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Some remarks on the power function of Wilcoxon's test for the problem of two samples

I and II

BY

H. R. VAN DER VAART

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SOME REMARKS ON THE POWER FUNCTION OF WILCOXON's TEST FOR THE PROBLEM OF TWO SAMPLES. I

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H. R. VAN DER VAART

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CONTENTS

(See further Proceedings 53, *507—520 (1950)).*

1. INTRODUCTION

F. WILCOXON (9) gave a non-parametric solution of the problem of two samples of equal sizes. This solution was generalized and studied in detail by H. B. MANN and D. R. WHITNEY (2).

Let $\mathbf{x}_1,\ldots,\mathbf{x}_m$, $\mathbf{y}_1,\ldots,\mathbf{y}_n$ [1]) be $m+n$ independent random variables, the $\mathbf{x}_i$ all having the *continuous* (cumulative) distribution function $F(x)$ and the $\mathbf{y}_j$ all having the *continuous* (cumulative) distribution function $G(x)$.

For any set of values $E=(x_1,\ldots,x_m,\ y_1,\ldots,y_n)$ the variables can take [2]) let $U=U\,(x_1,\ldots,x_m,y_1,\ldots,y_n)=U(E)$ be defined as the number

[1]) The letter which denotes a variable is printed in bold type when stress is laid upon the random character of the variable, i.e. upon the fact the variable has a distribution function.

[2]) Such a set E is called a "*sample point*"; the set of all sample points E which eventually might be obtained is called the "*sample space*" W.

of pairs of integers (i, j) $(1 \leqq i \leqq m;\ 1 \leqq j \leqq n)$ with $x_i > y_j$, provided $x_i \neq y_j$ for every pair (i, j). Apparently U is zero or a positive integer. Then $\mathbf{U} = U(\mathbf{x}_1, \ldots, \mathbf{x}_m, \mathbf{y}_1, \ldots, \mathbf{y}_n) = U(\mathbf{E})$ is a discrete random variable, defined on the sample space W everywhere with the exception of a set of probability zero consisting of all points E with $x_i = y_j$ for any pair (i, j) $(1 \leqq i \leqq m;\ 1 \leqq j \leqq n)$.

The statistic $U(E)$ was introduced by H. B. MANN and D. R. WHITNEY for testing the hypothesis that $G(x) = F(x)$ holds for all x (shortly: $G = F$) against the alternative that $G(x) < F(x)$ holds for all x (shortly: $G < F$).

The distribution function of $\mathbf{U}$ under the hypothesis $G = F$ has been computed by MANN and WHITNEY for $m \leqq 8$, $n \leqq 8$, and by the Computing Department of the Mathematical Centre at Amsterdam for $m \leqq 10$, $n \leqq 10$. For $m \to \infty$ and $n \to \infty$ the distribution of $(\mathbf{U} - \frac{1}{2}mn) \cdot [{}^1/_{12} \cdot mn\ (m + n + 1)]^{-\frac{1}{2}}$ tends to a normal (0,1)-distribution.

The test, as given by MANN and WHITNEY, consists in rejecting the hypothesis $G = F$ on the level of signifiance α if and only if $P[\mathbf{U} \leqq U(E) | G = F] \leqq \alpha$, where E is the sample point corresponding with the empirical data.

As for the power function of this test, MANN and WHITNEY remarked that it presents formidable difficulties. They proved however that their test is consistent [3]) with respect to the class of alternatives $G < F$.

2. THE POWER FUNCTION OF A STATISTICAL TEST IN GENERAL

2. 1. *Critical region.*

A statistical hypothesis H_0 is tested (against an alternative hypothesis H) by dividing the sample space W into two parts, w and $(W - w)$, and applying the rule that H_0 is rejected if the sample point E falls in w and that H_0 is not rejected if E falls in $(W - w)$. w is called the "*critical region*" of the test. (cf. J. NEYMAN and E. S. PEARSON (4) or J. NEYMAN (3)).

2. 2. *The concept of power function.*

The power function of a test, a concept introduced by NEYMAN and PEARSON (5) (cf. also J. NEYMAN (3)), is the probability $\alpha(H) = P[\mathbf{E} \in w | H]$, that the sample point $\mathbf{E}$ falls in the critical region w, calculated under any admissible hypothesis H. H_0 denoting the hypothesis tested (to which there may be an infinity of alternatives), the critical region w is chosen so that:

$$\alpha(H_0) = P[\mathbf{E} \in w \mid H_0] \text{ is equal to } \alpha.$$

---

[3]) A test is called "*consistent*" according to A. WALD and J. WOLFOWITZ (8), if and only if the probability of rejecting the hypothesis tested (here $G = F$) when it is false, tends to 1 as the sample size tends (here: sizes tend) to infinity.

Here α is a given positive number, the so-called level of significance. Apparently α is the probability that H_0 will be rejected (cf. the definition of w in 2. 1), when H_0 is true. Usually α is chosen small, as will be easily understood from I below.

Clearly, if among the many possible critical regions one would exist for which:

$$\text{I.} \quad P[\mathbf{E} \in w \mid H_0] = 0$$
$$\text{II.} \quad P[\mathbf{E} \in w \mid H] = 1 \text{ for all } H \neq H_0,$$

this one would be preferred for testing H_0.

Critical regions for which I and II hold cannot be realized, however, as soon as all sample points have a positive probability under both H_0 and H.

> Remark. w depends, of course, on the number of dimensions N of the sample space W. Making this dependence explicit by writing w_N for w, we see that a test is consistent (cf. the note at the end of 1) if and only if:
>
> $$\lim_{N\to\infty} P[\mathbf{E} \in w_N \mid H] = 1 \quad \text{for all} \quad H \neq H_0.$$
>
> Hence this asymptotic relation corresponding with II defines consistent tests.

2. 3. *The use of the power function.*

The power function is instrumental to judge the "goodness" of a test and to compare several tests.

α) A critical region w with $P[\mathbf{E} \in w \mid H_0] > P[\mathbf{E} \in w \mid H]$ for some alternative H, leads to the hypothesis H_0 being rejected more often under H_0 than under this alternative H. A test based on such a critical region is called "*biased*" (cf. NEYMAN and PEARSON (5) or NEYMAN (3)). An unbiased test is based on a critical region w with $P[\mathbf{E} \in w \mid H_0] < P[\mathbf{E} \in w \mid H]$ for *all* admissible alternatives H. Clearly an unbiased test is in general preferable to a biased one.

β) When two critical regions, w_1 and w_2 with $P[\mathbf{E} \in w_1 \mid H_0] = P[\mathbf{E} \in w_2 \mid H_0]$, both give unbiased tests, w_1 is a closer approximation to the ideal case as sketched in 2. 2 under II, than w_2 if:

$$P[\mathbf{E} \in w_1] \mid H] > P[\mathbf{E} \in w_2 \mid H] \text{ for } \textit{all} \text{ admissible } H \neq H_0.$$

Then the test based on w_1 is called *uniformly more powerful* than the test based on w_2.

> Remark. H specifies the joint distribution function of the N random coordinates of $\mathbf{E}$. So the power function is a function defined on a function space in the most general case, when all sorts of alternative H's are admitted. The class of admissible hypotheses H can be restricted to various degrees. In the most simplified case the only difference between the hypotheses H consists in the value of one parameter. Denote this parameter

by μ and the powerfunction $P[\mathbf{E} \in w \mid \mu]$ by $\alpha_w(\mu)$. Let H_0 consist in $\mu = 0$ and let μ be variable in an interval containing $\mu = 0$. If the first and second derivatives of $\alpha_w(\mu)$ for $\mu = 0$ exist and are denoted by $\alpha'_w(0)$ and $\alpha''_w(0)$ respectively, then a necessary condition for unbiasedness with respect to the alternatives $\mu > 0$ is, if $\alpha'_w(0) \neq 0$, that $\alpha'_w(0) > 0$ (or, if $\alpha'_w(0) = 0$, that $\alpha''_w(0) > 0$, etc.), whereas a necessary condition for unbiasedness with respect to the alternatives $\mu \neq 0$ is, that $\alpha'_w(0) = 0$ and, if $\alpha''_w(0) \neq 0$, that $\alpha''_w(0) > 0$ (or, if $\alpha''_w(0) = 0$, that $\alpha^{(3)}_w(0) = 0$ and $\alpha^{(4)}_w(0) > 0$, if these derivatives exist, etc.).

A necessary condition for w_1 being more powerful than w_2 is, with respect to the alternatives $\mu > 0$, that $\alpha'_{w_1}(0) > \alpha'_{w_2}(0)$, if $\alpha'_{w_1}(0) \neq \alpha'_{w_2}(0)$ (or, if $\alpha'_{w_1}(0) = \alpha'_{w_2}(0)$, that $\alpha''_{w_1}(0) > \alpha''_{w_2}(0)$, etc.), and with respect to the alternatives $\mu \neq 0$, if $\alpha''_{w_1}(0) \neq \alpha''_{w_2}(0)$, that $\alpha''_{w_1}(0) > \alpha''_{w_2}(0)$ (etc.).

3. THE POWER FUNCTION OF WILCOXON'S TEST

3. 1. *General remarks.*

Power functions of WILCOXON's test will be investigated under the following general restrictions on the *distribution functions* $F(x)$ and $G(x)$ (cf. 1):

a) $F(x)$ and $G(x)$ have continuous derivatives, $f(x)$ and $g(x)$, respectively, for all x-values with the exception at most of those bounding the infinite intervals (when present) for which $f(x)$ or $g(x)$ are zero.

b) $G(x) = F(x - \mu)$; $g(x) = f(x - \mu)$.

These restrictions will be assumed valid throughout the rest of the paper unless the contrary is mentioned. According to the restriction *b* the hypotheses H specify μ-values, but F is left unspecified.

The *critical regions* considered are defined by 1, 2 and 3 respectively:

$$\left.\begin{array}{ll} 1) & \mathbf{U} \leqq U_\alpha \text{ with the level of significance } \alpha \\ 2) & \mathbf{U} \geqq mn - U_\alpha \text{ with the level of significance } \alpha \\ 3) & \left| \mathbf{U} - \tfrac{1}{2} mn \right| \geqq \tfrac{1}{2} mn - U_\alpha \text{ with the level of} \\ & \text{significance } 2\,\alpha \end{array}\right\} \begin{array}{l} \text{with } U_\alpha < \tfrac{1}{2} mn. \\ \text{(for } U_\alpha \text{ cf 3. 2)} \end{array}$$

The first region, the only one considered by MANN and WHITNEY, serves to test the hypothesis $\mu = 0$ against $\mu > 0$ and the second region serves to test $\mu = 0$ against $\mu < 0$ (one-sided sets of alternative hypotheses, shortly: "one-sided alternatives"), whereas the third region serves to test $\mu = 0$ against $\mu \neq 0$ ("two-sided alternatives"). Sometimes mathematical difficulties require restrictions on m and n together with restrictions on α and on U_α.

In order to compare the power function of WILCOXON's test with that of STUDENT's test for the difference of two means (under conditions which allow the use of STUDENT's test), in addition to *a*) and *b*) the further restriction on the distribution functions involved:

c) $$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2},$$

will be imposed in a part of the paper.

