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On the geometrical representation of ele-
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between geometry and physics

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ON THE GEOMETRICAL REPRESENTATION OF ELEMENTARY PHYSICAL OBJECTS AND THE RELATIONS BETWEEN GEOMETRY AND PHYSICS

BY

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(Amsterdam)

Dedicated to Prof. Dr J. A. SCHOUTEN, on the occasion of his 70th birthday

1. *Introduction.* Perhaps the most characteristic feature of SCHOUTEN's methods throughout his work is his perfect matching of a notational system to geometric intuition. He never adopts such a system for its formal beauty alone, and thereby escapes the danger, pointed out by HERMANN WEYL, to fall into "Orgien des Formalismus". On the other hand he is never content with a geometrical investigation which does not reflect in its description the true geometrical background, i.e. its most general properties of invariance.

Every mathematical method of great generality becomes too clumsy if it is applied to very simple problems only. In the same way, as long as *only* linear and quadratic forms occur, ordinary matrix calculus has some undeniable advantages, in particular from a purely formal point of view, although it does not reveal the geometric background by not distinguishing e.g. in space point-point transformations from point-plane-transformations (e.g. polarities) T_{ij} . Also, as long as only (or mainly) skew-commutative differential forms occur, CARTAN's ω -methods have the advantage of great conciseness. But for the field of geometry *as a whole*, including the more intricate cases where geometric objects with all kinds of symmetry and all kinds of invariance occur, SCHOUTEN's "nucleus-index-method" has not been surpassed.

One of the simplest results of SCHOUTEN's methods is his classification of the quantities in a Euclidean n -dimensional space E_n under the affine group.

The purpose of the present paper is: to show that this classifica-

tion, the importance of which in geometrical and mathematical-physical researches is well known, is useful for a better understanding of the quantities occurring in *elementary physics* also, in particular from an epistemological point of view.

For simplicity we shall restrict ourselves in the main to classical (non-relativistic and non-quantum-mechanical) physics. For the same reason we shall not assume anything to be known of RICCI calculus, and we shall not use SCHOUTEN's nucleus-index-method, but sketch roughly and genetically a system of notation, which is appropriate for an elementary introduction, but which — as will be seen by the reader — becomes too clumsy when we pass to the less simple physical relations, and then can better be replaced by SCHOUTEN's notation, to which it gives a rather natural access.

The present considerations do not have the pretention to be preferable for the purposes of everyday physical research or instruction to the ordinary development, in which a free use is made of metrical geometry and the corresponding vector- and/or tensor-calculus. For such purposes the ordinary methods are quite satisfactory. But it will be found interesting to look, how far we can come in physics *without* making use of any metrical assumptions at all, and restrict the geometrical tools used to those provided by affine geometry. From a philosophical point of view the replacement of metrical by affine geometry, of course, might seem a rather half-hearted procedure, but it serves here only as an elementary introduction to a more consistent sifting out of metrical elements, which may be based on similar principles, but requires a more intimate knowledge of RICCI-calculus techniques than we presuppose here.

Also in order to keep things as simple as possible we shall work with ordinary (threedimensional) Euclidean space, and assume the concepts of affine geometry to be known (straight lines, planes, parallel lines and/or planes; ratio of parallel directed line-segments, ratio of areas of bounded oriented parts of parallel planes, etc., and to satisfy the ordinary axioms.

2. *Examples of affine treatment of physical concepts. a. Vectors; displacements.* A contravariant vector (shortly: a vector) in Euclidean space is defined ¹⁾ as an ordered pair of points determined except for a simultaneous translation of both points.

¹⁾ SCHOUTEN [1], I, § 2, [2], p. 10.

We shall consider each physical quantity to belong to a definite point of a material body (or, more generally, a definite point in space or space-time). In the present case we may identify it then with the initial point of the ordered pair, so that the latter is completely determined by its endpoint; the vectors then correspond one to one with the points of space (their endpoints). A vector is often denoted by a symbol with an arrow above it.

It is then trivial that a *displacement* of a material point corresponds with and can be represented by a vector \vec{dx} . As in classical physics time occurs as a scalar, it follows immediately that, after choice of a unit of time, the *velocity* \vec{dx}/dt and the *acceleration* $\vec{d^2x}/(dt)^2$ of a material point become vectors also. In relativistic mechanics the same is true for the proper velocity and acceleration \vec{dx}/ds and $\vec{d^2x}/(ds)^2$.

b) **F o r c e s ; c o v e c t o r s.** In classical mechanics the concept of force usually is introduced in connection with the so-called law of NEWTON $\vec{K} = m\vec{a}$. The current treatment suffers from many logical deficiencies, in particular in statics, where no accelerations occur at all. Although this difficulty can be overcome by introducing fictitious accelerations, some difficulties remain in connection with frictional forces which, when working on resting bodies can cause no accelerations at all.

From the point of view of modern physics it seems therefore more natural to introduce the concept of energy first, and then to define a force working e.g. on a material point by means of the amount ΔW of work done if the point suffers a displacement $\vec{\Delta x}$. Then a force determines a scalar ΔW with respect to every (sufficiently small) displacement $\vec{\Delta x}$. Assuming, as usual, this relation to be linear ²⁾, the force³⁾ becomes a linear vector function. Hence, the initial point of the displacement being given, a force can be represented, after choice of a unit of energy, by the plane of all endpoints of displacements for which ΔW has the value 1, or also, if an initial plane parallel to the other one through the initial posi-

²⁾ Here and further we neglect, for simplicity of expression, second order quantities, i.e. e.g. those caused by dependence of the force on place or time.

tion of the material point is drawn, by an ordered pair of planes. This is — apart from the common translations of the planes, which we omit — SCHOUTEN's definition ([1], I, § 2, [2], p. 1) of a covariant vector, or, as we shall shortly call it, a *covector*. We may denote it by a symbol with an arrow below it. In order, however, to simplify the writing, we may notice that the “bars” of the arrows are superfluous, and that it is sufficient to use their heads alone. We shall therefore further denote vectors and covectors by means of arrowheads only: \vec{v} , \vec{F} .

If \vec{F} is a covector and \vec{v} a vector, their *transvection* (or scalar product) is defined as the ratio of the “length” of the vector and the intercept of its working line with the two planes of the covector, taking account of their signs (orientations). The occurrence of the metrical term “length” here is only apparent (and could be avoided), the whole procedure being affinely invariant.

Hence from our point of view the natural representation of a force is by a covector, not a vector. This could also be seen immediately from the Hamiltonian theory.

