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On the Relation Between Geometry and Physics and the Concept of Space-time

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On the Relation Between Geometry and Physics and the Concept of Space-time

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1. Since olden times it has been assumed that the concepts and theorems of geometry are prerequisite to those used in mathematical models of other parts of physics [1]. The reasons for this priority relation, however, seem to be of a historical and traditional rather than of a logical nature. This holds for Euclidean and for Riemannian geometry, introduced by EINSTEIN as a model for gravitation, as well as for the later five-dimensional and projective generalizations, and the more recent general linear connexions, used by EINSTEIN and SCHROEDINGER. It is not quite clear which logical or epistemological advantage there is in interpreting a part of a geometrical object as an electromagnetic field, say, and not vice versa.

2. In rational mechanics the motion of a system is completely described by means of the Hamiltonian function, which has a direct physical meaning. Geometry enters only implicitly in the equations, in general through the kinetic as well as the potential part of the Hamiltonian H, i. e. through the identity, linking energy with momentum (together with the coordinates). As long as this relation is not specified, the equations remain independent of any special geometry.

3. It was found that a similar situation is present in other parts of physics (electromagnetism, thermo-hydrodynamics). The complete set of equations can be split into a) a set of 'fundamental equations' which describe relations between the physical quantities without intervention of geometry, and b) a set of 'linking equations', linking energies and momenta or their kinetic or potential parts. As long as the latter remain unspecified (i. e. contain unspecified functions), we have a kind of 'generalized physics', analogous with Hamiltonian dynamics; their specification will in general require geometrical, and even metrical assumptions.

4. The 'linking equations' are often of a less general character than the 'fundamental equations' [9]. E. g. in classical pointmechanics they express the proportionality of the (kinetic) momentum vector (NEWTON'S

'impetus') with the velocity; in relativistic point-mechanics the proportionality of the (kinetic) momentum energy vector with the relativistic velocity $i^h = dx^h/ds$. In relativistic quantum dynamics, however, the former is (taking $c = 1$) the vector operator $p_i - e\,\varphi_i$, whilst the latter is the vector operator γ^i and these operators are entirely unconnected. In most cases the metrical nature of the linking equations depends upon implicit assumptions of a metrical nature, e. g. assumptions of isotropy, and lose this character if the isotropy is violated. Even the isotropy of the vacuum might get lost in the presence of a directed beam of radiation, of neutrinos or of mesons, say; i. e. the linking equations in vacuo depend on the 'nature' of this vacuum and have their usual metrical form only if special (though usually valid) assumptions of a metrical nature are satisfied.

For these reasons one might be inclined to consider metrics as describing some 'normal' state of matter (inclusive radiation) and to give it a *statistical* interpretation as some kind of average of physical characteristics of surrounding events, instead of laying it at the base of the whole of physics. Also the fact that e. g. measurement of length requires rigid bodies, i. e. large numbers of particles, points to a statistical interpretation. It is, however, not yet known, *how* such a statistical interpretation of metric can be obtained[1]).

Such a statistical interpretation of metrics does not, of course, deny its physical reality (like in the case of temperature), which hardly will be denied by anyone who ever has been pricked by a needle, i. e. who has *felt* its rigidity and the smallness of its curvature.

5. In electromagnetism [2], [3] the Maxwell equations themselves can be written in a form independent of geometry, by means of 'natural differential invariants' only, whereas one form of the linking equations [3.5] is obtained by writing the expression of the potential covector φ_i at a world-point P by means of retarded potentials formally as a four dimensional integral of the current vector density $\mathfrak{s}^{j'}(P')$ (actually it is degenerate in the case of electromagnetism; it vanishes not only in the exterior, but also in the interior of the light cone of the past; this is not so in the case of meson theory)

$$\varphi_i(P) = \int \gamma_{i j'}(P, P')\,\mathfrak{s}^{j'} dU' \qquad (1)$$

where P and P' are two 'worldpoints' (points of space-time), φ_i are the retarded potentials, $\mathfrak{s}^{j'}$ the current-density, dU' a (4-dimensional) element