3. 2. *The critical region of Wilcoxon's test.*

I. The critical region defined by $\mathbf{U} \leqq U_\alpha$ ($U_\alpha < \frac{1}{2}mn$) consists of all sample points $\mathbf{E} = (\mathbf{x}_1, \ldots, \mathbf{x}_m, \mathbf{y}_1, \ldots, \mathbf{y}_n)$ satisfying the inequality $\mathbf{U} \leqq U_\alpha$, where U_α is the maximum of all integers U with $P[\mathbf{U} \leqq U \mid G = F] \leqq \alpha$.

This critical region, w, is the sum of $U_\alpha + 1$ disjoint regions w_k ($0 \leqq k \leqq U_\alpha$), where w_k is the set of all sample points for which $U = k$. Each region w_k consists of $p'_{m,n}(k)$ disjoint subregions $w_{k,q}$ ($q = 1, \ldots, p'_{m,n}(k)$) with constant $x - y$ — arrangement [4]). For $p'_{m,n}(k)$ Mann and Whitney gave the recurrence relation

$$(3.2,1)\quad \begin{cases} p'_{m,n}(k) = p'_{m-1,n}(k-n) + p'_{m,n-1}(k) \\ \left(p'_{i,j}(k) = 0 \text{ if } k < 0;\ p'_{i,0}(k) = p'_{0,i}(k) = \begin{cases} 0, \text{ if } k \neq 0 \\ 1, \text{ if } k = 0 \end{cases}\right) \end{cases}$$

It is seen that $p'_{m,n}(0) = 1 = p'_{m,n}(1)$ if $m \neq 0$, and $n \neq 0$.

Clearly each subregion $w_{k,q}$ is built up out of $m!\,n!$ disjoint sub-subregions, the points of which are characterized by a constant permutation of their m x-coordinates and their n y-coordinates respectively in the constant $x - y$-arrangement corresponding to $w_{k,q}$.

Because of the continuity of the distribution functions of $\mathbf{x}_i$ and $\mathbf{y}_j$ the probability of $\mathbf{E}$ lying in the boundary of one of the above-mentioned regions is zero, so that the boundaries of these regions may be included without changing any probability calculated.

II. The critical region defined by $\mathbf{U} \geqq m\,n - U_\alpha$ ($U_\alpha < \frac{1}{2}m\,n$) is the sum of $U_\alpha + 1$ disjoint regions w_{mn-k} ($0 \leqq k \leqq U_\alpha$). The regions w_{mn-k} consist of subregions $w_{(mn-k),q}$ with constant $x - y$-arrangement, which are built up out of sub-subregions of constant permutations, as described for the region $\mathbf{U} \leqq U_\alpha$.

a) By the substitutions $x_i = -x'_{i'}$ ($i, i' = 1, \ldots, m$); $y_j = -y'_{j'}$ ($j, j' = 1, \ldots, n$) a one-to-one correspondence is established between the points $E = (x_1, \ldots, x_m, y_1, \ldots, y_n)$ of the space W and the points $E' = (x'_1, \ldots, x'_m, y'_1, \ldots, y'_n)$ of the space W'. Here a pair (i, j) with $x_i \gtrless y_j$ corresponds to a pair (i', j') with $x'_{i'} \lessgtr y'_{j'}$. Hence a point E with $U(E) = m\,n - k$ corresponds to a point E' with $U(E') = k$ ($U(E')$ being defined as the number of pairs (i', j') with $x'_{i'} > y'_{j'}$), and a region w_{mn-k} corresponds to a region w'_k, while the region in W defined by $\mathbf{U} \geqq m\,n - U_\alpha$ corresponds to the region in W' defined by $\mathbf{U} \leqq U_\alpha$.

Remark 1. It is easily seen that $p'_{m,n}(k) = p'_{m,n}(mn - k)$.

Remark 2. The points E of W have m x-coordinates and n y-coordinates.

The points E' of W' have m x'-coordinates and n y'-coordinates.

To make this fact explicit one can write $W_{m,n}$ instead of W and $W'_{m,n}$ instead of W'.

---

[4]) By this term is meant, that the same ordered arrangement of x and y corresponds to every point E of $w_{k,q}$, when the coordinates x_i ($i = 1, \ldots, m$) and y_j ($j = 1, \ldots, n$) of E are arranged in order according to increasing magnitude and the suffices of x and y are omitted.

β) A one-to-one correspondence between the points $E = (x_1, \ldots, x_m, y_1, \ldots, y_n)$ of $W_{m,n}$ and the points $E' = (x'_1, \ldots, x'_n, y'_1, \ldots, y'_m)$ of $W'_{n,m}$ is established by the substitutions $x_i = y'_{j'} + \mu$ $(i, j' = 1, \ldots, m)$ and $y_j = x'_{i'} + \mu$ $(j, i' = 1, \ldots, n)$. Now a pair (i, j) with $x_i \gtrless y_j$ corresponds to a pair (i', j') with $x'_{i'} \lessgtr y'_{j'}$. By the same argument as under α) it is seen that the region of $W_{m,n}$ defined by $\boldsymbol{U} \geqq m\,n - U_\alpha$ corresponds to the region of $W'_{m,n}$ defined by $\boldsymbol{U} \leqq U_\alpha$.

Remark 3. Clearly $p'_{n,m}(k) = p'_{m,n}(m\,n - k)$. Because of remark 1. it is seen that $p'_{n,m}(k) = p'_{m,n}(k)$.

III. The critical region defined by $|\boldsymbol{U} - \frac{1}{2}\,m\,n\,| \geqq \frac{1}{2}\,m\,n - U_\alpha$ $(U_\alpha < \frac{1}{2}\,m\,n)$ is the sum of the two disjoint regions defined by $\boldsymbol{U} \leqq U_\alpha$ (cf. 3. 2, I) and by $\boldsymbol{U} \geqq m\,n - U_\alpha$ (cf. 3. 2, II), respectively.

3. 3. *General expressions for the power function of Wilcoxon's test.*

Under the restrictions a) and b) of 3.1 imposed on $F(x)$ and $G(x)$, the power function of Wilcoxon's test is given by

$$\alpha_+(\mu) = \int_{U \leqq U_\alpha} \!\!\cdots\! \int \prod_{i=1}^{m} \{f(x_i)\,dx_i\} \prod_{j=1}^{n} \{f(y_j - \mu)\,dy_j\} \tag{3.3, 1}$$

for testing $\mu = 0$ against $\mu > 0$,

$$\alpha_-(\mu) = \int_{U \geqq mn - U_\alpha} \!\!\cdots\! \int \prod_{i=1}^{m} \{f(x_i)\,dx_i\} \prod_{j=1}^{n} \{f(y_j - \mu)\,dy_j\} \tag{3.3, 2a}$$

$$= \int_{U \leqq U_\alpha} \!\!\cdots\! \int \prod_{i=1}^{m} \{f(-x_i)\,dx_i\} \prod_{j=1}^{n} \{f(-y_j - \mu)\,dy_j\} \tag{3.3, 2b}$$

$$= \int_{U \leqq U_\alpha} \!\!\cdots\! \int \prod_{i=1}^{n} \{f(x_i)\,dx_i\} \prod_{j=1}^{m} \{f(y_j + \mu)\,dy_j\} \tag{3.3, 2c}$$

for testing $\mu = 0$ against $\mu < 0$,

((3. 3, 2b) follows from (3. 3, 2a) by the substitutions of 3. 2, II, α); (3. 3, 2c) from (3. 3, 2a) by 3. 2, II, β)) and

$$\alpha_\pm(\mu) = \alpha_+(\mu) + \alpha_-(\mu) \tag{3.3, 3}$$

for testing $\mu = 0$ against $\mu \neq 0$.

When $U_\alpha = 0$, one finds, denoting in this case $\alpha_+(\mu)$ by $\alpha_+^{(0)}(\mu)$, that

$$\left\{\begin{array}{l} \alpha_+^{(0)}(\mu) = m!\,n! \displaystyle\int_R \cdots \int \prod_{i=1}^{m} \{f(x_i)\,dx_i\} \prod_{j=1}^{n} \{f(y_j - \mu)\,dy_j\} \\ \text{where } R \text{ is defined by the inequalities:} \\ -\infty < x_1 < x_2 < \ldots < x_m < y_1 < \ldots < y_n < +\infty. \end{array}\right. \tag{3.3, 4}$$

From (3. 3, 2c) it follows that

$$(3.3,5a)\quad \begin{cases} \alpha_{-}^{(0)}(\mu) = m!\,n! \int \cdots \int\limits_{R} \prod\limits_{i=1}^{n} \{f(x_i)\,dx_i\} \prod\limits_{j=1}^{m} \{f(y_j+\mu)\,dy_j\}, \\ \text{with } R \text{ defined by: } -\infty < x_1 < \ldots < x_n < y_1 < \ldots < y_m < +\infty \end{cases}$$

or, by replacing x_i by $x_i - \mu$ and y_j by $y_j - \mu$, that

$$(3.3,5b)\quad \begin{cases} \alpha_{-}^{(0)}(\mu) = m!\,n! \int \cdots \int\limits_{R} \prod\limits_{i=1}^{n} \{f(x_i-\mu)\,dx_i\} \prod\limits_{j=1}^{m} \{f(y_j)\,dy_j\} \\ \text{with } R \text{ defined as in (3.3, 5a).} \end{cases}$$

Remark. Even without the restriction b) of 3. 1, writing again $g(y)$ for $f(y-\mu)$, the $(m+n)$-fold integral (3. 3, 4) can be reduced to a single one. In fact, writing x for x_m, it is seen that $\alpha_{+}^{(0)}(\mu) =$

$$= m!\,n! \int\limits_{-\infty}^{+\infty} f(x)\,dx \int\limits_{-\infty<x_1<\ldots<x_{m-1}<x} \cdots\cdots \int \prod_{i=1}^{m-1} \{f(x_i)\,dx_i\} \int\limits_{x<y_1<\ldots<y_n<+\infty} \cdots\cdots \int \prod_{j=1}^{n} \{g(y_j)\,dy_j\}$$

Because of

$$\int\limits_{a<x_1<\ldots<x_k<b} \cdots\cdots \int \prod_{i=1}^{k} \{\varphi(x_i)\,dx_i\} = \frac{1}{k!} \left[\int_a^b \varphi(x)\,dx \right]^k$$

(cf. (7)) one finds:

$$(3.3,6)\quad \begin{cases} \alpha_{+}^{(0)}(\mu) = m \int\limits_{-\infty}^{+\infty} f(x) \cdot F^{m-1}(x) \cdot [1-G(x)]^n\,dx = \\ = \int\limits_{-\infty}^{+\infty} [1-G(x)]^n \cdot d\,[F^m(x)] = -\int\limits_{-\infty}^{+\infty} F^m(x) \cdot d\,[\{1-G(x)\}^n]. \end{cases}$$

In the same way it is found from (3. 3, 5b) that

$$(3.3,7)\quad \begin{cases} \alpha_{-}^{(0)}(\mu) = n \int\limits_{+\infty}^{+\infty} g(x) \cdot G^{n-1}(x) \cdot [1-F(x)]^m\,dx = \\ = \int\limits_{-\infty}^{+\infty} [1-F(x)]^m \cdot d\,[G^n(x)] = -\int\limits_{-\infty}^{+\infty} G^n(x) \cdot d\,[(1-F(x)\}^m] \end{cases}$$