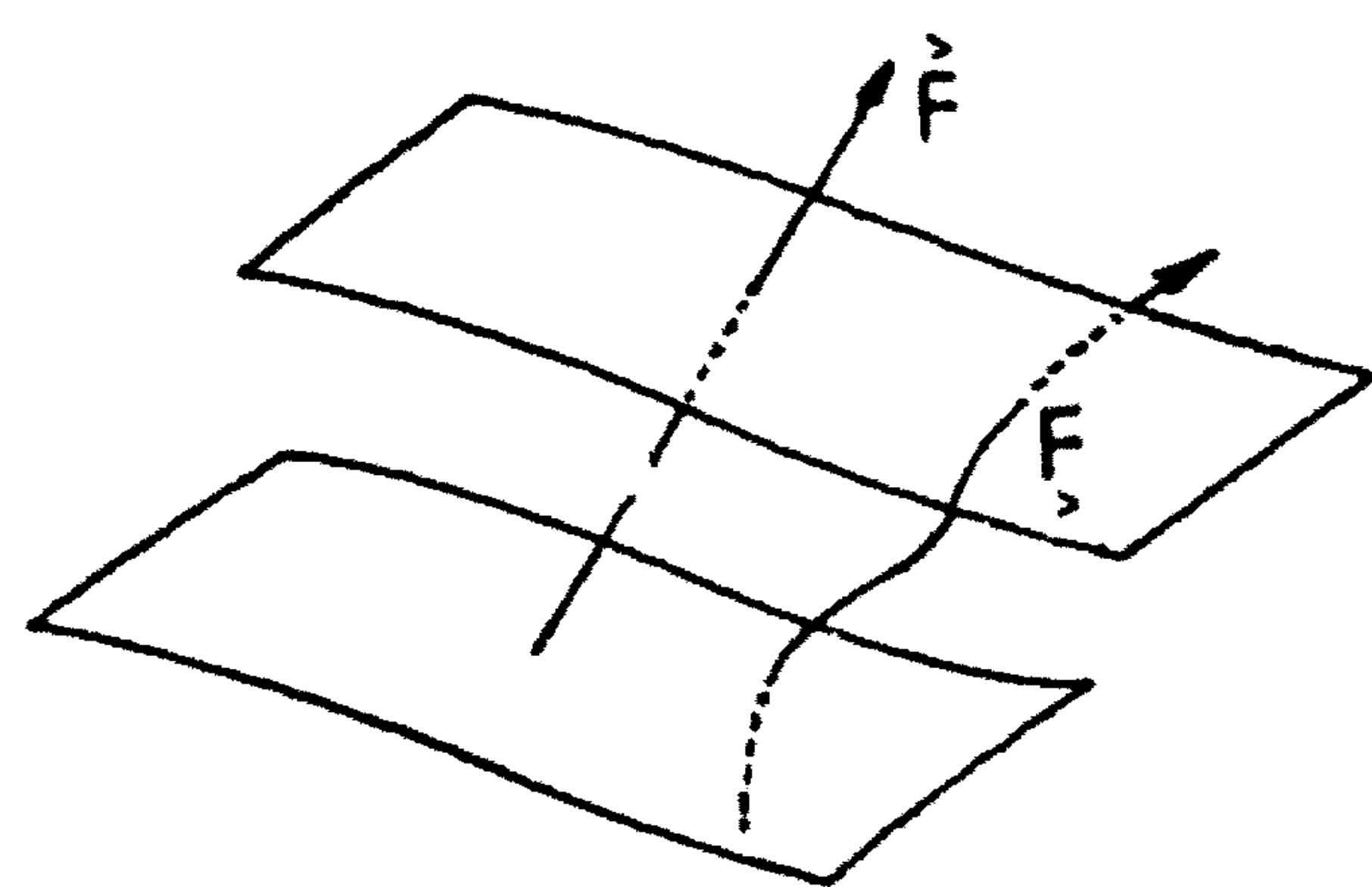


Fig. 1.

The relation of the covector \vec{F} now representing a force and the vector \vec{F} by which it usually is represented is the following one (fig. 1):

- 1°. \vec{F} is orthogonal to \vec{F} and has the same orientation;
- 2°. the length of \vec{F} is the reciprocal value of the distance of the two planes of \vec{F} .

Clearly both relations are of a metrical nature.

We shall show by some examples that the representation of a force by a covector is in several respects (not in all) far more satisfactory and closer to “physical reality” than that by a vector.

a) The work W done by a force under a displacement \vec{r} of a material point is usually formulated as: the product of the displacement and the projection of the force (force-vector) \vec{F} on it, i.e. $|\vec{F}| |\vec{r}| \cos \vartheta$ if ϑ is the angle between the two vectors (fig. 2). By means of \vec{F} the description is far simpler and not dependent on the metric (fig. 3): W is the ratio (provided with a sign according

to the correspondence or non correspondence of the orientations) of \vec{r} and the intercept of the line along which it takes place with the two planes of \vec{F} .

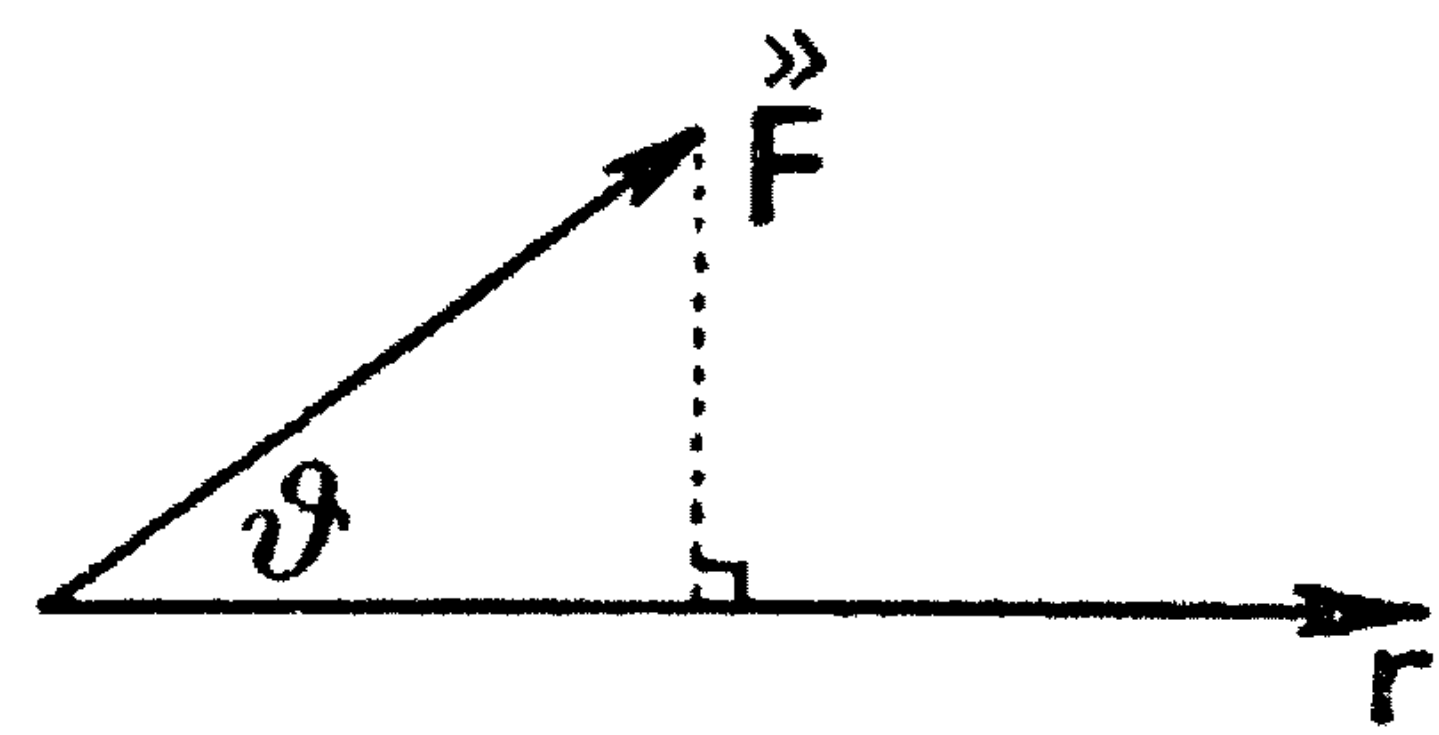


Fig. 2.