[1]) In particular I cannot see H. G. Küssner's considerations, who in his book *Principia Physica*, has kindly reported several of my ideas, as a fulfillment of this program. I also believe his appraisal of my ideas to be exaggerated, and I do not subscribe to his criticisms of Einstein, whose ideas doubtless have been fundamental for the whole subsequent development of physics.

at P' and $\gamma_{ij'}(P,P')$ a 'two point quantity', transforming as a covector (= covariant vector) at P as well as at P'. Metrical specialization in the case of an electromagnetic field in empty space gives (in HEAVISIDE-LORENTZ units)

$$\gamma_{ij'}(P, P') = -g_{ij} \frac{\delta(r - ct + ct')}{4\pi r} \tag{2}$$

where $t = t_P$, $t' = t_{P'}$, r is the special distance ($r > 0$) between P and P', and δ is the Dirac function, so that $\gamma_{ij'}$ is $\neq 0$ (and singular) only if P' is on the light cone of the past of P. Formally it is not relativistically invariant, but it can also be written as

$$\gamma_{ij'}(P, P') = g_{ij} \Box \frac{\iota(r - ct + ct')}{8\pi} \tag{3}$$

(where $\iota(x) = \begin{cases} 1 \; if \; x \geqq 0 \\ 0 \; if \; x < 0 \end{cases}$ is HEAVISIDE's 'unit function'), showing that it is invariant under the 'half' Lorentz-group, leaving the two halves of the light cone ('past' and 'future') each separately invariant. (Interchange of these interchanges retarded and advanced potentials). It is not clear, how the condition of invariance under the *full* Lorentz group (including reversal of time) is justified. It seems rather to lead to several difficulties in modern physics, as e.g. the occurrence of numerous 'spook-particles' (antiparticles).

The fundamental nature of the quantities $\gamma_{ij'}$ can also be seen from the fact [3.5] that quantization of the field according to BOHR-ROSENFELD yields equations which, in the metrical specialization used there, are equivalent with

$$[\varphi_i(P), \varphi_{j'}(P')] = -\hbar c i \{\gamma_{ij'}(P, P') - \gamma_{j'i}(P', P)\}. \tag{4}$$

The differences in the right hand member are invariant under the full Lorentz-group, and can be expressed also by the Jordan-Pauli D-function.

In thermo-hydrodynamics [4] the fundamental quantities describing the macroscopic motion of homogeneous matter with respect to any 3-dimensional element dV with components $d\mathfrak{V}_i$ are: the number of particles $N^{dV} = \mathfrak{N}^h d\mathfrak{V}_h$ whose worldlines intersect dV and their momentum and energy $P_i^{dV} = \mathfrak{P}_i^h \, d\mathfrak{V}_h$. In the force-free relativistic specialization $(-\mathfrak{g})^{-1/2} g_{hj} \, \mathfrak{P}_i^h$ equals the stress tensor T_{ij}. Putting for simplicity $k = c = 1$ the temperature T enters into the theory in the form of the fundamental invariant differential $d\tau = T\,dt$, or also of the temperature vector $\vartheta^i = dx^i/d\tau$, the time component of which is $1/T$, whilst its space-components are $1/T$ times the ordinary (macro-) velocity of the fluid. In the relativistic specialization $d\tau = T_0\,ds$ and $\vartheta^h = 1/T_0 \, i^h$, T_0 being the proper temperature. Space lacks to go into the hydrodynamical equation [5] or into the kinetic gas theory [6].