For $\alpha_{+}^{(0)}(\mu)$ one can write instead of (3. 3, 4):

$$(3.3,8)\quad \alpha_{+}^{(0)}(\mu) = m!\,n! \int\limits_{-\infty}^{+\infty} f(x_1)\,dx_1 \int\limits_{x_1}^{+\infty} f(x_2)\,dx_2 \int\limits_{x_2}^{+\infty} \cdots \int\limits_{x_m}^{+\infty} f(y_1-\mu)\,dy_1 \int\limits_{y_1}^{+\infty} \cdots \int\limits_{y_{n-1}}^{+\infty} f(y_n-\mu)\,dy_n.$$

This expression will be written more shortly by the use of some operators and notations which will be defined in (3. 3, 9, 9a, 9b and 10):

$$(3.3,9)\quad \begin{cases} \int\limits_{y}^{+\infty} \varphi_1(x_1)\,dx_1 \int\limits_{x_1}^{+\infty} \varphi_2(x_2)\,dx_2 \int\limits_{x_2}^{+\infty} \cdots \int\limits_{x_{p-1}}^{+\infty} \varphi_p(x_p)\,dx_p = \\ = [(I\varphi_1)(I\varphi_2) \ldots (I\varphi_p)](y) = \left[\prod\limits_{\nu=1}^{p} (I\varphi_\nu) \right](y). \end{cases}$$

If $\varphi_\nu(x) = f(x)$ $(\nu = 1, \ldots, k)$, the following abbreviation is used:

$$\left[\prod_{\nu=1}^{p} (I\varphi_\nu)\right](y) = [(If)^p](y). \tag{3.3, 9a}$$

If $\varphi_\nu(x) = f(x)$ for all integers $\nu \neq i$ with $1 \leqq \nu \leqq p$ and $\varphi_i(x) = \{f(x)\}^\omega$, then the abbreviation used is:

$$\left[\prod_{\nu=1}^{p} (I\varphi_\nu)\right](y) = [(If)^{i-1}(If^\omega)(If)^{p-i}](y) \tag{3.3, 9b}$$

Furthermore, the following notations are used:

$$\begin{cases} f(x-\mu) = f_{-\mu} \;;\; f(x+\mu) = f_{+\mu} \;;\; f'(x) = f' \;; \\ f'(x-\mu) = f'_{-\mu} \;;\; f'(x+\mu) = f'_{+\mu} \text{ and } f''(x) = f''. \end{cases} \tag{3.3, 10}$$

With (3. 3, 9, 9*a* and 10) the expression (3. 3, 8) becomes:

$$\alpha_+^{(0)}(\mu) = m!\,n!\,[(If)^m(If_{-\mu})^n](-\infty) \tag{3.3, 11}$$

$$= m!\,n!\,[(If_{+\mu})^m(If)^n](-\infty) \tag{3.3, 11a}$$

((3. 3, 11*a*) follows from (3. 3, 11) by the substitutions

$$x_i = x_i' + \mu \; (1 \leqq i \leqq m) \text{ and } y_j = y_j' + \mu \; (1 \leqq j \leqq n).)$$

In a similar way one finds for $\alpha_+(\mu)$, for any (integer) value of U which is $\leqq \frac{1}{2}mn$:

$$\alpha_+(\mu) = m!\,n! \sum_{U=0}^{U_a} \left[\prod_{h=1}^{r} \{(If)^{m_h}(If_{-\mu})^{n_h}\}\right](-\infty) \tag{3.3, 12}$$

$$= m!\,n! \sum_{U=0}^{U_a} \left[\prod_{h=1}^{r} \{(If_{+\mu})^{m_h}(If)^{n_h}\}\right](-\infty) \tag{3.3, 12a}$$

$$\begin{cases} \text{Here} \quad m_1 \geqq 0 \text{ and } m_h > 0 \text{ for } h > 0, \\ \qquad n_h > 0 \text{ for } h < r \text{ and } n_r \geqq 0\,; \\ \sum_{h=1}^{r} m_h = m\,,\; \sum_{h=1}^{r} n_h = n \text{ and } U = \sum_{i=1}^{r} n_i \sum_{j=i+1}^{r} m_j \end{cases} \tag{3.3, 12b}$$

(3. 3, 12*c*) The summations (over U) in (3. 3, 12) and (3. 3, 12*a*) are to be extended over *all* combinations of r, and of m_h and n_h $(1 \leqq h \leqq r)$ which give a value of $U \leqq U_a$.

A similar expression follows for $\alpha_-(\mu)$ by the use of (3. 3, 2*c*); $\alpha_\pm(\mu)$ then follows from (3. 3, 3).

4. SOME PROPERTIES OF THE POWER FUNCTION OF WILCOXON'S TEST WHEN ALTERNATIVES ARE ONE-SIDED

4.1. *General theorems.*

4.11. *A theorem on the unbiasedness of the test.*

Theorem 1. *Under the restrictions a) and b) of* 3. 1 *the power function*

$\alpha_+(\mu)$ given by (3. 3, 1), *when α, m and n are constant, is a monotonous non-decreasing function of μ.*

Proof.[5]) By the substitutions $x_i = x'_{i'}$; $y_j = y'_{j'} + \mu$ $(i, i' = 1, \ldots, m;$ $j, j' = 1, \ldots, n)$ a one-to-one correspondence is established between the points $E = (x_1, \ldots, x_m, \; y_1, \ldots, y_n)$ of the space W and the points $E' = (x'_1, \ldots, x'_m, \; y'_1, \ldots, y'_n)$ of the space W'. When $U = U(E)$ is defined as in 1. and $U_\mu(E')$, for each point E' with $x'_{i'} \neq y'_{j'} + \mu$ for each pair (i', j'), is defined as the number of pairs (i', j') $(1 \leqq i' \leqq m; 1 \leqq j' \leqq n)$ with $x'_{i'} > y'_{j'} + \mu$, then $U(E) = U_\mu(E')$ if E and E' are corresponding points.

Hence

$$(4.1,1) \qquad \alpha_+(\mu) = \int_{U(E) \leqq U_\alpha} \!\!\cdots\cdots\!\! \int \prod_{i=1}^{m} \{f(x_i)\, dx_i\} \prod_{j=1}^{n} \{f(y_j - \mu)\, dy_j\}$$

$$(4.1,2) \qquad = \int_{U_\mu(E') \leqq U_\alpha} \!\!\cdots\cdots\!\! \int \prod_{i'=1}^{m} \{f(x'_{i'})\, dx'_{i'}\} \prod_{j'=1}^{n} \{f(y'_{j'})\, dy'_{j'}\}.$$

Now the integrand of (4. 1, 2) is independent of μ and non-negative for every E'. From the definition of $U_\mu(E')$ follows that, in a fixed point E', $U_\mu(E')$ is a monotonous non-increasing function of μ. Hence the set of all points with $U_\mu(E') \leqq U_\alpha$ cannot decrease when μ increases. So $\alpha_+(\mu)$ is a monotonous non-decreasing function of μ.

Corollary. For all distribution functions F and G satisfying *a*) and *b*) of 3. 1, the critical region $\boldsymbol{U} \leqq U_\alpha$ provides an *unbiased* test of the hypothesis $\mu = 0$ against all alternatives $\mu > 0$ (cf 2. 3, *a*).

Remark. The same holds for the critical region $\boldsymbol{U} \geqq m\,n - U_\alpha$ when the alternatives are given by $\mu < 0$.

4. 12. *On the interchangeability of m and n in the power function.*

Theorem 2. *Under the restrictions a) and b) of* 3.1 *the power functions $\alpha_+(\mu)$ as well as $\alpha_-(\mu)$ given by* (3. 3, 1) and by (3. 3, 2) *respectively, when α is constant, are identical with those obtained when m and n are interchanged, provided $f(x - c)$ is an even function of x for some c.*

Proof. Without loss of generality one may suppose $c = 0$. Now the expression (3. 3, 2*b*) is equal to the expression (3. 3, 2*c*), when $f(x)$ is an even function. The proof of this equality given in 3. 2, II, *α*) and 3. 2, II, *β*), holds good when $+\mu$ is replaced by $-\mu$. The theorem can also be proved directly from (3. 3, 1) by the substitutions $x_i = -y'_{j'} + \mu$ $(i, j' = 1, \ldots, m)$ and $y_j = -x'_{i'} + \mu$ $(j, i' = 1, \ldots, n)$.

4. 2. *Calculation of $\alpha'_+(0)$ for Wilcoxon's test with specialization to the normal distribution.*

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[5]) This proof is due to Mr J. Hemelrijk.

From the expression (3. 3, 12a) for $\alpha_+(\mu)$, $\alpha'_+(\mu) = \frac{d\alpha_+(\mu)}{d\mu}$ is easily calculated.

Defining the operator I^{-1} by

$$I^{-1}\varphi(x) = I^{-1}\varphi = -\varphi'(x) = -\varphi' \tag{4.2, 1}$$

it follows from the definition of the operator I in (3. 3, 9) that

$$I^{-1}(I\varphi) = \varphi. \tag{4.2, 2}$$

Furthermore defining

$$[\varphi_1(I\varphi_2)](y) = \varphi_1(y) \int^{+\infty} \varphi_2(x)\,dx \tag{4.2, 3}$$

and

$$[(I\varphi_2)\varphi_1](y) = \int_y^{+\infty} \varphi_2(x)\,\varphi_1(x)\,dx \tag{4.2, 4}$$

it is seen that

$$\left\{\begin{aligned} &[(If'_{+\mu})(I\varphi)](y) = -[f_{+\mu}(I\varphi)](y) + [If_{+\mu})\varphi](y) = \\ &= -[I^{-1}(If_{+\mu})(I\varphi)](y) + [(If_{+\mu})I^{-1}(I\varphi)](y). \end{aligned}\right. \tag{4.2, 5}$$

Hence

$$\left\{\begin{aligned} &\left[\frac{\partial}{\partial\mu}\{(If_{+\mu})^m\}\chi\right](y) = \sum_{l=0}^{m-1}[(If_{+\mu})^l(If'_{+\mu})(If_{+\mu})^{m-l-1}\chi](y) = \\ &= -\sum_{l=0}^{m-1}[If_{+\mu})^l I^{-1}(If_{+\mu})^{m-l}\chi](y) + \\ &\qquad + \sum_{l=0}^{m-1}[(If_{+\mu})^{l+1}I^{-1}(If_{+\mu})^{m-l-1}\chi](y) = \\ &= -[I^{-1}(If_{+\mu})^m\chi](y) + [(If_{+\mu})^m I^{-1}\chi](y). \end{aligned}\right. \tag{4.2, 6}$$

Now

$$\alpha'_+(\mu) = m!\,n! \sum_{U=0}^{U_a}\sum_{k=1}^{r}\left[\prod_{h=1}^{k-1}\{(If_{+\mu})^{m_h}(If)^{n_h}\}\cdot \right.$$
$$\left.\cdot\frac{\partial}{\partial\mu}\{(If_{+\mu})^{m_k}\}(If)^{n_k}\prod_{h=k+1}^{r}\{(If_{+\mu})^{m_h}(If)^{n_h}\}\right](-\infty).$$

Hence, by means of (4. 2, 6):

$$\alpha'_+(\mu) = \left\{\begin{aligned} &-m!\,n!\sum_{U=0}^{U_a}\sum_{k=1}^{r}\left[\prod_{h=1}^{k-1}\{(If_{+\mu})^{m_h}(If)^{n_h}\}I^{-1}(If_{+\mu})^{m_k}(If)^{n_k}\prod_{h=k+1}^{r}\{(If_{+\mu})^{m_h}(If)^{n_h}\}\right](-\infty) \\ &+m!\,n!\sum_{U=0}^{U_a}\sum_{k=1}^{r}\left[\prod_{h=1}^{k-1}\{(If_{+\mu})^{m_h}(If)^{n_h}\}(If_{+\mu})^{m_k}I^{-1}(If)^{n_k}\prod_{h=k+1}^{r}\{(If_{+\mu})^{m_h}(If)^{n_h}\}\right](-\infty). \end{aligned}\right. \tag{4.2, 7}$$

If the expression between [ ] begins or ends with I^{-1} for some k, then the corresponding term is to be considered as zero.