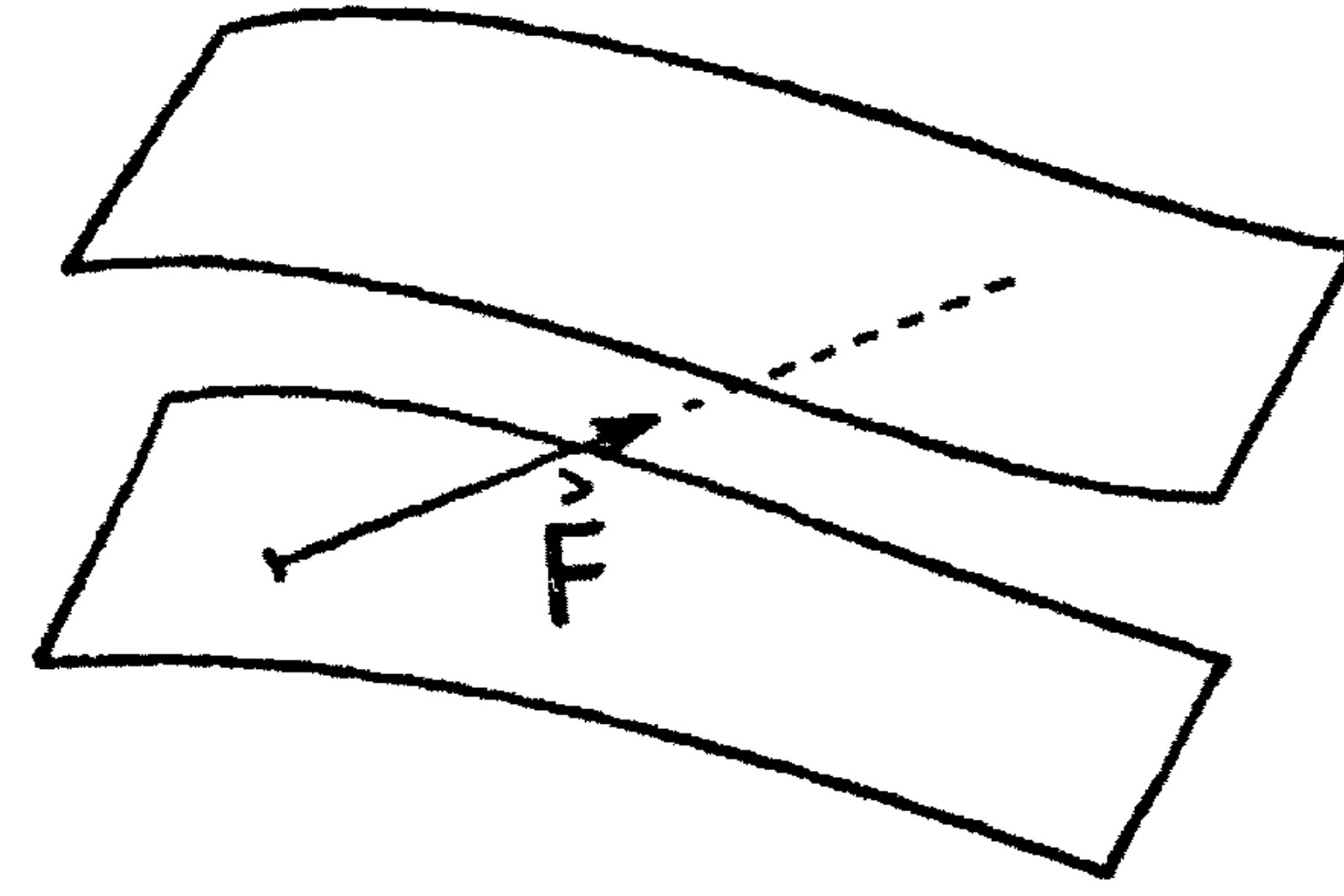


Fig. 3.

β) A conservative field of force can be represented by its equipotential surfaces. Assuming — for simplicity of the verbal description — that the unit of energy is chosen sufficiently small and that the forces vary sufficiently slowly, the force covector is immediately visualized (fig. 4) by a pair of tangent planes in two neighbouring points where the potential differs one unit, i.e. essentially by the equipotential surfaces themselves. The representation by vectors \vec{F} lacks visual evidence.

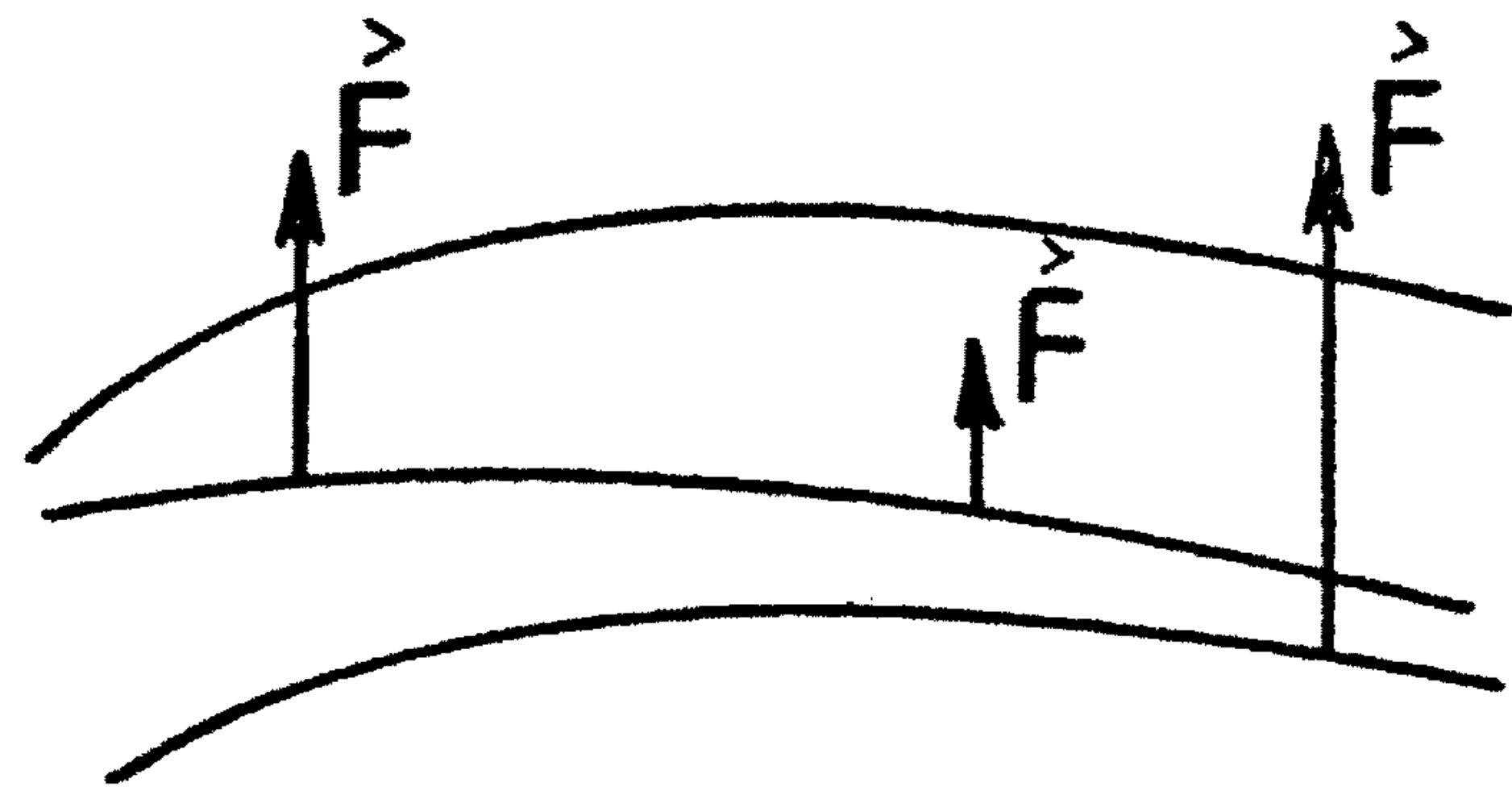


Fig. 4 2a).

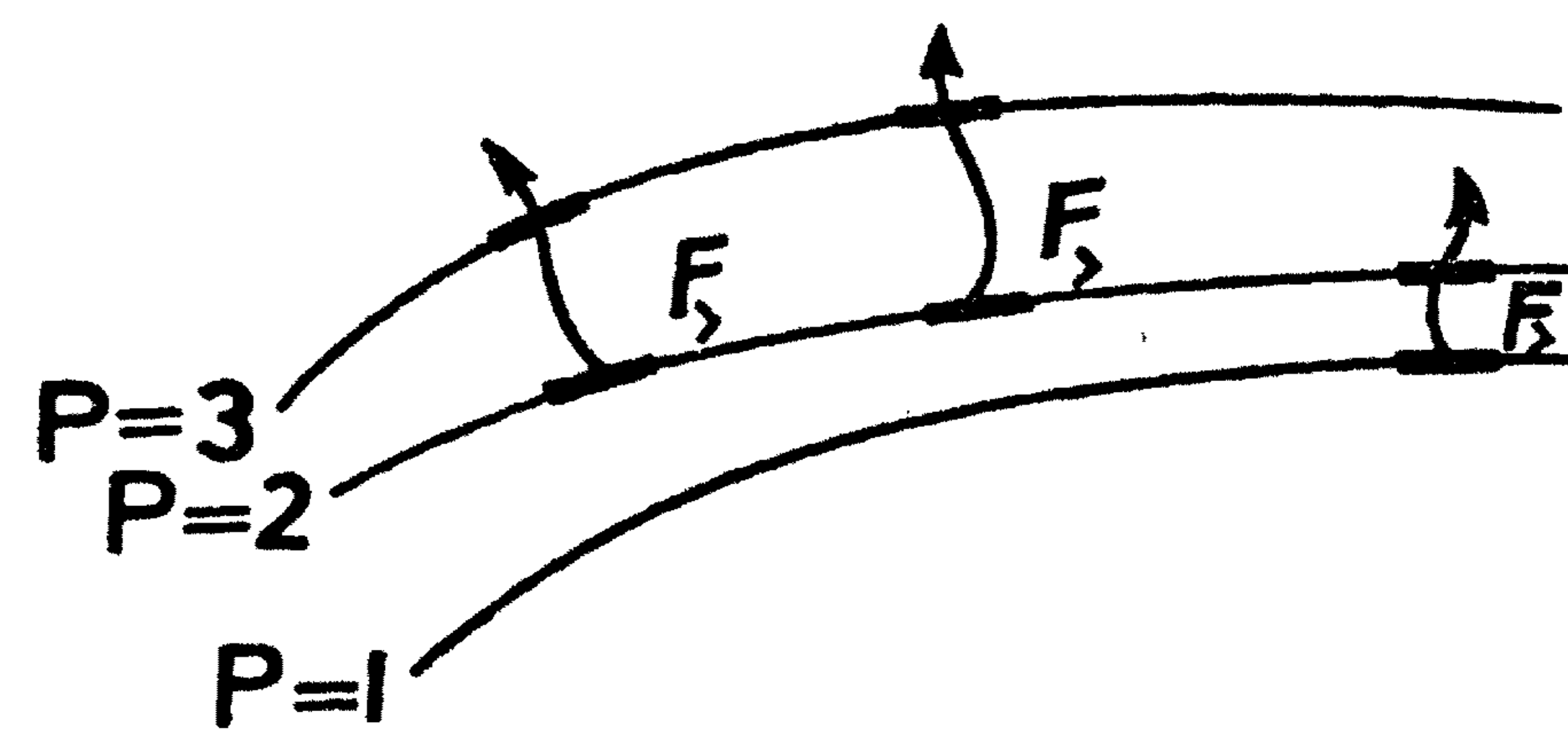


Fig. 5.

