MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM
STATISTISCHE AFDELING

Leiding: Prof. Dr D. van Dantzig
Chef van de Statistische Consultatie: Prof. Dr J. Hemelrijk

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A remark on representation of dependent random variables as functions of independent ones

by

Prof.Dr D. van Dantzig

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This note contains a remark which, although an only slightly less general remark has been made by P. Lévy [1], seems not to be generally known. A few simple consequences concerning stochastic processes are also mentioned.

§ 1

As usual two random variables $\underline{x}$ and $\underline{y}$ [2] are said to be stochastically dependent if they have a common distribution. A particular case is the one where they are functionally dependent, e.g. where $\underline{y}$ is a function of $\underline{x}$: $\underline{y} = \varphi(\underline{x})$, $\varphi(x)$ being a measurable function. A more general case is the one where the ordinary function $\varphi(x)$ of one variable is replaced by a random function, e.g. a function $\varphi(x, \theta_1, \theta_2, \cdots)$ depending on one or more or even an infinity of parameters for which random variables $\underline{\alpha}_1, \underline{\alpha}_2, \cdots$ which are stochastically independent of $\underline{x}$ are substituted :

$$\underline{y} = \varphi(\underline{x}, \underline{\alpha}_1, \underline{\alpha}_2, \cdots)$$

At first sight one might assume this still to be a special case. This, however, is not so, and, moreover, one parameter is sufficient.

In fact, let $\underline{x}$ and $\underline{y}$ be stochastically dependent and let us assume that the conditional distribution function $F(y|x)$ of $\underline{y}$, given $\underline{x}$, exists spr 0 [3], and is measurable with respect to x and y separately as well as simultanuously. We take here the distribution function continuous from the left:

$$(1) \qquad F(y|x) \overset{\text{def}}{=} P\{\underline{y} < y \mid x\} \qquad 4)$$

We introduce the discontinuous function

$$(2) \qquad p(y|x) \overset{\text{def}}{=} P\{\underline{y} = y \mid x\}$$

which vanishes everywhere, except on a denumerable set. Moreover, we put for $0 \le \alpha \le 1$

$$(3) \qquad \varphi(x,\alpha) \overset{\text{def}}{=} \sup\{y \mid F(y|x) < \alpha\}$$

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1) P. Lévy, Théorie de l'addition des variables aléatoires, 1e ed., Paris, 1937, p. 71-73, 122-123.

2) Random variables are denoted by underlined symbols.

3) spr α (salva probabilitate α) means: except for a probability α Hence spr 0 is equivalent with: almost surely.

4) $\overset{\text{def}}{=}$ denotes an equality defining the <u>left</u> hand member.

Then, with $\underline{\lambda}$ stochastically independent of $\underline{x}$ and $\underline{y}$ and $H(0,1)$ (i.e. homogeneously distributed on $(0,1)$); putting

(4) $$\underline{q} \overset{\text{def}}{=} F(\underline{y}|\underline{x}) + \underline{\lambda}\, p(\underline{y}|\underline{x})$$

we have

$$P\{\underline{q} \leq c \,|\, x\} = c$$

for every c with $0 \leq c \leq 1$ and almost every x, i.e. $\underline{q}$ is stochastically independent of $\underline{x}$ and

(5) $$\underline{y} = \varphi(\underline{x}, \underline{q}) \quad \text{spr } 0 .$$

Hence we have:

I <u>For any pair of stochastically dependent random variables $\underline{x}, \underline{y}$ a measurable function $\varphi(x,q)$ of two variables can be found such that (5) holds with $\underline{q}$ $H(0,1)$ and stochastically independent of $\underline{x}$.</u>

We notice that

a) the assumption that $\underline{x}$ is one-dimensional enters nowhere.
b) the case of stochastic independence of $\underline{x}$ and $\underline{y}$ enters as the special case where φ does not depend on its first argument.
c) the case of functional dependence of $\underline{y}$ on $\underline{x}$ enters as the special case, where φ does not depend on its second argument.
d) the above proof is identical with the one (obtained by putting $\underline{x} = 0$, spr 0) for the known fact that any one-dimensional random variable $\underline{y}$ is a function of a $H(0,1)$ distributed random variable.

Using these facts we obtain:

II <u>If $\underline{x}_1, \cdots, \underline{x}_n$ have a common distribution function and are one-dimensional, then for every integer k $(1 \leq k \leq n)$ a measurable function φ_k of k variables exists, such that for all $k \in \{1, \cdots, n\}$</u>

(6) $$\underline{x}_k = \varphi_k(\underline{y}_1, \cdots, \underline{y}_k) \quad \text{spr } 0$$

<u>where $\underline{y}_1, \cdots, \underline{y}_n$ are $H(0,1)$ and (completely) stochastically independent.</u>

§ 2

Let $\underline{x}(t)$ be a stochastic process on a set T of elements t. If T is finite or enumerable the result II may be applied. In the latter case, taking for T the set of positive integers, we get for each $t \in T$ $\quad \underline{x}(t) = \varphi(t, \underline{y}_1, \cdots, \underline{y}_t)$, the $\underline{y}_t$ being all $H(0,1)$ and stochastically independent.

Assume now, for an arbitrary T, that a topology is defined in T, that $T_0 \overset{\text{def}}{=} \{t_1, t_2, \cdots\}$ is everywhere dense in T, and that the process is everywhere continuous spr o. We again have for every $t_k \in T_0$ a representation of the form (6) for $\underline{x}(t_k)$. For any $t \in T$ there is a subsequence $t_{\nu_1}, t_{\nu_2}, \cdots$ converging towards t, and

(7) $\underline{x}(t) = \underline{x}(t_{\nu_1}) + \sum_{n=1}^{\infty} (\underline{x}(t_{\nu_{n+1}}) - \underline{x}(t_{\nu_n}))$ spr o, so that for every $t \in T$ a representation of the form

$$(8) \qquad \underline{x}(t) = \varphi(t, \underline{u}_1, \underline{u}_2, \cdots)$$

exists, the function φ being measurable with respect to the u_n, but not necessarily so with respect to t. A similar result holds if the condition of absolute continuity is replaced by an other one, admitting a representation spro of $\underline{x}(t)$ as a function of the $\underline{x}(t_\nu)$.

In the most general case measure theoretic difficulties of the customary type arise. If, however, $\underline{x}(t)$ for every $t \in T$ has a conditional probability distribution, given $\underline{x}(t_1), \underline{x}(t_2), \cdots$, I again can be applied, so that $\underline{x}(t)$ for $t \in T - T_0$ can be expressed by means of the $\underline{u}(t_n)$ $(= \underline{u}_n)$ and new independent $H(0,1)$ variables $\underline{u}(t)$. It seems therefore worthwhile to try to find out, under exactly which conditions a representation of the form

$$(9) \qquad \underline{x}(t) = \varphi(t, \underline{u}(s))$$

exists, where s runs through a set $S \subset T$ and $\underline{u}(s)$ are $H(0,1)$ for all $s \in S$, and independent in the sense that for any function $f(s)$ on S such that $0 < f(s) \leq 1$ and $f(s) = 1$ except for at most at a countable number of elements $s_1, s_2, \cdots \in S$,

$$P\{0 \leq \underline{u}(s) \leq f(s)\} = \prod_{i=1}^{\infty} f(s_i).$$