From (4. 2, 7) it follows that:

$$\alpha'_+(0) = \left\{\begin{aligned} &-m!\,n!\sum_{U=0}^{U_a}\sum_{k=1}^{r}[(If)^{L_k-1}(If^2)(If)^{m_k+n_k-1+M_k}](-\infty) \\ &+m!\,n!\sum_{U=0}^{U_a}\sum_{k=1}^{r}[(If)^{L_k+m_k-1}(If^2)(If)^{n_k-1+M_k}](-\infty). \end{aligned}\right. \tag{4.2, 8}$$

Here $L_k = \sum_{h=1}^{k-1} (m_h + n_h)$ and $M_k = \sum_{h=k+1}^{r} (m_h + n_h)$; for m_h, n_h, r and $\sum_{U}$ cf. (3. 3, 12*b*) and (3. 3, 12*c*). Those terms in the sums of (4. 2, 8) in which the first or the last symbolic power has a negative exponent are to be considered as zero.

The expression (4. 2, 8) for $\alpha'_+(0)$ will be calculated for $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ (cf. *c*) in 3. 1) from the Appendix; cf. (A. 1, 1) and (A. 1, 9). It is seen from A. 1, Remark 2, that the calculations will be restricted to $\tau \leqq 4$, i.e. to $m + n \leqq 5$. Furthermore, only relatively low values of α are considered (cf. 2. 2). If $U_\alpha = 0$ determines an $\alpha > 0{,}15$, then only $U_\alpha = 0$ is considered, otherwise $U_\alpha = 1$, too, will be considered. Cases with $m = 1$ or $n = 1$ are omitted as being trivial. By means of (5. 2, 8), (A. 1, 1) and (A. 1, 9) the following results are obtained:

$$(4.2,9)\quad \begin{cases} \textit{When } U_\alpha = 0,\ \alpha'_+(0) = m!\,n!\,[(If)^{m-1}(If^2)(If)^{n-1}](-\infty) = \\ \qquad = \dfrac{m!\,n!\,(m+n-1).\ \Gamma\left(\frac{m+n-1}{2}\right)}{2^2\,.\,\pi^{\frac{1}{2}(m+n)}} \cdot V_{1^{m-1},2,1^{n-1}} \end{cases}$$

$$(4.2,10)\quad \begin{cases} \textit{When } U_\alpha = 1,\ \alpha'_+(0) = \\ = +m!\,n!\,[(If)^{m-2}(If^2)(If)^{n}](-\infty) + m!\,n!\,[(If)^{m}(If^2)(If)^{n-2}](-\infty) = \\ = \dfrac{m!\,n!\,(m+n-1)\,.\,\Gamma\left(\frac{m+n-1}{2}\right)}{2^2\,.\,\pi^{\frac{1}{2}(m+n)}} \cdot \{V_{1^{m-2},2,1^{n}} + V_{1^{m},2,1^{n-2}}\} \end{cases}$$

From (4. 2, 9) and (4. 2, 10) the following results are obtained by means of (A. 3, 2) and (A. 4, 4):

m	n	U_α	α	$\alpha'_+(0)$
2	2	0	$\frac{1}{6}$	$\frac{3}{2\pi\sqrt{\pi}} V_{1,2,1} = \frac{1}{\pi\sqrt{\pi}} \text{arc cos } \frac{1}{3} = 0{,}22106$
2	3	0	$\frac{1}{10}$	$\frac{12}{\pi^2\sqrt{\pi}} V_{1,2,1\,1} = \frac{3}{2\pi\sqrt{\pi}} \text{arc sin } \frac{\sqrt{3}}{3} = 0{,}16580$
2	3	1	$\frac{1}{5}$	$\frac{12}{\pi^2\sqrt{\pi}} (V_{2,1,1,1} + V_{1,1,2,1}) =$

$$= \frac{3}{2\pi\sqrt{\pi}} \left(\text{arc sin } \frac{\sqrt{18} - \sqrt{3}}{6} + \text{arc sin } \frac{\sqrt{3}}{3}\right) = \frac{1}{2\sqrt{\pi}} = 0{,}28209.$$

Remark 1. For the values of α cf. Mann and Whitney or 3. 2.
Remark 2. Because of theorem 2 (4. 12) it is not necessary to calculate separately the case $m = 3$, $n = 2$.

4. 3. *A comparison with $\alpha'_+(0)$ for Student's test for the difference of two means.*

The alternatives to the hypothesis tested ($\mu = 0$) are $\mu > 0$ (therefore the notation α_+ is used, cf. (3. 3, 1)). Let $\mathbf{x}_1, \ldots, \mathbf{x}$, $\mathbf{y}_1 \ldots, \mathbf{y}_n$ be

$m+n=N$ independent random variables, the $\mathbf{x}_i$ all having the distribution function $F(x)$ and the $\mathbf{y}_j$ all having the distribution function

$F(x-\mu)$ with $F(x)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{x} e^{-\frac{1}{2}\xi^2}\, d\,\xi$. The critical region for testing, according to STUDENT's test, the hypothesis $\mu=0$ against $\mu>0$ at the level of significance α is given by $\mathbf{t}\geqq t_\alpha$.

Here t is defined by

$$t=\sqrt{\frac{mn\,(N-2)}{N}}\cdot\frac{\bar{y}-\bar{x}}{\sqrt{ms_x^2+ns_y^2}},$$

where

$$\bar{y}=\frac{1}{n}\sum_{j=1}^{n} y_j;\;\; \bar{x}=\frac{1}{m}\sum_{i=1}^{m} x_i;\;\; m\,s_x^2=\sum_{i=1}^{m}(x_i-\bar{x})^2$$

and $n\,s_y^2=\sum\limits_{j=1}^{n}(y_j-\bar{y})^2$, whereas t_α is defined by:

$$\alpha=\frac{1}{\sqrt{(N-2)\,\pi}}\cdot\frac{\Gamma\left(\frac{N-1}{2}\right)}{\Gamma\left(\frac{N-2}{2}\right)}\int\limits_{t_\alpha}^{-\infty}\left(1+\frac{x^2}{N-2}\right)^{-\frac{N-1}{2}} dx.$$

t_α^1 can be found from the tables of STUDENT's distribution. For the values of $\alpha=1/6$, $1/10$ and $1/5$ t_α has been tabulated in 3 decimal places only, or not at all. Therefore the quantity $\eta_\alpha=\frac{N-2}{N-2+t_\alpha^2}$ was calculated directly from

$$\alpha=\tfrac{1}{2}\,I_{\eta_\alpha}\left(\frac{N-2}{2},\frac{1}{2}\right), \tag{4.3,1}$$

where $I_x(p,q)$ is the incomplete B-function tabulated by K. PEARSON (6).

The power function of the critical region $\mathbf{t}\geqq t_\alpha$ is, analagously to P. L. HSU (1):

$$\alpha_+(\mu)=\begin{cases}\dfrac{e^{-\frac{1}{2}\frac{mn}{N}\mu^2}}{2\sqrt{\pi}\,.\,\sqrt{N-2}\cdot\Gamma\left(\frac{N-2}{2}\right)}\cdot\sum\limits_{k=0}^{\infty}\left[\dfrac{2\,mn}{N\,(N-2)}\right]^{\frac{k}{2}}\cdot\\[2ex] \cdot\dfrac{\mu^k\,.\,\Gamma\left(\frac{k+N-1}{2}\right)}{k!}\int\limits_{t_\alpha^2}^{\infty} x^{\frac{k-1}{2}}\left(1+\dfrac{x}{N-2}\right)^{-\frac{k+N-1}{2}} dx\end{cases} \tag{4.3,2}$$

Hence

$$\alpha'_+(0)=\sqrt{\frac{mn}{2\,\pi\,N}}\cdot\left(\frac{N-2}{N-2+t_\alpha^2}\right)^{\frac{N-2}{2}}=\sqrt{\frac{mn}{2\,\pi\,N}}\cdot\eta_\alpha^{\frac{N-2}{2}} \tag{4.3,3}$$

From (4. 3, 3) and (4. 3, 1) the following results are obtained:

m	n	α	η_α	$\alpha'_+(0)$	$\alpha'_+(0)_{St} - \alpha'_+(0)_{Wi}$
2	2	1/6	5/9	0,22163	0,00057
2	3	1/10	0,527963	0,16765	0,00185
2	3	1/5	0,758072	0,28845	0,00636

In order to facilitate the comparison with $\alpha'_+(0)$ for WILCOXON's test, a column is added containing the difference of $\alpha'_+(0)$ for the test of STUDENT and for the test of WILCOXON. It is seen that STUDENT's test and WILCOXON's test satisfy a necessary condition that STUDENT's test is more powerful than WILCOXON's test ($\alpha'_+(0)_{St} - \alpha'_+(0)_{Wi} > 0$, cf. the end of the remark in 2. 3), but clearly the difference is very small.

SOME REMARKS ON THE POWER FUNCTION OF WILCOXON's TEST FOR THE PROBLEM OF TWO SAMPLES. II

BY

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5. SOME PROPERTIES OF THE POWER FUNCTION OF WILCOXON'S TEST WHEN ALTERNATIVES ARE TWO-SIDED

5. 1. *General theorems when alternatives are two-sided.*

5. 11. *Properties of symmetry of the power function.*

Theorem 3. *Under the restrictions a) and b) of* 3. 1 *the power function* $\alpha_{\pm}(\mu)$ *given by* (3. 3, 3), *when* α, m *and* n *are constant, is an even function of* μ, *either when* 1) $f(x-c)$ *is an even function of* x *for some* c, *or when* 2) $m=n$ (*or when both conditions are satisfied, of course*).

Proof.

1. From (3. 3, 3, 1 and 2b) one obtains:

$$\alpha_{\pm}(\mu)=\int_{U\leq U_\alpha}\!\!\cdots\int \prod_{i=1}^{m}\{f(x_i)\,dx_i\}\prod_{j=1}^{n}\{f(y_j-\mu)\,dy_j\}+$$

$$+\int_{U\leq U_\alpha}\!\!\cdots\int \prod_{i=1}^{m}\{f(-x_i)\,dx_i\}\prod_{j=1}^{n}\{f(-y_j-\mu)\,dy_j\}.$$

Without loss of generality c may be supposed to be equal to zero.