The equipotential surfaces possess an immediate physical reality: they are the only surfaces on which, if they are replaced by smooth material surfaces, a material point, not subject to other forces, can remain in rest, or move freely.

γ) A material point on a smooth plane suffers a “normal force” (fig. 5). Here, like in the previous cases, the arrow orthogonal to the surface has nothing to do with physical reality, but is just a mathematical device, useful for those who are better acquainted with metrical than with affine geometry. The direction of the force-covector is not orthogonal, but parallel to, i.e. the same as that of the plane.

^{2a)} The vectors in fig. 4 should be orthogonal to the corresponding curves in their initial points, instead of vertical.

δ) What happens if the surface is rough? Then the reaction force-vector \vec{F} is no longer normal to the surface, but makes an angle φ with it, so that $\operatorname{tg} \varphi = f$ is the friction coefficient. Hence under varying circumstances \vec{F} describes a cone of revolution, the axis of which is orthogonal to the surface. Evidently both concepts: “cone of *revolution*” and “orthogonal” are metrical. The force-covector F envelops a cone having the same two properties. How it is possible that, just by making the plane rough instead of smooth, we have introduced metric?

Evidently this can only be so because we have unconsciously introduced metrical assumptions. The two above-mentioned properties of the cone do *not* follow from the assumption of roughness alone, i.e. do not hold for every rough surface, but only if the roughness is isotropic, i.e. is the same for all directions. If f is different, e.g. in two orthogonal directions, the first property is lost; if it is different in two opposite directions, then also the second property is lost. Examples are: a rough board, or a brush with skew implanted hairs, e.g. a cat.

Finally we mention a few cases where the representation of a force by a covector is *not* the most natural one.

α) Of course we may not expect the distinction between vectors, covectors, etc. to remain tenable if we have to do with rigid bodies, the concept of rigidity being itself of a metrical nature.

β) If a force has a definite “working line” (e.g. a force working on a point of a rigid body), it can better be represented by a vector than by a covector, and thereby requires the introduction of a metric.

γ) The concept of a *central* field of force, dealing with the working lines, is of a metrical nature.

c) M o m e n t u m- a n d w a v e- c o v e c t o r s. Historically NEWTON’s law did not state that the force is the product of mass and acceleration, but that it is the fluxion of the “impetus” (kinetic momentum), which is the product of mass and velocity. Without going into a detailed discussion we may state here the fact that the kinetic (and also the potential and the total) momentum which we denote by $\underset{\sim}{j}$, behaves as a covector. Then NEWTON’s law states that $F = \underset{\sim}{j}$, whereas the relation between momentum and velocity:

$$\underset{\sim}{j} = m\vec{v}$$

is a “linking equation” and implies metric. As it is equivalent with

$\vec{j} = \partial T / \partial \vec{v}$ or with $\vec{v} = \partial T / \partial \vec{j} = \partial H / \partial \vec{p}$ where $T = \frac{1}{2}mv^2 = j^2/2m$ is the kinetic and H the total energy, we can also say that the linking equation expresses the relation between kinetic energy and kinetic momentum. It loses its validity α) in special relatively theory as soon as we have to do with systems consisting of more than one point (because there the concepts of point of gravity and of resulting velocity get lost), and β) in relativistic quantum mechanics, where the relativistic velocity corresponds with DIRAC's matrix vector \vec{a} and \vec{j} with $\vec{p} - \frac{e}{c} \vec{\varphi}$. Between these two quantities no relation at all exists.

Also the wave "vector" in optics is rather a covector. Its planes are tangent to the surfaces of equal phase, and are made visible in interference experiments. The direction of a light-ray has not directly to do with the wave-covector κ (it is orthogonal to κ under conditions of isotropy), but with the energy-current, i.e. POYNTING'S "vector" (which rather is a vector-density cf. e , f); a ray is rather a tube than a line). The quantum theoretical relation $\vec{p} = \hbar \kappa$ does not depend on metric.

The laws of impact (e.g. of a material point against a perfectly elastic wall), of reflection and of refraction consist of two parts: 1°. a condition expressing the conservation of energy and thereby implying metrical geometry, 2°. a "geometrical condition", not of a metrical nature, although it usually is expressed in a metrical form. In fact, one usually says that (in the case of a light-ray) the "normal component" of the wave-"vector" is continuous on the surface of continuity, and, in the case of impact, that the difference of the velocities (momenta) of the point before and after the collision is "orthogonal" to the reflecting surface (fig. 6). Here again the direction orthogonal to the surface is quite foreign to the physical phenomena, which are more appropriately described by

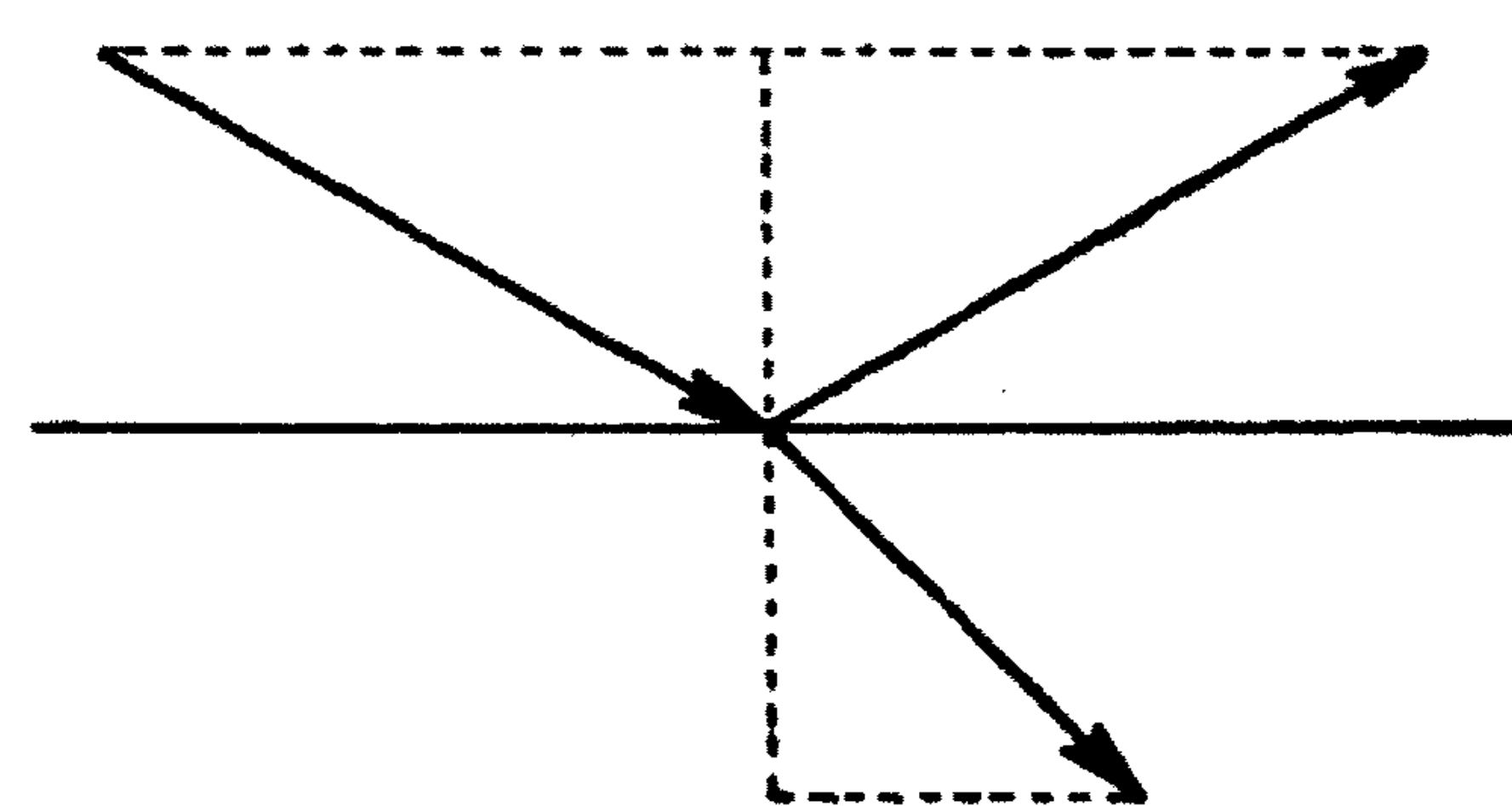


Fig. 6.