7. Although the above considerations are of an epistemological nature rather than of practical physical importance, two more or less concrete results may be mentioned. Firstly the following small, though in principle measurable, new relativistic effect was discovered [4]. If a fluid is 'perfect', i.e. has negligible internal friction if observed by an observer in rest with respect to its relative motion, but non-negligible heat-conductivity, then it will in general not appear as a perfect fluid to a moving observer. If e.g. it flows with high velocity between two walls having different temperatures, then the flow of heat caused by the temperature-gradient, i.e. the energy current, will be accompanied by a momentum-current, i.e. an apparent internal frictional force, retarding the hotter part of the fluid relative to the colder part. Fluids which are perfect with respect to *every* observer, however moving, were called 'perfectly perfect'. These are the fluids which had hitherto been considered in R.T., and represented by their stress tensor

$$T_{ij} = p\, g_{ij} - (\varrho + p)\, i_i\, i_j \qquad \left(i^h = \frac{dx^h}{ds}\right) \tag{6}$$

Their equations of motion — leaving the entropy constant — have been studied in greater generality [5].

Secondly an old question of interpretation could be settled decisively [7]. SCHWARZSCHILD interpreted ϱ in (6), i.e. $-T_{44}$ if $i_4 = 1$, $i_1 = i_2 = i_3 = 0$, as the 'proper density' of the fluid. EDDINGTON, interpreting the term 'proper density' as the *particle* density multiplied (for $c = 1$) with the proper mass m_0, criticized SCHWARZSCHILD and stated that this density were equal to $-T_i^i = \varrho - 3\,p$. SYNGE showed that the latter result by EDDINGTON was wrong and returned to SCHWARZSCHILD's formula. The non-metrical theory leads to the unequivocal result that for a general fluid the particle density is not determined at all by T_{ij} alone, i.e. that both EDDINGTON and SYNGE were right in their *critical* parts, but (like SCHWARZSCHILD) wrong in their positive affirmation. In general the particle density can only be expressed by T_{ij} together with the proper temperature. For the case, however, of an ideal gas of not excessively high temperature, the proper density is in first approximation just the mean of the values proposed by SCHWARZSCHILD-SYNGE and by EDDINGTON. The difference $3\,p/2$ between ϱ and the proper density is that between the proper mass density of the fluid and the (smaller) density of the sum of the proper masses of the molecules constituting it, i.e. in first approximation the kinetic energy density as seen by an observer moving with the fluid.

8. The foregoing considerations have lead to a program of considerably wider scope. This is based upon the following facts. Firstly it is impossible, not only to *measure* physical quantities with unrestricted accuracy, but already to *define* them in such a way, so that infinitesimals and even real numbers in the mathematical sense, hence also differential equations, can enter only into a highly idealized or, rather, simplified model of observable phenomena. Secondly, whereas spatial and temporal *relations* between observable phenomena have certainly an empirical background, this is not the case with the concepts of space and time (or space-time) themselves; these form a non-empirical kind of 'duplication' of the set of observable events. Thirdly the relativistic relationship between space and time is somewhat disturbed by the spatial atomicity together with the temporal continuity of matter. These reasons make it desirable to strive for a more realistic model of physics in the form of a so-called 'flash-model' [1], [8], where matter is represented by a *finite* number of finite groups of elementary events, called flashes, where the finite groupings represent the momentum energy as well as the spatio-temporal relations.

The program of eliminating from the foundations of mathematical physics the concept of a space-time continuum and replacing it by a finite set of discrete events with space-time relations between them can be supported by the same type of argument which originally lead EINSTEIN to the special and the general theory of relativity: physics does not provide us with any means of defining empirically the elements of space-time, i.e. the world-points, i.e. possible events with coordinates to be defined with *infinite* accuracy. For this purpose the replacement of the differential equations of physics by the equivalent integral equations (from which they often have been derived, and which alone have a direct physical meaning) is important, as the integrals can easily be interpreted as mathematical idealizations of sums over a large but finite number of events ('flashes').

The correct appraisal of the role of metrics in physics is the only and preliminary part of the program which hitherto could be carried out to a certain extent. Although this could be considered as a bad omen, it is the author's present conviction that this is due to the many gaps in his knowledge of physics rather than to an essential defect of the program as such.

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