Then $f(x)$ is an even function. Hence:

$$(5.\ 1,\ 1)\quad \begin{cases} \alpha_{\pm}(\mu) = \int\limits_{U \leqq U_a} \cdots \int \prod\limits_{i=1}^{m} \{f(x_i)\, dx_i\} \prod\limits_{j=1}^{n} \{f(y_j - \mu)\, dy_j\} + \\ \qquad + \int\limits_{U \leqq U_a} \cdots \int \prod\limits_{i=1}^{m} \{f(x_i)\, dx_i\} \prod\limits_{j=1}^{n} \{f(y_j + \mu)\, dy_j\}. \end{cases}$$

By changing $+\mu$ into $-\mu$ the first term of the second member of (5. 1, 1) passes into the second one and vice versa. Hence $\alpha_{\pm}(-\mu) = \alpha_{\pm}(+\mu)$, q.e.d.

2. From (3. 3, 3, 1 and 2c) one obtains:

$$\alpha_{\pm}(\mu) = \int\limits_{U \leqq U_a} \cdots \int \prod\limits_{i=1}^{m} \{f(x_i)\, dx_i\} \prod\limits_{j=1}^{n} \{f(y_j - \mu)\, dy_j\} +$$

$$+ \int\limits_{U \leqq U_a} \cdots \int \prod\limits_{i=1}^{n} \{f(x_i)\, dx_i\} \prod\limits_{j=1}^{m} \{f(y_j + \mu)\, dy_j\}.$$

When $m = n$, it follows that:

$$(5.\ 1,\ 2)\quad \begin{cases} \alpha_{\pm}(\mu) = \int\limits_{U \leqq U_a} \cdots \int \prod\limits_{i=1}^{m} \{f(x_i)\, dx_i\} \prod\limits_{j=1}^{m} \{f(y_j - \mu)\, dy_j\} + \\ \qquad + \int\limits_{U \leqq U_a} \cdots \int \prod\limits_{i=1}^{m} \{f(x_i)\, dx_i\} \prod\limits_{j=1}^{m} \{f(y_j + \mu)\, dy_j\}. \end{cases}$$

By changing $+\mu$ into $-\mu$ the first term of the second member of (5. 1, 2) passes into the second one and vice versa. Hence $\alpha_{\pm}(-\mu) = \alpha_{\pm}(+\mu)$, q.e.d.

Corollary. If the conditions of theorem 3 are satisfied and $\alpha'_{\pm}(0)$ exists, then $\alpha'_{\pm}(0) = 0$.

When $f(x)$ is a non-symmetric function and $m \neq n$, then $\alpha_{\pm}(\mu)$, defined by (3. 3, 3), need not be an even function of μ, as follows from theorem 4

Theorem 4. *Under the restrictions a) and b) of 3. 1 a non-symmetric function $f(x)$ and a value of U_a can be given, such that $\alpha'_{\pm}(0)$ exists and is $\neq 0$, if $m \neq n$.*

Proof.

Let $F(x) = \begin{Bmatrix} 0 & \text{for } x \leqq 0 \\ 1 - e^{-x} & \text{for } x \geqq 0 \end{Bmatrix}$, $G(x) = F(x - \mu)$ and $U_a = 0$.

Then by somewhat laborious calculations it can be proved, using (3. 3, 6 and 7), that:

$$(5.1, 3)\quad \alpha_{\pm}(\mu) = \begin{cases} \dfrac{m!\, n!}{(m+n)!}\, e^{n\mu} + (1 - e^{\mu})^n + n e^{-m\mu} \int\limits_0^{e^{\mu}} (1-x)^{n-1} x^m\, dx, & \text{if } \mu < 0. \\ \dfrac{m!\, n!}{(m+n)!}\, e^{-m\mu} + (1 - e^{-\mu})^m + m e^{n\mu} \int\limits_0^{e^{-\mu}} (1-x)^{m-1} x^n\, dx, & \text{if } \mu > 0. \end{cases}$$

Hence

$$\alpha'_{\pm}(0) = \frac{m!\,n!}{(m+n)!}(n-m) \neq 0, \text{ q. e. d.} \tag{5. 1, 4}$$

5. 12 *A theorem on possible biasedness of the test when alternatives are two-sided.*

As the critical region given by $|\boldsymbol{U} - \frac{1}{2}mn| \geqq \frac{1}{2}mn - U_\alpha$ cannot possibly provide an unbiased test for the hypothesis $\mu = 0$ against all alternatives $\mu \neq 0$, when $\alpha'_{\pm}(0)$ exists and is $\neq 0$, one should, if possible, take $m = n$ in the applications of the test. In this way one secures the symmetry of the power function. One might hope that unbiasedness then would be secured, too. One might think that as general a theorem would hold good for two-sided alternatives, as theorem 1 and its corollary proved in 4. 1, for onesided alternatives. That this is impossible in such a generality, is shown by theorem 5:

Theorem 5. *Under the restrictions a) and b) of* 3. 1 *for every m and $n = m$ an even function $f(x)$ and a value of U_α can be given, such that $\alpha''_{\pm}(0)$ exists and is* < 0.

Proof. Let $U_\alpha = 0$. By (3. 3, 3, 1 and 2*b*) the power function $\alpha_{\pm}(\mu)$, when $f(x)$ is an even function and $m = n$, is equal to (5. 1, 2), where now "$U \leqq U_\alpha$" is to replaced by "$U = 0$". By a reduction as described in the remark of 3. 3 (under (3. 3, 5*b*)) it is found that:

$$\begin{cases} \alpha_{\pm}(\mu) = \int\limits_{-\infty}^{+\infty} [F^m(x)]' \cdot [(1 - F(x-\mu))^m]\, dx + \\ + \int\limits_{-\infty}^{+\infty} [F^m(x)]' \cdot [(1 - F(x+\mu))^m]\, dx. \end{cases} \tag{5. 1, 5}$$

Hence, writing $\alpha''_{\pm}(\mu)$ for $\frac{d^2\alpha_{\pm}(\mu)}{d\mu^2}$,

$$\alpha''_{\pm}(0) = 2\int\limits_{-\infty}^{+\infty} [F^m(x)]' \cdot [(1 - F(x))^m]''\, dx, \tag{5. 1, 6}$$

the primes denoting differentiation with respect to x. Because of $f(x)$ being even, $F(-y) = 1 - F(y)$. Hence:

$$\int\limits_{-\infty}^{0} [F^m(x)]' \cdot [(1 - F(x))^m]''\, dx = -\int\limits_{0}^{+\infty} [(1 - F(y))^m]' \cdot [F^m(y)]''\, dy, \tag{5. 1, 7}$$

as is seen by the substitution $x = -y$. (The primes in the second member of (5. 1, 7) denote differentiation with respect to y). By partial integration the second member of (5. 1, 7) is seen to be equal to

$$+\int\limits_{0}^{+\infty} [F^m(y)]' \cdot [(1 - F(y))^m]''\, dy,$$

so that (cf. (5. 1, 6) and (5. 1, 7))

$$\alpha''_{\pm}(0) = 4\int\limits_{0}^{+\infty} [F^m(x)]' \cdot [(1 - F(x))^m]''\, dx. \tag{5. 1, 8}$$

Now $[F^m(x)]' \geqq 0$ for every x with $0 \leqq x \leqq \infty$. Hence a sufficient condition for $\alpha''_{\pm}(0)$ being < 0 is that $[(1 - F(x))^{m}]'' \leqq 0$ for $0 \leqq x \leqq \infty$, where the equality-sign does not hold in the whole interval $(0, \infty)$.

Such a function $F(x)$, the derivative $f(x)$ of which is an even function, is defined by

$$(5.\,1,\,9) \qquad F(x) = \left\{\begin{array}{ll} 0 & \text{for} \quad x \leqq -1 \\ \frac{1}{2}(1-x^2)^{1/m} & \text{for} \; -1 \leqq x \leqq 0 \\ 1-\frac{1}{2}(1-x^2)^{1/m} & \text{for} \quad 0 \leqq x \leqq +1 \\ 1 & \text{for} \quad x \geqq +1 \end{array}\right\}$$

For $[(1-F(x))^m]'' = 2^{-m}(1-x^2)'' = -2^{1-m}$, if $0 \leqq x \leqq +1$. Hence $\alpha''_{\pm}(0) = -2^{3-m}\left(1-\frac{1}{2m}\right) < 0$.

Conclusion. The necessary condition for unbiasedness (consisting in $\alpha''_{\pm}(0)$ not being < 0, cf. 2. 3, remark) is not satisfied in general. Hence the critical region defined by $|\boldsymbol{U} - \frac{1}{2}mn| \geqq \frac{1}{2}mn - U_\alpha$ does not provide an unbiased test for the hypothesis $\mu = 0$ against $\mu \neq 0$ without further restrictions being imposed upon F and G.

5. 13. *On the interchangeability of m and n in the power function when alternatives are two-sided.*

Theorem 6. *Under the restrictions a) and b) of 3.1 the power function $\alpha_{\pm}(\mu)$, given by (3. 3, 3), when α is constant, is identical with the power function $\alpha_{\pm}(\mu)$ obtained when m and n are interchanged, provided $f(x-c)$ is an even function of x for some c.*

Proof. (3. 3, 3) and theorem 2.

5. 2. *Calculation of $\alpha''_{\pm}(0)$ for Wilcoxon's test with specialization to the normal distribution.*

Throughout 5. 2 $f(x)$ is assumed to be an even function. After (5. 2, 8b) $f(x)$ is taken to be equal to $\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$. From (3. 3, 3, 1 and 2b) it follows that ($f(x)$ being an even function): $\alpha_{\pm}(\mu) = \alpha_{+}(\mu) + \alpha_{-}(\mu) =$

$$= \int_{U \leqq U_\alpha} \cdots \int \prod_{i=1}^{m} \{f(x_i)\,dx_i\} \prod_{j=1}^{n} \{f(y_j - \mu)\,dy_j\} +$$

$$+ \int_{U \leqq U_\alpha} \cdots \int \prod_{i=1}^{m} \{f(x_i)\,dx_i\} \prod_{j=1}^{n} f(y_j + \mu)\,dy_j\}.$$

Hence $\alpha_{\pm}(\mu) = \alpha_{+}(\mu) + \alpha_{+}(-\mu)$, so that $\alpha_{\pm}(0) = 2\,\alpha_{+}(0) = 2\,\alpha$
$\alpha'_{\pm}(0) = \alpha'_{+}(0) - \alpha'_{+}(0) = 0$ and $\alpha''_{\pm}(0) = 2\,\alpha''_{+}(0)$.