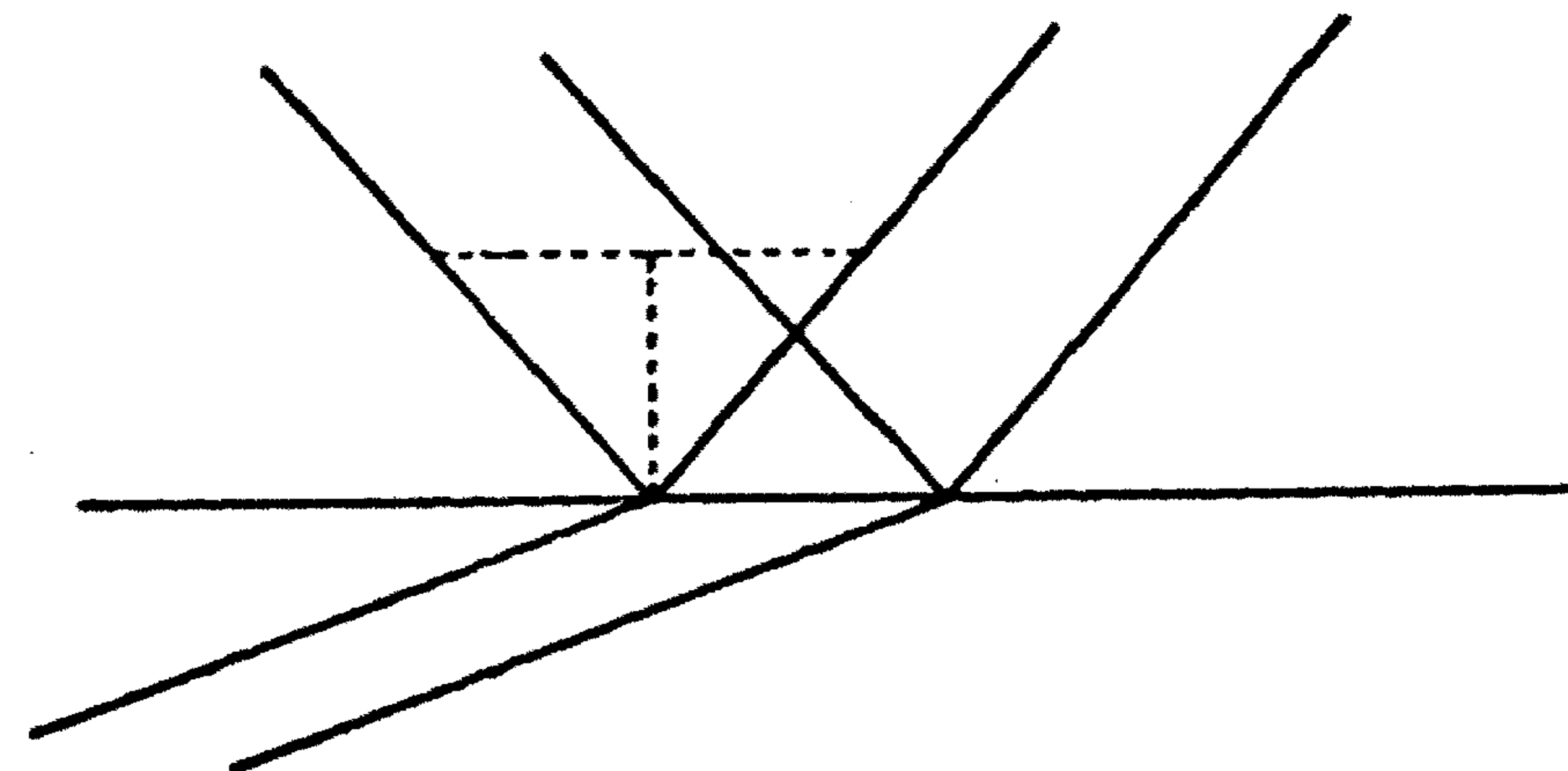


Fig. 7.

the wave- or momentum-covector, which, when passing the discontinuity surface changes by a covector *parallel* to the surface (fig. 7). Hence the *direction* of the difference between the reflected or refracted and the ingoing covector is determined by the geometrical nature of the discontinuity surface alone, and that independent of metrical geometry, whereas its *magnitude* depends on the linking equation, e.g. the energy law. The equality of the angles of the surface with the ingoing and reflected covectors, as well as SNELLIUS' defraction law depends, of course, on the latter, and are thereby of a metrical nature.

d) Electric field and magnetic induction (bivector). An electric field \vec{E} is defined as a force per unit of charges, and can therefore, after choice of a unit of charge, be represented as a covector. Line integrals $\int \vec{E} d\vec{s}$ occurring in the theory of MAXWELL have therefore an invariant meaning (cf. the "transvection", defined under b)).

The magnetic induction has the nature of a bivector (or covector). This is defined ³⁾ as a cylindrical tube, the direction of the generating lines being given, as well as the area and orientation of the intersection with one (and then with any) plane, not parallel to the generating lines (fig. 8).

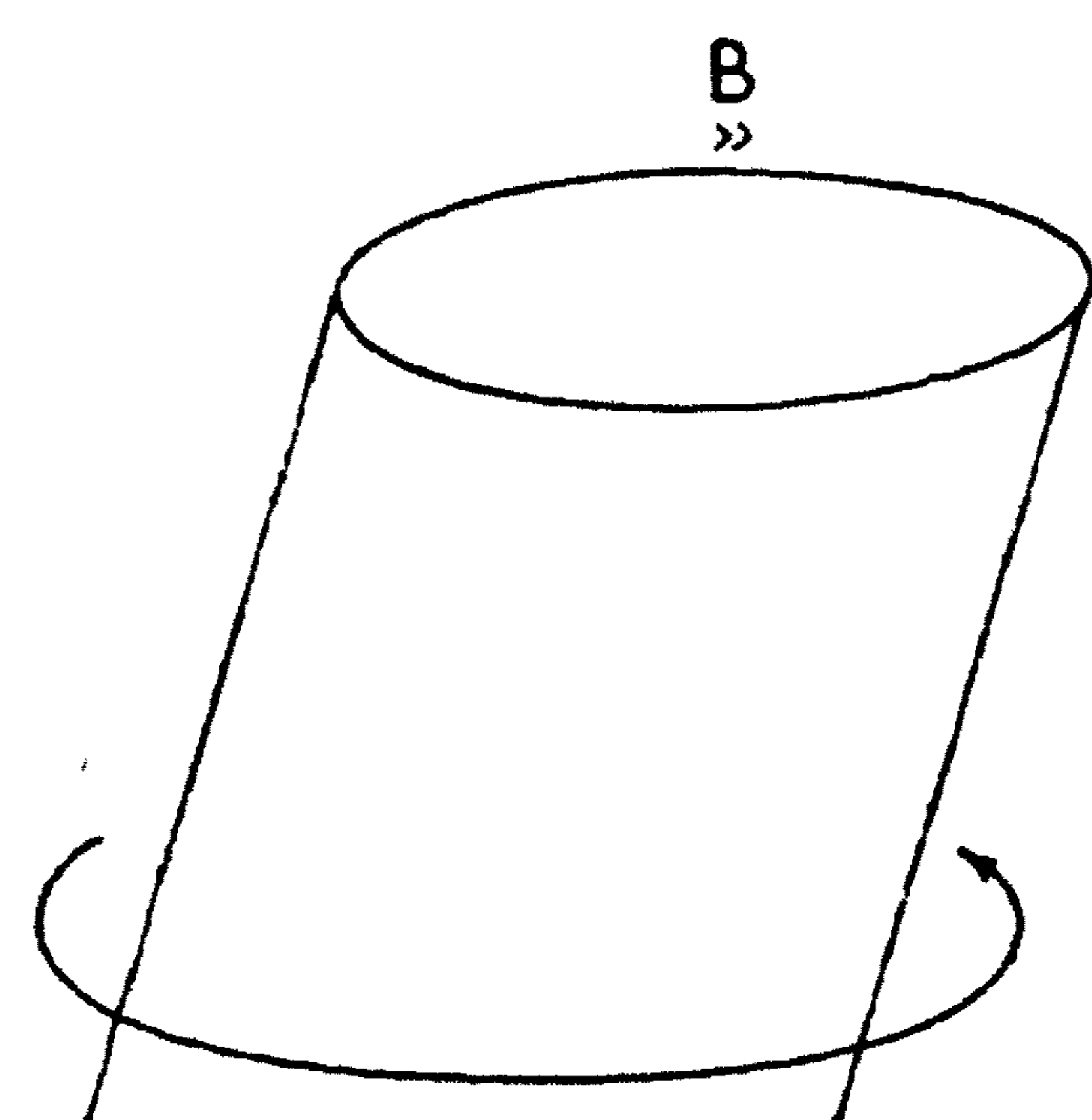


Fig. 8.

