_2 < t_3 < \ldots < t_\varrho$ be the integers > 1 and $< m$ belonging to at least one of the systems $A_1, \ldots, A_n$. Choose $t_{\varrho+1} = m$. Inequality

(19), applied successively with $t_2, t_3, \ldots, t_{\varrho+1}$ instead of m gives the ϱ following inequalities

$$\lambda_1 \psi(t_1) \geq \sum_{h=1}^{t_2-1} \chi(h) \psi(h);$$

$$\lambda_1 \psi(t_1) + \lambda_2 \psi(t_2) \geq \sum_{h=1}^{t_3-1} \chi(h)\psi(h);$$

- - - - - - - - - - - - - - - - - - -

$$\lambda_1 \psi(t_1) + \ldots + \lambda_\varrho \psi(t_\varrho) \geq \sum_{h=1}^{t_{\varrho+1}-1} \chi(h)\psi(h);$$

here λ_ν denotes the number of systems $A_1, \ldots, A_n$ containing t_ν, so that the left hand side of (20) is equal to

$$1 = \lambda_1\psi(t_1+u) + \lambda_2\psi(t_2+u) + \ldots + \lambda_\varrho\psi(t_\varrho+u).$$

In order to obtain for this sum an appropriate lower bound, I multiply the sides of the ϱ inequalities respectively by

$$\frac{\psi(t_1+u)}{\psi(t_1)} - \frac{\psi(t_2+u)}{\psi(t_2)}; \ldots; \frac{\psi(t_{\varrho-1}+u)}{\psi(t_{\varrho-1})} - \frac{\psi(t_\varrho+u)}{\psi(t_\varrho)}; \frac{\psi(t_\varrho+u)}{\psi(t_\varrho)}$$

All these factors are ≥ 0 since it follows from (18) that

$$\frac{\psi(t+1)}{\psi(t)} - \frac{\psi(t+2)}{\psi(t+1)} \geq 0.$$

Adding we find

$$1 \geqq \frac{\psi(t_1+u)}{\psi(t_1)} \sum_{h=t_1}^{t_2-1} \chi(h)\,\psi(h) + \ldots$$

$$\ldots + \frac{\psi(t_\varrho+u)}{\psi(t_\varrho)} \sum_{h=t_\varrho}^{t_{\varrho+1}-1} \chi(h)\,\psi(h).$$

In the first sum we have according to (18)

$$\frac{\psi(t_1+u)}{\psi(t_1)}\,\psi(h) \geqq \psi(h+u),$$

in the second sum

$$\frac{\psi(t_2+u)}{\psi(t_2)}\,\psi(h) \geqq \psi(h+u),$$

and so on, so that

$$1 \geqq \sum_{h=t_1}^{t_2-1} \chi(h)\,\psi(h+u) + \ldots + \sum_{h=t_\varrho}^{t_{\varrho+1}} \chi(h)\,\psi(h+u) =$$

$$= \sum_{h=1}^{m-1} \chi(h)\,\psi(h+u),$$

since $t_1 = 1$ and $t_{\varrho+1} = m$. This establishes the proof.

This lemma enables us to deduce the following theorem.

Theorem.

Suppose that A_ν $(\nu = 1,\ldots,n)$ is formed by integers $\geqq 0$ and contains the integer zero. Assume

<u>for each positive integer m</u>

(21) $\psi(m+1) \geqq \psi(m) > 0; \ \psi^2(m+1) \geqq \psi(m)\,\psi(m+2);$

$\chi(m) \geqq \chi(m+1) \geqq 0.$

Let k denote an integer $\geqq 2$. If

(22) $$\sum_{0<a_1<m} \psi(a_1) + \sum_{0<a_2<m} \psi(a_2) + \ldots + \sum_{0<a_n<m} \psi(a_n) \geqq$$

$$\geqq \sum_{h=1}^{m-1} \chi(h)\,\psi(h) \qquad (m = 2,3,\ldots,k),$$

then the sumsystem $S = A_1 + \ldots + A_n$ satisfies the inequalities

(23) $$\sum_{0<s<m} \psi(s) \geqq \sum_{h=1}^{m-1} \chi(h)\psi(h) \qquad (m = 1,2,\ldots,k)$$

and even for each integer $u \geqq 0$

(24) $$\sum_{0<s<m} \psi(s+u) \geqq \sum_{h=1}^{m-1} \chi(h)\psi(h+u) \qquad (m = 1,2,\ldots,k).$$

The proof runs as follows. According to the lemma we have for each integer $u \geqq 0$

$$\sum_{0<a_1<m} \psi(a_1+u) + \ldots + \sum_{0<a_n<m} \psi(a_n+u) \geqq$$

$$\geqq \sum_{h=1}^{m-1} \chi(h)\,\psi(h+u) \geqq \sum_{h=1}^{m-1} \chi(h+u)\,\psi(h+u),$$

since $\chi(h) \geqq \chi(h+u)$ by (21) This inequality is obvious for $m = 1$, because then the right hand side is equal to zero. In this way we find

$$(25) \qquad \sum_{0 < a_1 < m} \psi(a_1+u) + \ldots + \sum_{0 < a_n < m} \psi(a_n+u) \geqq$$

$$\geqq \varphi(m+u) - \varphi(u),$$

where

$$\varphi(m) = \sum_{h=1}^{m-1} \chi(h)\, \psi(h).$$

We apply this result for each element u of the system H_n formed by the integers $u = u_2+\ldots+u_n$, where u_ν $(2 \leqq \nu \leqq n)$ is the sum of different elements of A_ν; therefore $H_1 = 0$. Inequality (23) is obvious for $n = 1$, so that I may assume that $n \geqq 2$, that A_n contains at least one positive element and that the required property has already been proved for $n-1$ instead of n. According to the last theorem (applied with $H =$ $= H_{n-1}$ and with $\omega(u) = 0$) the systems $B_1,\ldots,B_{n-1}$, constructed in the principal theorem, satisfy for each element u of $A_1+\ldots+A_{n-1}$ and for $m = 1,2,\ldots,k$ the inequality

$$\sum_{0 < b_1 < m} \psi(b_1+u) + \ldots + \sum_{0 < b_{n-1} < m} \psi(b_{n-1}+u) \geqq$$

$$\geqq \varphi(m+u) - \varphi(u).$$

According to our induction hypothesis the sum-system $T = B_1+\ldots+B_{n-1}$ satisfies therefore the inequality

$$\sum_{0<t<m} \psi(t) \geq \varphi(m) - \varphi(0) = \varphi(m) = \sum_{h=1}^{m-1} \chi(h)\, \psi(h).$$

Since T is a subset of $S = A_1+\ldots+A_n$, this last system satisfies certainly the required inequality (23).

The preceding lemma, applied with $n = 1$ and with S instead of A_1 gives immediately formula (24)

This completes the proof.