Now $\alpha''_{+}(0)$ is calculated, starting from (4. 2, 7):

The operator I^{-2} is defined by

$$(5.\,2,\,1) \qquad I^{-2}\varphi(x) = I^{-1}I^{-1}\varphi = -I^{-1}\varphi' = +\varphi''(x) = +\varphi''.$$

From the definitions in 4. 2 it then follows that

$$(5.\,2,\,2)\quad \begin{cases} \left[\frac{\partial}{\partial\mu}\{I^{-1}(If_{+\mu})^{m}\}\,\chi\right](y) = \left[I^{-1}\frac{\partial}{\partial\mu}\{(If_{+\mu})^{m}\}\,\chi\right](y) = (by\ (4.\,2,6)) = \\ = -[I^{-2}(If_{+\mu})^{m}\chi](y) + [I^{-1}(If_{+\mu})^{m}I^{-1}\chi](y). \end{cases}$$

Furthermore from (5. 2, 1) and (4. 2, 6):

$$(5.\,2,3)\quad \left[\frac{\partial}{\partial\mu}(If_{+\mu})^{m}I^{-1}\chi\right](y) = -[I^{-1}(If_{+\mu})^{m}I^{-1}\chi](y) + [(If_{+\mu})^{m}I^{-2}\chi](y).$$

From (4. 2, 7), by means of (4. 2, 6), (5. 2, 2) and (5. 2, 3), one obtains: $\alpha''_{\pm}(0) = 2\,\alpha''_{+}(0) =$

$$(5.2,4) = \begin{cases} 4\,m!\,n!\sum_{U=0}^{U_a}\sum_{z=2}^{2r-1}\sum_{w=1}^{z-1}[(-1)^{w+z}(If)^{P_w}I^{-1}(If)^{Q_{w,z}}I^{-1}(If)^{R_z}](-\infty) \\ +\,2\,m!\,n!\sum_{U=0}^{U_a}\sum_{w=1}^{2r-1}[(If)^{P_w}I^{-2}(If)^{S_w}](-\infty). \end{cases}$$

Here:

$$P_w = \begin{cases} \sum_{h=1}^{\frac{1}{2}w}(m_h + n_h), & \text{if} \quad w \text{ is even.} \\ \sum_{h=1}^{\frac{1}{2}(w-1)}(m_h + n_h) + m_{\frac{1}{2}(w+1)}, & \text{if} \quad w \text{ is odd.} \end{cases}$$

$$R_z = \sum_{h=1}^{r}(m_h + n_h) - P_z = \begin{cases} \sum_{h=\frac{1}{2}z+1}^{r}(m_h + n_h), & \text{if} \quad z \text{ is even.} \\ n_{\frac{1}{2}(z+1)} + \sum_{h=\frac{1}{2}(z+3)}^{r}(m_h + n_h), & \text{if} \quad z \text{ is odd.} \end{cases}$$

$$Q_{w,z} = \sum_{h=1}^{r}(m_h + n_h) - P_w - R_z.$$

$$S_w = \sum_{h=1}^{r}(m_h + n_h) - P_w.$$

If in (5. 2, 4) the expression between [ ] begins or ends with I^{-1} or I^{-2} for some w or z, then the corresponding term is to be considered as zero.

> Remark 1. The summation variables w and z should not be confounded, of course, with the w of 2. 1 (critical region) and the z (coordinates) of the Appendix.

As to the meaning of m_h, n_h and r, cf. (3. 3, 12*b*).

The summations over U in (5. 2, 4) are to be extended over *all* combinations of r, and of m_h and n_h ($1 \leqq h \leqq r$) which give a value of $U \leqq U_a$. If $U = 0$, then $r = 1$ and the sum over z in (5. 2, 4) is empty. Hence in the first term of (5. 2, 4) $\sum_{U=0}^{U_a}$ can be replaced by $\sum_{U=1}^{U_a}$.

Remark 2. Introducing the notations:

$$d_{2h-1} = m_h,\ d_{2h} = n_h, \text{ and}$$
$$f_{\mu,j} = \begin{cases} f \quad (= f(x)), & \text{if } j \text{ is even,} \\ f_{+\mu}\ (= f(x+\mu)), & \text{if } j \text{ is odd,} \end{cases}$$

$\alpha''_{\pm}(\mu)$ can be written as:

$$(5.\,2,\,5)\quad \begin{cases} 4\,m!\,n! \sum\limits_{U=1}^{U_a} \sum\limits_{z=2}^{2r-1} \sum\limits_{w=1}^{z-1} \left[\prod\limits_{j=1}^{w} \{(If_{\mu,j})^{d_j}\}\, I^{-1} \prod\limits_{j=w+1}^{z} \{(If_{\mu,j})^{d_j}\}\, I^{-1} \prod\limits_{j=z+1}^{2r} \{(If_{\mu,j})^{d_j}\} \right] (-\infty) \\ + \, 2\,m!\,n! \sum\limits_{U=0}^{U_r} \sum\limits_{w=1}^{2r-1} \left[\prod\limits_{j=1}^{w} \{(If_{\mu,j})^{d_j}\}\, I^{-2} \prod\limits_{j=w+1}^{2r} \{(If_{\mu,j})^{d_j}\} \right] (-\infty) \text{ (end of Remark 2.)} \end{cases}$$

In the applications of (5. 2, 4 or 5) the following reductions are to be used:

$$(5.\,2,\,6)\quad \begin{cases} [(If)^K I^{-1} (If)^L I^{-1} (If)^M] (-\infty) = \\ = \begin{cases} [(If)^{K-1} (If^2) (If)^{L-2} (If^2) (If)^{M-1}] (-\infty), & \text{if } L \geqq 2, \\ [(If)^{K-1} (If^3) (If)^{M-1}] (-\infty), & \text{if } L = 1 \end{cases} \end{cases}$$

and: $[(If)^K I^{-2} (If)^L] (-\infty) =$

$$(5.\,2,\,7)\quad \begin{cases} = [(If)^K I^{-1} f (If)^{L-1}] (-\infty) = \\ = -[(If)^K f' (If)^{L-1}] (-\infty) + [(If)^K ff (If)^{L-2}] (-\infty) = \\ = -[(If)^{K-2} (If) (Iff') (If)^{L-1}] (-\infty) + [(If)^{K-1} (If^3) (If)^{L-2}] (-\infty) = \\ = +\tfrac{1}{2} [(If)^{K-2} (If^3) (If)^{L-1}] (-\infty) + \tfrac{1}{2} [(If)^{K-1} (If^3) (If)^{L-2}] (-\infty). \end{cases}$$

By means of (5. 2, 6) and (5. 2, 7) it follows from (5. 2, 4) that $\alpha''_{\pm}(0) =$

$$(5.\,2,\,8)\quad \begin{cases} 4\,m!\,n! \sum\limits_{U=1}^{U_a} \sum\limits_{z=2}^{2r-1} \sum\limits_{w=1}^{z-1} [(-1)^{w+z} (If)^{P_w-1} (If^2) (If)^{Q_{w,z}-} (If^2) (If)^{R_z-1}] (-\infty) \\ + \, m!\,n! \sum\limits_{U=0}^{U_a} \sum\limits_{w=1}^{2r-1} [(If)^{P_w-2} (If^3) (If)^{S_w-1}] (-\infty) \\ + \, m!\,n! \sum\limits_{U=0}^{U_a} \sum\limits_{w=1}^{2r-1} [(If)^{P_w-1} (If^3) (If)^{S_w-2}] (-\infty). \end{cases}$$

For P_w, $Q_{w,z}$, R_z and S_w and for the summations over U see the indications under (5. 2, 4).

If $Q_{w,z} = 1$ for some w or z, then the corresponding term in the first sum of the second member of (5. 2, 8) is to be replaced by:

$$(5.\,2,\,8a)\qquad 4\,m!\,n! [(-1)^{w+z} (If)^{P_w-1} (If^3) (If)^{R_z-1}] (-\infty).$$

In (5. 2, 6), (5. 2, 7) and (5. 2, 8) those terms in which the first or the last symbolic power has a negative exponent, are to be considered as zero.

The expression (5. 2, 8) for $\alpha''_{\pm}(0)$ will be calculated for $f(x) = \frac{1}{\sqrt{2\pi}} e^{-1/2 x^2}$ (cf. c) in 3. 1) by means of the Appendix; cf. (A. 1, 1) and (A. 1, 9). It is seen from A. 1, Remark 2, that the calculations will be restricted to $\tau \leqq 4$, i.e. to $m + n \leqq 6$. Furthermore, only relatively low values of 2α are considered (cf. 2. 2; $2\alpha = \alpha_{\pm}(0)$, see 5. 2, some lines before (5. 2, 1)). If $U_a = 0$ determines a value of $2\alpha > 0{,}15$, then only $U_a = 0$ is considered, otherwise $U_a = 1$, too, will be considered. Cases with $m = 1$ or $n = 1$ are omitted as being trivial. By means of (5. 2, 8), (A. 1, 1) and A. 1, 9) the following results are obtained:

$$(5.2,9)\quad \begin{cases} \textit{When } U_a = 0,\ \alpha''_{\pm}(0) = \begin{cases} + m!\,n!\,[(If)^{m-2}(If^3)(If)^{n-1}](-\infty) + \\ + m!\,n!\,[(If)^{m-1}(If^3)(If)^{n-2}](-\infty) = \end{cases} \\ = \dfrac{m!\,n!\,(m+n-2)\,.\,\Gamma\left(\frac{m+n-2}{2}\right)}{2^2\,.\,\pi^{\frac{m+n}{2}}\,.\,\sqrt{3}}\,[V_{1^{m-2},3,1^{n-1}} + V_{1^{m-1},3,1^{n-2}}]. \end{cases}$$

$$(5.2,10)\quad \begin{cases} \textit{When } U_a = 1,\ \alpha''_{\pm}(0) = m!\,n!\,\{+ 4\,[(If)^{m-2}(If^2)^2(If)^{n-2}](-\infty)\} + \\ + m!\,n!\,\{+ [(If)^{m-3}(If^3)(If)^{n}](-\infty) - [(If)^{m-2}(If^3)(If)^{n-1}](-\infty)\} + \\ + m!\,n!\,\{- [(If)^{m-1}(If^3)(If)^{n-2}](-\infty) + [(If)^{m}(If^3)(If)^{n-3}](-\infty)\} = \\ = \dfrac{m!\,n!\,(m+n-2)\,.\,\Gamma\left(\frac{m+n-2}{2}\right)}{2^2\,.\,\pi^{\frac{m+n}{2}}}\,[2\cdot V_{1^{m-2},2,2,1^{n-2}} + \dfrac{1}{\sqrt{3}}\cdot V_{1^{m-3},3,1^{n}} - \\ - \dfrac{1}{\sqrt{3}}\cdot V_{1^{m-2},3,1^{n-1}} - \dfrac{1}{\sqrt{3}}\cdot V_{1^{m-1},3,1^{n-2}} + \dfrac{1}{\sqrt{3}}\cdot V_{1^{m},3,1^{n-3}}]. \end{cases}$$

Those terms in the last members of (5. 2, 9) and (5. 2, 10) in which a symbolic power has a negative exponent, are to be considered as zero.