We shall denote bi-covectors by symbols with two arrowheads below them, e.g. $\vec{\vec{B}}$.

In metrical geometry it is replaced by a vector \vec{B} parallel to the generating lines, and having a length equal to the area of a cross-section orthogonal to them (and correspondingly directed). Now it is evident that FARADAY's picture of the lines (or tubes) of magnetic force is far better represented by the bi-covector $\vec{\vec{B}}$ than by

the corresponding vector \vec{B} . The orientation around the tube has an immediate physical meaning: it is the orientation of a moving positive charge which might generate the magnetic field.

If $\vec{\vec{O}}$ is a bivector (i.e. an oriented bounded part of a plane, two

³⁾ SCHOUTEN [1], I, § 7, [2], p. 26. Here the term "covariant bivector" is used.

bivectors being identical if their planes are parallel, and the areas and orientations equal), then the transvection $\overset{\gg}{\vec{O}}\vec{B}$ equals the ratio of the area of the intersection of the tube of \vec{B} with the plane of $\overset{\gg}{\vec{O}}$ to the area of $\overset{\gg}{\vec{O}}$, taking account of the two orientations. The surface-integrals occurring in MAXWELL'S theory are of the type $\iint \vec{B} d\overset{\gg}{\vec{O}}$, hence invariant.

e) Mass- and other densities; pressure. Whereas in the metrical form mass-densities occur as scalars, they occur in the non-metrical invariant form as scalar densities ⁴⁾ of weight 1, i.e. under transformation of coordinates they are multiplied with the reciprocal absolute value of the JACOBIAN determinant of the transformation. Geometrically a scalar density of weight +1 or -1 is represented by a bounded part of space, provided with the sign +1 or -1 according to the sign of the scalar density (not of the weight). If the weight is -1 or +1, the value of the scalar density is directly and inversely proportional respectively with the volume of the part of space. Densities of weight +1 and -1 will be denoted by symbols with a tilde respectively above and below it. A volume-element is a scalar density of weight -1. The terminology becomes more closely related to the geometrical representation, if we remind that a scalar-, vector-, bivector- and trivector-density is equivalent with a pseudo-cotriple, -cobivector, -covector and -scalar respectively, and use the latter terminology instead of that of the densities ⁵⁾.

In the present case this means that a mass density $\tilde{\varrho}$ is affinely invariantly described by a part of space of any form, containing just one unit of mass. Hence $\tilde{\varrho}$ and a volume-element dV are represented by the same part of space if $\tilde{\varrho} dV = 1$.

A completely analogous argument holds for a charge-density $\tilde{\varrho}$ and (in non-relativistic physics) an energy-density \tilde{E} , although here the sign must be taken into account.

It is somewhat more surprising that also the pressure \tilde{p} (say of

⁴⁾ We consider here WEYLIAN densities *only*, and therefore drop the letter *W* used by SCHOUTEN [2], p. 31.

⁵⁾ Usually one will prefer to call $\tilde{\varrho}$ a scalar-density rather than a pseudo-co-trivector, but to call a line-element ds a pseudo-vector rather than a bicovector-density.

a gas) is a scalar-density of weight $+1$ and depends on the unit of energy only. This can be seen as follows. The pressure is defined as a force on a unit of area. Let the gas be contained in a closed container, a portion dO ⁶⁾ of the wall of which is replaced by a plane piston in a cylinder. Let the piston be allowed to move backward so far that the pressure exerts one unit of energy. Then the initial and the final position of the piston are the two planes representing the force $F^{dO} = \tilde{p} dO$ exerted on dO , and the pressure itself is represented by the volume contained between these two positions of the piston and the walls of the cylinder (cf. fig. 9). Whereas the “distance” of the planes of K depends on dO , viz. is inversely proportional to it, the volume representing \tilde{p} is independent of it.

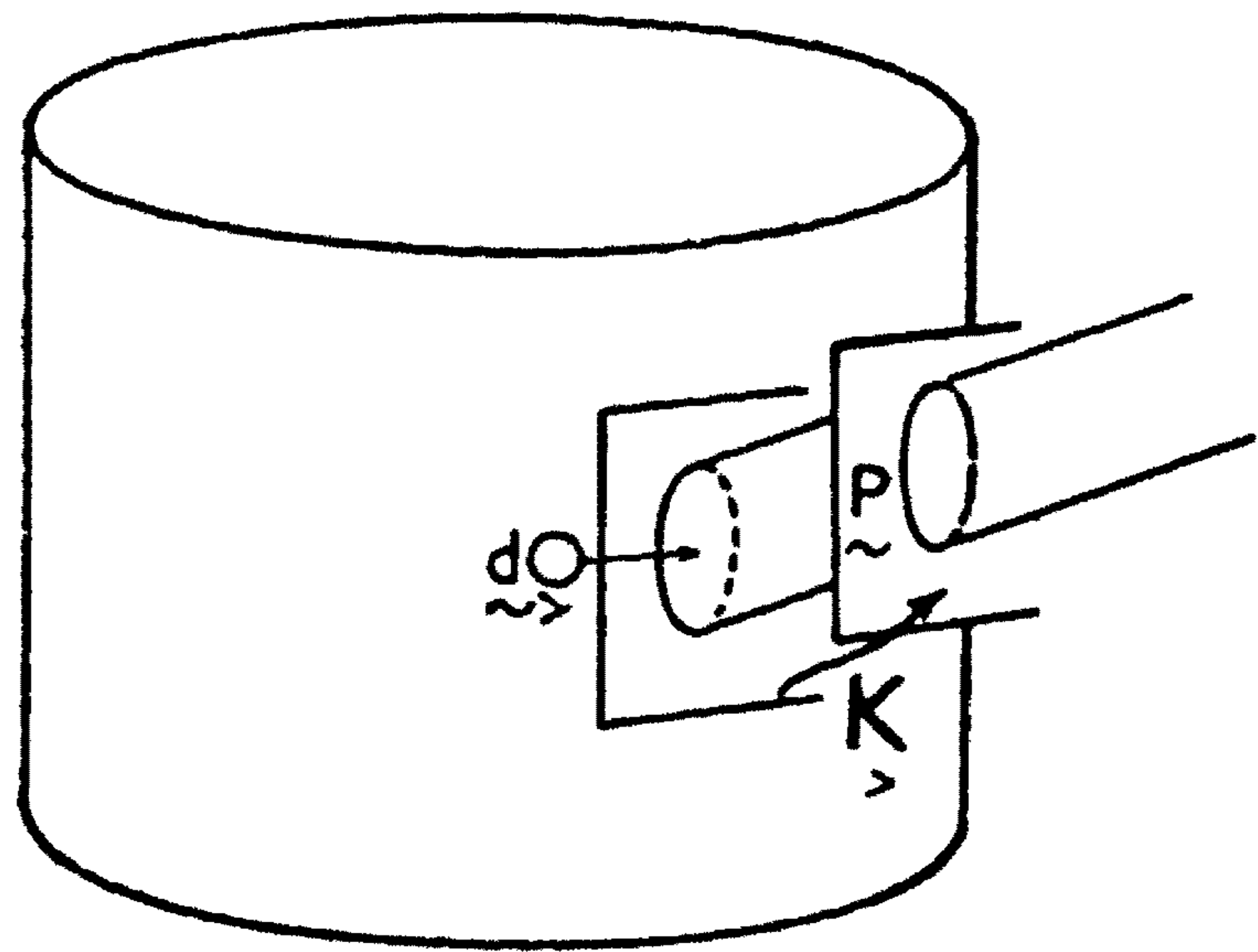


Fig. 9.