From (5. 2, 9) and (5. 2, 10) the following results are obtained by means of (A. 2, 1), (A. 3, 2) and (A. 4, 4):

m	n	U_a	2α	$\alpha''_{\pm}(0)$
2	2	0	$\frac{1}{3}$	$\frac{2}{\pi^2\sqrt{3}}[V_{3,1} + V_{1,3}] = \frac{2}{\pi\sqrt{3}} = 0{,}36755$
2	3	0	$\frac{1}{5}$	$\frac{3\sqrt{3}}{2\pi^2}[V_{3,1,1} + V_{1,3,1}] = \frac{\sqrt{3}}{\pi^2}\left[\text{arc cos}\,\frac{\sqrt{6}}{4} + \text{arc cos}\,\frac{1}{4}\right]$ $= \frac{\sqrt{3}}{\pi^2}\left[\pi - \text{arc cos}\,\frac{\sqrt{6}}{4}\right] = 0{,}39132$
2	4	0	$\frac{2}{15}$	$\frac{16\sqrt{3}}{\pi^3}[V_{3,1,1,1} + V_{1,3,1,1}] =$ $= \frac{2\sqrt{3}}{\pi^2}\left[\text{arc sin}\,\frac{\sqrt{30}-\sqrt{6}}{8} + \text{arc sin}\,\frac{\sqrt{6}}{4}\right] =$ $= \frac{2\sqrt{3}}{\pi^2}\,\text{arc cos}\,\frac{1}{2} = \frac{2\sqrt{3}}{3\pi} = 0{,}36755$

m	n	U_α	2α	$\alpha''_{\pm}(0)$
2	4	1	$\frac{4}{15}$	$\frac{6}{\pi^2}\left[2\,V_{2,2,1,1}-\frac{1}{\sqrt{3}}\,V_{3,1,1,1}\right]= =\frac{6}{\pi^2}\left[2 \text{ arc sin } \frac{\sqrt{20}-\sqrt{2}}{6}-\frac{1}{\sqrt{3}} \text{ arc sin } \frac{\sqrt{30}-\sqrt{6}}{8}\right] = 0{,}51398.$
3	3	0	$\frac{1}{10}$	$\frac{24\sqrt{3}}{\pi^3}\,V_{1,3,1,1}=\frac{3\sqrt{3}}{\pi^2} \text{ arc sin } \frac{\sqrt{6}}{4}=0{,}34698.$
3	3	1	$\frac{1}{5}$	$\frac{72}{\pi^2}\left[V_{1,2,2,1}+\frac{1}{\sqrt{3}}\,V_{3,1,1,1}-\frac{1}{\sqrt{3}}\,V_{1,3,1,1}\right]= =\frac{9}{\pi^2}\left[\text{arc sin } \frac{2}{3}+\frac{1}{\sqrt{3}} \text{ arc sin } \frac{\sqrt{30}-\sqrt{6}}{8}-\frac{1}{\sqrt{3}} \text{ arc sin } \frac{\sqrt{6}}{4}\right]=0{,}52282.$

Remark 3. For the values of α cf. MANN and WHITNEY (2) or 3. 2.

Remark 4. Because of theorem 6 (5. 13) it is not necessary to calculate separately the case $\left\{\begin{matrix} m=3, & n=2 \\ m=4, & n=2 \end{matrix}\right\}$.

5. 3. *A comparison with $\alpha''_{\pm}(0)$ for Student's test for the difference of two means.*

The alternatives to the hypothesis tested ($\mu=0$) are $\mu \neq 0$ (therefore the notation $\alpha_{\pm}$ is used, cf. (3. 3, 3)). Under the same assumptions about $\mathbf{x}_1, \ldots, \mathbf{x}_m, \mathbf{y}_1, \ldots, \mathbf{y}_n$ as in 4. 3, the critical region for testing, according to STUDENT's test, the hypothesis $\mu=0$ against $\mu \neq 0$ at the level of significance 2α is given by $|\mathbf{t}| \geqq t_\alpha$. Here t and t_α are defined as in 4. 3. As t_α for the α-values needed has been tabulated in 3 decimal places only or not at all, the quantity $\eta_\alpha=\frac{N-2}{N-2+t_\alpha^2}$ $(N=m+n)$ was calculated directly from

$$\alpha=\tfrac{1}{2}\,I_{\eta_\alpha}\left(\frac{N-2}{2},\tfrac{1}{2}\right), \tag{5. 3, 1}$$

where $I_x\,(p,q)$ is the incomplete B-function tabulated by K. PEARSON (6).

The power function of the critical region $|\mathbf{t}| \geqq t_\alpha$ is (cf. P. L. HSU (1)):

$$\alpha_{\pm}(\mu)=\left\{\begin{aligned} &\frac{e^{-\frac{1}{2}\frac{mn}{N}\mu^2}}{\sqrt{N-2}\,.\,\Gamma\left(\frac{N-2}{2}\right)}\cdot\sum_{k=0}^{\infty}\left[\frac{mn}{2N(N-2)}\right]^k\cdot \\ &\cdot\frac{\mu^{2k}\,.\,\Gamma\left(\frac{2k+N-1}{2}\right)}{k!\,\Gamma(k+\frac{1}{2})}\int_{t_\alpha^2}^{\infty} x^{k-\frac{1}{2}}\left(1+\frac{x}{N-2}\right)^{-k-\frac{N-1}{2}} dx. \end{aligned}\right. \tag{5. 3, 2}$$

Hence $\alpha''_{\pm}(0)=$

$$=\frac{mn\,.\,\Gamma\left(\frac{N-1}{2}\right)}{N\sqrt{N-2}\,.\,\sqrt{\pi}\,.\,\Gamma\left(\frac{N-2}{2}\right)}\cdot$$

$$\cdot\left[\frac{N-1}{N-2}\int_{t_\alpha^2}^{\infty} x^{\frac{1}{2}}\left(1+\frac{x}{N-2}\right)^{-\frac{N+1}{2}} dx-\int_{t_\alpha^2}^{\infty} x^{-\frac{1}{2}}\left(1+\frac{x}{N-2}\right)^{-\frac{N-1}{2}} dx\right].$$

By the substitution $1+\frac{x}{N-2}=y^{-1}$ it easily follows that:

$$(5.\,3,\,3)\quad \left\{ \begin{aligned} \alpha''_{\pm}(0) &= \frac{mn\,.\,\Gamma\left(\frac{N-1}{2}\right)}{N\sqrt{\pi}\,.\,\Gamma\left(\frac{N-2}{2}\right)}\left[(N-1)\int_0^{\eta_a} y^{\frac{N-4}{2}}(1-y)^{\frac{1}{2}}\,dy-\int_0^{\eta_a} y^{\frac{N-4}{2}}(1-y)^{-\frac{1}{2}}\,dy\right] \\ &= \frac{mn}{N}\cdot\frac{2\,\Gamma\left(\frac{N-1}{2}\right)}{\sqrt{\pi}\,.\,\Gamma\left(\frac{N-2}{2}\right)}\cdot\eta_a^{\frac{N-2}{2}}\cdot(1-\eta_a)^{\frac{1}{2}}. \end{aligned} \right.$$

Here $N = m + n$.

In the cases, for which $\alpha''_{\pm}(0)$ was calculated for Wilcoxon's test (cf. 5. 2, at the end) the following results are obtained for Student's test from (5. 3, 3) and (5. 3, 1):

m	n	$2\,\alpha$	η_a	$\alpha''_{\pm}(0)$	$\alpha''_{\pm}(0)_{St}-\alpha''_{\pm}(0)_{Wi}$
2	2	1/3	5/9	0,37037	0,00282
2	3	1/5	0.527963	0,40270	0,01138
2	4	2/15	0,530988	0,38618	0,01863
2	4	4/15	0,706294	0,54070	0,02672
3	3	1/10	0,468123	0,35959	0,01261
3	3	1/5	0,629850	0,54306	0,02024

In order to facilitate the comparison with $\alpha''_{\pm}(0)$ for Wilcoxon's test, a column is added containing the difference of $\alpha''_{\pm}(0)$ for the test of Student and for the test of Wilcoxon. This comparison shows that under the conditions which allow the use of Student's test,[6] a necessary condition for Student's test being more powerful than Wilcoxon's one is satisfied (cf. the remark in 2. 3), but in the cases investigated the difference is very small.

Remark. The following provisional result for large m and n was obtained for the power function of Wilcoxon's test when alternatives are two-sided:

$$\alpha''_{\pm}(0)_{Wi} \approx \frac{\sqrt{2}}{\sqrt{\pi}}\cdot e^{-\frac{1}{2}\zeta_a^2}\cdot\zeta_a\cdot\frac{6\,mn}{2\pi\,(m+n)},$$

[6]) When $f(x)=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}$ and $g(x)=\frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-\mu)^2}$ and a critical region $|\mathbf{t}| \geqq t_a$ is defined as in the beginning of 5.3, then $\alpha''_{\pm}(\mu)$ depends on σ. Denoting $\alpha_{\pm}(0)$ by $\beta(\sigma^2)$ to make this dependence explicit, it follows from Hsu (1) that, when $m=2$, $n=3$, and $2\,\alpha=1/5$, : $\beta'(1)=0{,}0447$ for Student's test (the prime denotes differentiation with respect to σ^2), whereas calculations showed that the corresponding quantity for Wilcoxon's test is equal to 0,0357. So, in this example at least, Wilcoxon's test is less sensitive to the invalidity of the assumption $\sigma = 1$ than Student's test.

except for a relative error of the order $1/m$ and $1/n$. Here ζ_a is defined by:

$$a = \frac{1}{\sqrt{2\pi}} \int_{\zeta_a}^{\infty} e^{-\frac{1}{2}x^2}\, dx,$$

where $2a$ is the level of significance of the test.

For STUDENT's test it is obtained from (5. 3, 3) that:

$$a''_{\pm}(0)_{St} \approx \frac{\sqrt{2}}{\sqrt{\pi}} \cdot e^{-\frac{1}{2}\zeta_a^2} \cdot \zeta_a \cdot \frac{mn}{m+n},$$

(except for a relative error of the order $1/m$ and $1/n$). Hence $a''_{\pm}(0)_{Wi}/a''_{\pm}(0)_{St} \approx 3/\pi$, so that the difference in power between the two tests is not great indeed. And then, WILCOXON's test is more general in that $a(H_0)$ has the same value whatever $F(x)$, provided $F(x)$ is continuous.

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6. *Summary.*

Some properties of the critical region and the power function of WILCOXON's non-parametric solution of the problem of two samples are studied. Under the conditions which allow the use of STUDENT's test the difference in power between the two tests is investigated, as well for one-sided as for two-sided sets of alternative hypotheses, when the sum of the sample-sizes is $\leqq 5$ and $\leqq 6$ respectively. In these cases the difference is rather small. Indications are that for large sample sizes, too, the difference in power is not great. In an appendix the relation of the power function of WILCOXON's test with the volume of a spherical simplex is exposed, which shows the limitation of the sample sizes (to 5 and 6 resp.) to be relevant. In an introduction the concepts of critical region and of power function in general are exposed.

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A. APPENDIX

CALCULATION OF A CERTAIN MULTIPLE INTEGRAL

A. 1. *The relation of the integral with a spherical simplex.*

Let

(A. 1, 1) $$J = \left[\prod_{i=1}^{\tau} (If^{k_i})\right](-\infty) = \int_G \cdots \int \prod_{i=1}^{\tau} \{f^{k_i}(x_i)\, dx_i\}$$

where

$$k_i > 0,\ \sum_{i=1}^{\tau} k_i = N,\ f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

and the region G is defined by $-\infty < x_1 < x_2 < \ldots < x_\tau < +\infty$.

Then

(A. 1, 1a) $$J = \frac{1}{(2\pi)^{\frac{1}{2}N}} \int_G \cdots \int e^{-\frac{1}{2}\sum\limits_{i=1}^{\tau} k_i x_i^2} \prod_{i=1}^{\tau} dx_i$$

Be $\cos\theta_j$ denoted with c_j and $\sin\theta_j$ with s_j ($j = 1, \ldots, \tau-1$). Make the substitutions:

(A. 1, 2) $$x_i = \frac{1}{\sqrt{k_i}} \cdot R \cdot s_{\tau-i} \cdot \prod_{j=\tau-i+1}^{\tau-1} c_j \qquad (i = 1, \ldots, \tau)$$

with $s_0 = 1$; $0 \leqq \theta_1 \leqq 2\pi$; $0 \leqq \theta_j \leqq \pi$ ($j = 2, \ldots, \tau-1$).

Then

$$\frac{\partial(x_1, x_2, \ldots, x_{\tau-1}, x_\tau)}{\partial(R, \theta_1, \ldots, \theta_{\tau-2}, \theta_{\tau-1})} = \frac{1}{\sqrt{\prod\limits_{i=1}^{\tau} k_i}} R^{\tau-1} \cdot \prod_{j=1}^{\tau-1} c_j^{j-1}.$$

Hence

(A. 1, 3) $$J = \frac{1}{(2\pi)^{\frac{1}{2}N} \cdot \sqrt{\prod\limits_{i=1}^{\tau} k_i}} \int_0^\infty e^{-\frac{1}{2}R^2} \cdot R^{\tau-1}\, dR \cdot \mathfrak{L} = \frac{\Gamma\left(\frac{\tau}{2}\right)}{2^{\frac{1}{2}(N-\tau-2)}\, \pi^{\frac{1}{2}N} \sqrt{\prod\limits_{i=1}^{\tau} k_i}} \cdot \mathfrak{L}$$

Here

$$\text{(A. 1, 4)} \qquad \mathfrak{L} = \int_{G'} \cdots \int \prod_{j=1}^{\tau-1} \{\cos^{j-1}\theta_j \, d\theta_j\},$$

where the region G' is defined by:

$$-\frac{1}{\sqrt{k_1}} < \frac{1}{\sqrt{k_1}} s_{\tau-1} < \frac{1}{\sqrt{k_2}} s_{\tau-2} c_{\tau-1} < \ldots < \frac{1}{\sqrt{k_\tau}} \prod_{j=1}^{\tau-1} c_j < + \frac{1}{\sqrt{k_\tau}}.$$

Now the (τ-dimensional) volume of that part of the τ-dimensional [7]) unit sphere, the orthogonal Cartesian coordinates $z_i (i = 1, \ldots, \tau)$ of which satisfy the inequalities (defining the region G'') $\frac{z_1}{\sqrt{k_1}} < \frac{z_2}{\sqrt{k_2}} < \ldots < \frac{z_\tau}{\sqrt{k_\tau}}$ is given by

$$\text{(A. 1, 5)} \qquad \int_{G''} \cdots \int \prod_{i=1}^{\tau} dz_i = V_{k_1, k_2, \ldots, k_\tau} \quad \text{(say)}$$

(cf. Remark 1). Bij the substitutions $z_i = -z_i' \ (i = 1, \ldots, \tau)$ it is easily seen that

$$\text{(A. 1, 6)} \qquad V_{k_1, k_2, \ldots, k_{\tau-1}, k_\tau} = V_{k_\tau, k_{\tau-1}, \ldots, k_2, k_1}.$$

Moreover by the substitutions:

$$\text{(A. 1, 7)} \qquad z_i = R \cdot s_{\tau-i} \cdot \prod_{j=\tau-i+1}^{\tau-1} c_j \qquad (i = 1, \ldots, \tau)$$

with $s_0 = 1$; $0 \leqq \theta_1 \leqq 2\pi$; $0 \leqq \theta_j \leqq \pi \ (j = 2, \ldots, \tau-1)$ it is seen that

$$\text{(A. 1, 8)} \qquad V_{k_1, \ldots, k_\tau} = \int_0^1 R^{\tau-1} \, dR \cdot \mathfrak{L} = (1/n) \cdot \mathfrak{L}.$$

Thus by (A. 1, 3) and (A. 1, 8):

$$\text{(A. 1, 9)} \qquad J = \frac{\tau \cdot \Gamma\left(\frac{\tau}{2}\right)}{2^{\frac{1}{2}(N-\tau+2)} \pi^{\frac{1}{2}N} \sqrt{\prod_{i=1}^{\tau} k_i}} \cdot V_{k_1, \ldots, k_\tau}.$$

Remark 1. If in $V_{k_1, \ldots, k_\tau}$ $k_{i+1} = k_{i+2} = \ldots = k_{i+h} = k$, the sequence of suffices $k_{i+1}, \ldots, k_{i+h}$ will be denoted shortly with the symbolic power k^h.

Remark 2. $V_{k_1, k_2, \ldots, k_\tau}$ will be calculated below for $\tau = 2, 3, 4$.

A. 2. *The case* $\tau = 2$.

V_{k_1, k_2} is the area of that part of the unit circle the coordinates z_1, z_2 of which satisfy $\frac{z_1}{\sqrt{k_1}} < \frac{z_2}{\sqrt{k_2}}$. Clearly this is just half of the total area. So

$$\text{(A. 2, 1)} \qquad V_{k_1, k_2} = \pi/2 \text{ for every } k_1, k_2.$$

[7]) τ here is the number of dimensions of the underlying space.

A. 3. *The case* $\tau = 3$.

V_{k_1, k_2, k_3} is the volume of that part of the 3-dimensional unit sphere, the coordinates z_1, z_2, z_3 of which satisfy the inequalities

$$\frac{z_1}{\sqrt{k_1}} < \frac{z_2}{\sqrt{k_2}} < \frac{z_3}{\sqrt{k_3}}.$$

This region is bounded by the surface of the sphere and by two planes, the equations of which are given by

$$\text{(A. 3, 1)} \qquad \frac{z_1}{\sqrt{k_1}} - \frac{z_2}{\sqrt{k_2}} = 0 \quad \text{and} \quad \frac{z_2}{\sqrt{k_2}} - \frac{z_3}{\sqrt{k_3}} = 0.$$

The volume V_{k_1, k_2, k_3} is built up out of points which make both the first members of the equations (A. 3, 1) negative.

The corresponding angle between the two planes is given by $\pi - \varphi$, where φ is the angle between the positive normals.

$$\varphi = \arccos \frac{-\frac{1}{\sqrt{k_2}} \cdot \frac{1}{\sqrt{k_2}}}{\sqrt{\left(\frac{1}{k_1} + \frac{1}{k_2}\right)\left(\frac{1}{k_2} + \frac{1}{k_3}\right)}} = \pi - \arccos \frac{\sqrt{k_1 k_3}}{\sqrt{(k_1 + k_2)(k_2 + k_3)}}.$$

So

$$\text{(A. 3, 2)} \qquad V_{k_1, k_2, k_3} = \frac{\pi - \varphi}{2\pi} \cdot \tfrac{4}{3}\pi = \tfrac{2}{3} \arccos \frac{\sqrt{k_1 k_3}}{\sqrt{(k_1 + k_2)(k_2 + k_3)}}.$$

A. 4. *The case* $\tau = 4$.

V_{k_1, k_2, k_3, k_4} is the volume of that part of the 4-dimensional unit sphere, the coordinates z_1, z_2, z_3, z_4 of which satisfy the inequalities

$$\frac{z_1}{\sqrt{k_1}} < \frac{z_2}{\sqrt{k_2}} < \frac{z_3}{\sqrt{k_3}} < \frac{z_4}{\sqrt{k_4}}.$$

This region is bounded by the "surface" of the sphere and by three hyperplanes, the equations of which are given by

$$\text{(A. 4, 1)} \qquad \left\{ \begin{array}{llll} \frac{z_1}{\sqrt{k_1}} - \frac{z_2}{\sqrt{k_2}} & & & = 0 \quad (I) \\ & \frac{z_2}{\sqrt{k_2}} - \frac{z_3}{\sqrt{k_3}} & & = 0 \quad (II) \\ & & \frac{z_3}{\sqrt{k_3}} - \frac{z_4}{\sqrt{k_4}} & = 0 \quad (III) \end{array} \right.$$

The hyperplane $P \equiv \sum_{i=1}^{4} z_i \sqrt{k_i} = 0$ is perpendicular to the line

$$\frac{z_1}{\sqrt{k_1}} = \frac{z_2}{\sqrt{k_2}} = \frac{z_3}{\sqrt{k_3}} = \frac{z_4}{\sqrt{k_4}},$$

which is contained in each of the three bounding hyperplanes. The points, P has in common with V_{k_1, k_2, k_3, k_4}, fill up that part of the

3-dimensional unit sphere in the hyperplane P with $(0, 0, 0, 0)$ as a centre, which is common to the point sets for which

$$P=0;\ \frac{z_1}{\sqrt{k_1}}-\frac{z_2}{\sqrt{k_2}}<0;\ \frac{z_2}{\sqrt{k_2}}-\frac{z_3}{\sqrt{k_3}}<0;\ \frac{z_3}{\sqrt{k_3}}-\frac{z_4}{\sqrt{k_4}}<0.$$

Be v the volume of this part of the 3-dimensional unit sphere in the hyperplane P. Then:

(A. 4, 2) $$V_{k_1,k_2,k_3,k_4}=v\cdot\int_{-1}^{+1}(1-q^2)^{3/2}\,dq=\frac{3\pi}{8}\,v.$$

As to v, the following equality holds good:
$v/\frac{4}{3}\pi=\Phi/4\pi$, where Φ is the spherical excess of the (spherical) triangle defined by the three hyperplanes mentioned above; $\frac{4}{3}\pi$ is the volume and 4π the surface of the 3-dimensional unit sphere. So

(A. 4, 3) $$v=\tfrac{1}{3}\Phi=\tfrac{1}{3}\,[(\mathrm{I},\mathrm{II})+(\mathrm{II},\mathrm{III})+(\mathrm{III},\mathrm{I})-\pi]$$

where (I, II) is the angle between the hyperplanes I and II corresponding to the inequalities defining v and the same holds for (II, III) and (III, I). (I, II) is the supplement of the angle between the positive normals on I and II respectively; etc. So $(\mathrm{III},\mathrm{I})=\pi/2$, whereas

$$(\mathrm{I},\mathrm{II})=\arccos\frac{\sqrt{k_1k_3}}{\sqrt{(k_1+k_2)(k_2+k_3)}};\ (\mathrm{II},\mathrm{III})=\arccos\frac{\sqrt{k_2k_4}}{\sqrt{(k_3+k_3)(k_3+k_4)}}.$$

Hence

$$\Phi=\arcsin\frac{\sqrt{k_2k_3}\,[\sqrt{(k_1+k_2+k_3)(k_2+k_3+k_4)}-\sqrt{k_1k_4}]}{(k_2+k_3)\sqrt{(k_1+k_2)(k_3+k_4)}}$$

and, by (A. 4, 3) and (A. 4, 2) it follows that

(A. 4, 4) $$V_{k_1,k_2,k_3,k_4}=\frac{\pi}{8}\,\frac{\sqrt{k_2k_3}\,[\sqrt{(k_1+k_2+k_3)(k_2+k_3+k_4)}-\sqrt{k_1k_4}]}{(k_2+k_3)\sqrt{(k_1+k_2)(k_3+k_4)}}.$$