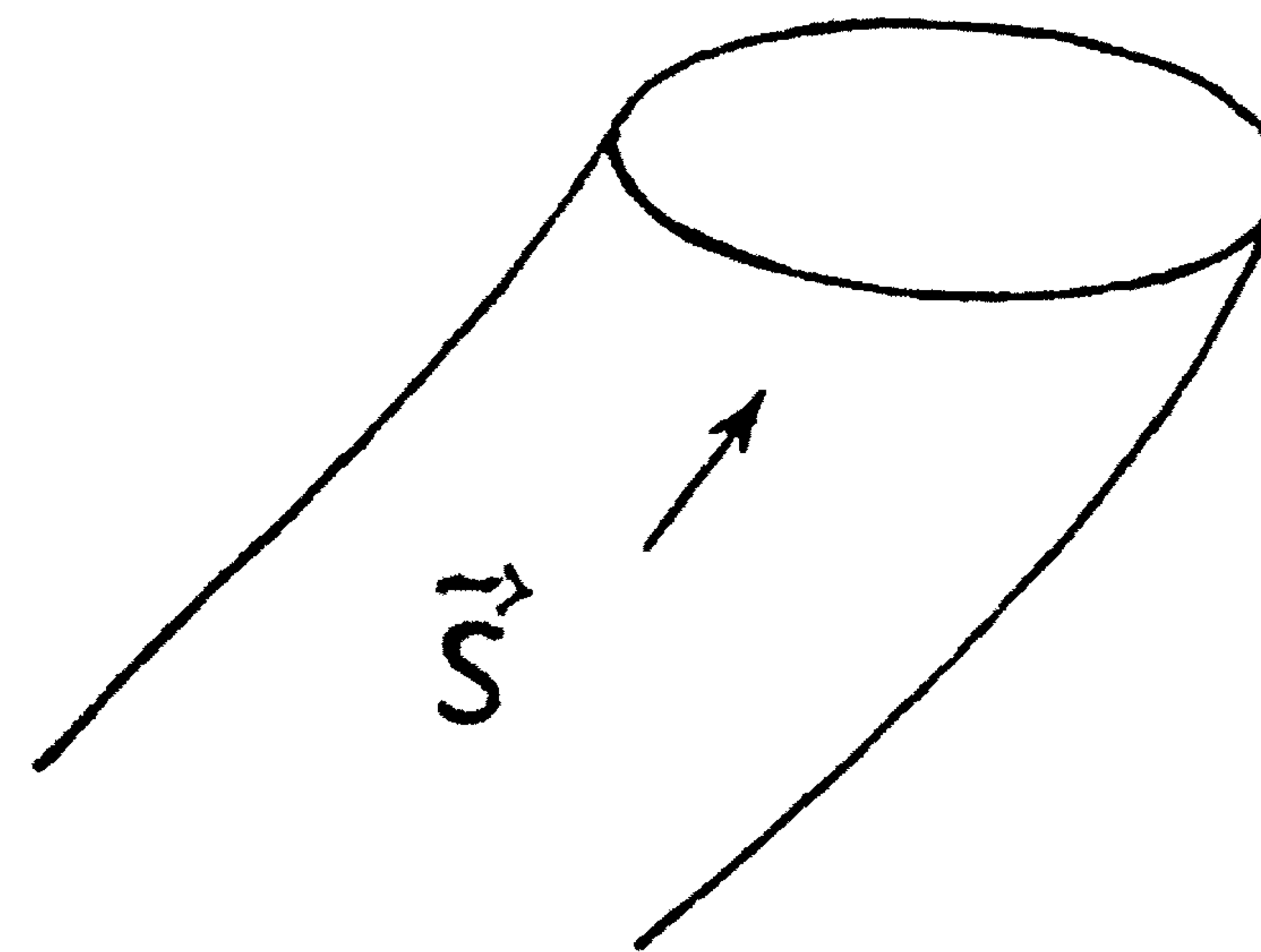


Fig. 10.

f) Current; vector-densities. A current density \vec{S} (be it a mass- or electric current, or — in non relativistic physics — an energy-current) is a vector-density represented by a tube like a co-bivector, but provided with an orientation *along* the generators, not around them (cf. fig. 10). Clearly $\vec{S} dO$, hence also $\iint \vec{S} dO$ is an invariant.

In electromagnetism also the dielectric displacement \vec{D} and — in non-relativistic physics — POYNTING'S vector \vec{P} , are vector-densities.

⁶⁾ A surface element dO is a covector-density; it is represented by a bounded portion of a plane provided with an orientation, not of the plane, but of any line crossing it, i.e. such that a “front” and a “back” side, or an “interior” and an “exterior” side is distinguished.

g) Magnetic field; bivector-densities. The magnetic field \tilde{H} is found to be a bivector-density (or, equivalently, a pseudo-covector), represented by a pair of parallel similarly oriented planes. It occurs together with a line-element ds with an orientation around (not along) it, i.e. a bicovector-density or pseudo-vector.

h) Some fundamental equations of physics. In order to illustrate the coherency of the system of concepts we introduced we write down some of the fundamental equations of physics. In order to show that the tensor-analytic calculus becomes more and more preferable to the ad hoc calculus we introduced here, as the equations become gradually more complicated, we write the equations down according to both formalisms without going into the precise distinction of the different multiplications necessary in our elementary formalism (which difficulties constitute its main disadvantage). (The suffixes i, j, k, l run independently through the numbers 1, 2, 3).

NEWTON:

$$\dot{j} = K \qquad \dot{j}_i = -K_i$$

PLANCK:

$$\dot{p} = \hbar \kappa \qquad p_i = \hbar \kappa_i$$

Pressure force:

$$F^{do} = \tilde{p} dO \qquad F_i^{do} = \tilde{p} dO_i$$

LAGRANGE:

$$\begin{aligned} \frac{\partial L}{\partial \dot{q}} &= \dot{p} & \frac{\partial L}{\partial \dot{q}} &= \dot{p} & \frac{\partial L}{\partial \dot{q}^i} &= p_i & \frac{\partial L}{\partial \dot{q}^i} &= \dot{p}_i \\ L &= -H + \dot{p} \cdot \dot{q} & L &= -H + p_i \dot{q}^i \end{aligned}$$

HAMILTON

$$\frac{\partial H}{\partial \dot{p}} = \dot{q} \qquad \frac{\partial H}{\partial \dot{q}} = -\dot{p} \qquad \frac{\partial H}{\partial p_i} = \dot{q}^i \qquad \frac{\partial H}{\partial \dot{q}^i} = -\dot{p}_i$$

MAXWELL I ⁷⁾ ⁸⁾ (differential form)

$$\begin{aligned} c \nabla \cdot \vec{\tilde{H}} - \vec{\tilde{D}} &= \vec{\tilde{I}} & c \partial_j H^{ij} - \tilde{D}^i &= I^i \\ \nabla \cdot \vec{\tilde{D}} &= \tilde{\rho} & \partial_j D^j &= \tilde{\rho} \end{aligned}$$

(integral form)

$$\begin{aligned} c \int_{S'} \vec{\tilde{H}} \cdot d\vec{s} - \int_{\tilde{S}} \vec{\tilde{D}} \cdot d\vec{O} &= \int_{\tilde{S}} \vec{\tilde{I}} \cdot d\vec{O} & \frac{1}{2} c \int_{S'} \tilde{H}^{ij} ds_{ij} - \int_{\tilde{S}} \tilde{D}^i dO_i &= \int_{\tilde{S}} \tilde{I}^j dO_j \\ \int_{\tilde{G}'} \vec{\tilde{D}} \cdot d\vec{O} &= \int \int_{\tilde{G}} \tilde{\rho} dV & \int_{\tilde{G}'} \tilde{D}^i dO_i &= \int \int_{\tilde{G}} \tilde{\rho} dV \end{aligned}$$

MAXWELL II (differential form)

$$\begin{aligned} c \nabla \times \vec{E} + \dot{\vec{B}} &= 0 & 2c \partial_{[i} E_{j]} + \dot{B}_{ij} &= 0 \\ \nabla \times \vec{B} &= 0 & \partial_{[i} B_{jk]} &= 0 \end{aligned}$$

(integral form)

$$\begin{aligned} c \int_{S'} \vec{E} \cdot d\vec{S} + \int_{\tilde{S}} \dot{\vec{B}} \cdot d\vec{O} &= 0 & c \int_{S'} E_i ds^i + \frac{1}{2} \int_{\tilde{S}} B_{ij} dO^{ij} &= 0 \\ \int_{\tilde{G}'} \vec{B} \cdot d\vec{O} &= 0 & \int \int_{\tilde{G}'} B_{ij} dO^{ij} &= 0 \end{aligned}$$

Potentials ⁹⁾

$$\vec{E} = -\nabla \Phi + c^{-1} \dot{\vec{A}} \quad E_i = -\partial_i \Phi + c^{-1} \dot{A}_i$$

Energy-density

$$\tilde{E} = \frac{1}{2} [\vec{E} \cdot \vec{D} + \frac{1}{2} \vec{B} \cdot \dot{\vec{H}}] \quad \tilde{E} = \frac{1}{2} [\tilde{E}_i D^i + \frac{1}{2} B_{ij} \dot{H}^{ij}]$$

POYNTING

$$\vec{P} = \vec{E} \times \vec{H} \quad \tilde{P}^i = E_j \dot{H}^{ij}$$

LORENTZ

$$\vec{F} = e [\vec{E} + c^{-1} \vec{B} \times \vec{v}] \quad F_i = e [E_i + c^{-1} B_{ij} v^j]$$

i) Linking equations and metric. In order to write down the most important equations, it is easier to use the

⁷⁾ Here $\vec{\tilde{I}}$ is the electromagnetic current-density; the integrals are to be extended over open two — or three dimensional domains S or G or their (complete) one — or two dimensional boundaries S' , G' respectively.

⁸⁾ The gradient is a covector-operator; $\partial_i = \frac{\partial}{\partial x^i}$. For the meaning of the square brackets cf. SCHOUTEN [2], p. 20.

⁹⁾ Here Φ denotes the scalar, \vec{A} the (co)vector potential.

symbolism of RICCI-calculus only, and to use the (spacial) fundamental tensor g_{ij} (its reciprocal and the square root of the absolute value of its determinant being as usual denoted by g^{ij} , and \tilde{g} respectively).

$$\begin{aligned} \text{NEWTON's impetus} \quad j_i &= m g_{ij} \dot{x}^j \\ \text{Kinetic energy} \quad T &= \frac{1}{2} m \dot{x}^i g_{ij} \dot{x}^j = (2m)^{-1} p_i g^{ij} p_j \\ \text{MAXWELL (in vacuo)} \quad B_{ij} &= \mu_0 g_{ik} g_{il} H^{kl} \\ \tilde{D}^i &= \varepsilon_0 \tilde{g} g^{ij} E_j. \end{aligned}$$

3. *Some remarks on the relations between physics and geometry.*

1. In 1917 A. EINSTEIN showed that under certain conditions Riemannian geometry could be used as a model for a class of physical phenomena, viz. gravitation theory. Since then a large number of authors have tried to find more general geometries, which could be used as models for a larger class of physical phenomena. Among these we mention only such names as J. A. SCHOUTEN, H. WEYL, TH. KALUZA, O. KLEIN, L. ROSENFELD, A. EDDINGTON, O. VEBLEN, A. EINSTEIN, E. SCHRÖDINGER and A. PAIS. All these efforts are based on a common fundamental idea: independent of definite physical phenomena space-time exists as a definite geometrical structure (a differentiable manifold, in which a connection is defined either by a fundamental tensor or in a more general way); physical phenomena are described by fields, determined by this structure by means of differentiation processes, i.e. they are "manifestations" of definite geometrical properties.

In the thirties this situation lead the present author to the question whether there really was an epistemological basis for this preponderant position of geometry. This question was answered by the author in the negative, and lead further in a natural way to the question, in how far physical phenomena can be described without introducing either a fundamental tensor or even a more general connection, but only by means of "natural invariants" (gradients of scalars, exterior derivatives of multi-covectors, etc.) applied to the physical field *themselves*. In a subsequent survey of several parts of physics this was found to be possible for an astonishingly large number of the fundamental equations of physics, but *not* for their totality.

2. This is not astonishing, for it stands to reason that the equations of physics can not *all* be invariant with respect to arbitrary

transformations of space, independent of some metrical quantities. For the existence (at least in the non-relativistic approximation) of rigid bodies and their rotations shows that metrical geometry has some kind of physical reality, which can not be disregarded. Anyone who ever has knocked his head against a sharp stone has *felt* the large curvature, i.e. metrical geometry.

This fact, however, does not prevent the possibility to bring the equations of physics in such a form that *some* of them are invariant with regard to affine (and even more general) transformations, whereas the metrical relations are contained in a second set of relations only. The author is inclined to consider the first ones as the more fundamental ones, although they become physically meaningful only after the second set, the "linking equations", have been added.

This can easily be seen by considering analytic dynamics. The Hamiltonian equations are invariant, even with regard to arbitrary contact transformations. They do not themselves imply any metrical assumption¹⁰⁾. As soon, however, as we apply the general theory to any special problem, we have to substitute for the Hamiltonian H a special function of the momenta p_i and the coordinates q^i , which in all important cases depend upon the metrical properties of space, 1°. through the kinetic energy being a quadratic function of the momenta, 2°. through the potential energy being a metrically determined function of the coordinates (e.g. by a NEWTON or COULOMB energy being proportional to r^{-1}). In this case one could consider the Hamiltonian (or the equivalent Lagrangian) equation as the fundamental ones, and the equation specifying H as a (metrical) function of the p_i and q^i as the "linking equation". In relativistic mechanics the situation is not essentially different, the main alteration being the replacement of the kinetic energy of any mass point $T = p^2/2m$ by $T = c\sqrt{m^2c^2 + p^2} - mc^2$; in both cases metric enters through the norm p^2 of the vector p_i . With appropriate alterations the same holds true for quantum mechanics, as is seen most clearly by SCHOUTEN's [3] beautiful analysis of DIRAC's equation by means of sedenion systems.

The program of splitting the equations of physics into a "fundamental" and a "linking" set, i.e. of "localizing" the occurrence of

¹⁰⁾ Of course, a generalized metric can be *derived* from them by means of the tensor $\partial^2 H / \partial p_i \partial p_j$.

metrical relations has to some extent been carried out by the author a number of years ago, a task which even to this degree could only be performed by making an extensive use of SCHOUTEN's methods.

3. In all cases hitherto studied the metrical relations could be "localized" in three places:

- a.* The equations linking the energy and the momenta, be it kinetic or potential;
- b.* Material constants of rigid or non-rigid matter (like dielectric constant, permeability, viscosity coefficient, conductivity, friction coefficients, etc.) behind which often implicit assumptions of isotropy are hidden;
- c.* The corresponding limiting case of "empty space".

The less fundamental nature of the "linking equations" reveals itself by their losing their validity in several cases, where the fundamental equations remain valid. With respect to the localization *a*, this is the case already under transition from classical to relativistic physics; with respect to *b*, under relief of the implicit assumptions of isotropy. Examples were given above.

In the case *c*. of empty space it is somewhat less easy to see that here also implicit assumptions of isotropy (which naturally are of a metrical nature) are hidden. In order to understand this we have to look for cases where the isotropy of empty space is released. The possibility that "empty space" may have definite properties like isotropy, is connected with the fact that the former absolute distinction between empty and non-empty space has been replaced by a more gradual one since quantum mechanics has shown that radiation is not essentially different from matter, the question whether the rest mass and the spin of some particles (photons) disappears or not, being rather irrelevant in this respect. As it seems rather doubtful whether any physical reality or even meaningfulness can be attributed to the concept of "absolutely empty" space, not even containing any radiation, it seems plausible that the isotropy of "empty" space (containing no ordinary matter) may be disturbed by letting non-isotropic radiation pass through it, e.g. a directed beam of strong γ -rays (or even an ordinary lightray)¹¹⁾.

¹¹⁾ Dr S. A. WOUTHUYSEN kindly suggested to me that a beam of neutrinos would suit the purpose better, as it would make the distinction clearer between the phenomenon to be described (the electromagnetic field) and the cause of the anisotropy (the directed beam).

Although I am not aware of any attempt to settle this point beyond doubt, it seems reasonable to expect that the equations, say of electromagnetism, in empty space, would have to be altered if this space were materially empty but anisotropic with respect to radiation. In particular the “linking equations”¹²⁾ $\vec{B} = \mu_0 \vec{H}$ $\vec{D} = \epsilon_0 \vec{E}$ would have to be altered in such a way as not to depend anymore on the abstract metrical geometry, but on the anisotropic stress tensor of the radiation.

4. If this conjecture were found to be correct, it would show that metrical geometry would enter the physics of “empty” space only through its stress tensor (which usually is isotropic with regard to the surrounding material bodies). This would lead to a *statistical* interpretation of metrical geometry in terms of the stress tensor of large assemblies of material particles (or photons), expressing properties of the moments of second order of the distribution or the assembly of some physical quantities attributable to the individual particles. The possibility to restrict the order of the moments to two because of the central limit theorem of probability theory would thereby lead to the quadratic nature of the metric.

It must be added that the restriction to general affine geometry (natural invariants in differential manifolds) (which in the present paper was narrowed down for purposes of simplicity to elementary affine geometry) is by no means essential and can be released without great difficulty. The introduction into physics of a space-time continuum, apart from the physical phenomena themselves, seems to be completely superfluous, although it appears implicitly in quantum physics also.

Apart from these, perhaps too abstract and too speculative, considerations the examples given in § 2, taken from elementary physics, may perhaps present some conceptual and educational interest of their own.

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¹²⁾ Cf. D. VAN DANTZIG [2].

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