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A CONFIDENCE INTERVAL FOR THE PROBABILITY THAT A NORMALLY DISTRIBUTED VARIABLE EXCEEDS A GIVEN VALUE, BASED ON THE MEAN AND THE MEAN RANGE OF A NUMBER OF SAMPLES

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§ 1. *Introduction.* In quality control the following problem often arises: A product is manufactured in mass production and a quality characteristic $\boldsymbol{x}$ *) is kept under control. The variable $\boldsymbol{x}$ may usually be assumed to have a probability distribution and in particular it is often supposed to be normally distributed with mean μ and variance σ^2. A specimen of the lot is said to be acceptable if x is smaller than a given tolerance limit a and to be rejected if x exceeds a. The fraction P of defective specimens in the lot is then given by

$$P = \frac{1}{\sigma\sqrt{2\pi}} \int_a^\infty e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{1}{\sqrt{2\pi}} \int_{(a-\mu)/\sigma}^\infty e^{-\frac{1}{2}t^2} dt. \tag{1.1}$$

If the parameters μ and σ are unknown, P also is unknown. The statistical problem is to devise an inspection plan for estimating P.

Johnson and Welch [2]) have developed a method to estimate P by means of a confidence interval $[\boldsymbol{p}_1, \boldsymbol{p}_2]$, based on the mean $\boldsymbol{m}$ and the variance $\boldsymbol{s}^2$ of a sample $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$, which are given by

$$\boldsymbol{m} = \frac{1}{n} \sum_{i=1}^n \boldsymbol{x}_i, \quad \boldsymbol{s}^2 = \frac{1}{n-1} \sum_{i=1}^n (\boldsymbol{x}_i - \boldsymbol{m})^2.$$

This confidence interval contains the true (unknown) value of P, except for a given probability α.

*) The random character of a variable is denoted by the use of a bold type symbol a special value, assumed by a random variable, is denoted by the same symbol in italics.

There is, however, one practical objection against this method: the computation of $\mathbf{s}^2$ may take much time. It is more convenient to use, instead of $\mathbf{s}$, another statistical quantity which can be computed more easily. Patnaik [8]) has shown that the mean range $\boldsymbol{w}_{k,l}$ of k samples, each of size l, can be used successfully instead of $\mathbf{s}$ in several cases. In this paper it will be shown that Patnaik's method can be employed for the problem, mentioned also in the beginning, i.e. a method will be given to find a confidence interval $[\boldsymbol{p}_1, \boldsymbol{p}_2]$ based on the mean $\boldsymbol{m}$ and the mean range $\boldsymbol{w}_{k,l}$.

For $k = 3$, $l = 8$ and $\alpha = 0.05$ the two methods (viz. using the standard deviation and the mean range respectively) are compared experimentally by means of a number of samples, taken from Wold's table [7]) of random normal deviates. It appears that both methods give practically the same results. For $k = 1(1)5$, $l = 4(2)10$ and $\alpha =$ $= 0.10$ nomograms are given for a determination of this interval from the mean and the mean range.

§ 2. *The theory of confidence intervals.* Let $\boldsymbol{x}$ be a random variable with a probability distribution which is known except for a parameter λ. This unknown parameter λ can be estimated by means of a confidence interval $[\boldsymbol{l}_1, \boldsymbol{l}_2]$, derived from a sample $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$. This interval has the property of containing the true (unknown) value of λ, except for a given probability α. The general principle for deriving confidence interval $\boldsymbol{I}$ is as follows. Let T be a test for the hypothesis $\lambda = \lambda_0$, then $\boldsymbol{I}$ is the set of all those values λ_0 which on a certain level of significance α are not rejected by applying the test T, using the sample $x_1, \ldots, x_n$. Then the confidence coefficient of $\boldsymbol{I}$, i.e. the probability that the random limite $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ include the true value of λ, is $1 - \alpha$.

§ 3. *The non-central t-distribution.* Because for both methods, indicated above, the non-central t-distribution is used for the computation of a confidence interval for P, we first consider this distribution function more closely.

A non-central t-distribution with parameter θ and f degrees of freedom is defined as the distribution of a variable

$$t = \frac{\boldsymbol{z} + \theta}{\sqrt{\boldsymbol{w}}},$$

where $\boldsymbol{z}$ is a variable distributed normally about zero with unit standard deviation, $\boldsymbol{w}$ a variable which is distributed independently as χ^2/f, where f is the number of degrees of freedom of χ^2 and θ is a constant.

Because the variables $\boldsymbol{z}$ and $\boldsymbol{w}$ are distributed independently, the simultaneous density function $f(z, \sqrt{w})$ is the product of the density functions $g(z)$ and $h(\sqrt{w})$, given by

$$g(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, \tag{3.1}$$

and

$$h(\sqrt{w}) = C\, e^{-\frac{1}{2}fw}\, w^{\frac{1}{2}(f-1)},$$

where

$$C = [\Gamma(\tfrac{1}{2}f)]^{-1}\, 2^{-(\frac{1}{2}f-1)}\, f^{\frac{1}{2}f}. \tag{3.2}$$

If we denote the cumulative distribution function of $\boldsymbol{t}$ by $F(t; \theta, f)$, we obtain:

$$F(t; \theta, f) = P[\boldsymbol{t} \leq t \mid \theta, f] =$$

$$= C \int_0^\infty e^{-\frac{1}{2}fw}\, w^{\frac{1}{2}f-1} \left[\frac{1}{\sqrt{2\pi}} \int_{-t\sqrt{w}+\theta}^{\infty} e^{-\frac{1}{2}u^2}\, \mathrm{d}u \right] \mathrm{d}(\sqrt{w}) \tag{3.3}$$

where C is given by (3.2).

§ 4. *The computation of a confidence interval for P, using the mean* $\boldsymbol{m}$ *and the standard deviation* $\boldsymbol{s}$. From (1.1) it follows, that the problem of determining a confidence interval for P is identical with the problem of determining a confidence interval for $\eta = (a - \mu)/\sigma$. If $[\boldsymbol{n}_1, \boldsymbol{n}_2]$ is a confidence interval for η with confidence coefficient $1 - \alpha$, then the interval $[\boldsymbol{p}_1, \boldsymbol{p}_2]$ with

$$\boldsymbol{p}_1 = \frac{1}{\sqrt{2\pi}} \int_{\boldsymbol{n}_2}^{\infty} e^{-\frac{1}{2}u^2}\, \mathrm{d}u, \qquad \boldsymbol{p}_2 = \frac{1}{\sqrt{2\pi}} \int_{\boldsymbol{n}_1}^{\infty} e^{-\frac{1}{2}u^2}\, \mathrm{d}u \tag{4.1}$$

is a confidence interval for P with the same confidence coefficient $1 - \alpha$.

A confidence interval for η can be derived from the mean $\boldsymbol{m}$ and the standard deviation $\boldsymbol{s}$ of the sample $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$, because the variable

$$\boldsymbol{t}_1 = \sqrt{n}\frac{a - \boldsymbol{m}}{\boldsymbol{s}} = \frac{\sqrt{n}(a-\mu)/\sigma - \sqrt{n}(\boldsymbol{m}-\mu)/\sigma}{\boldsymbol{s}/\sigma} \qquad (4.2)$$

has a non-central t-distribution with parameter $\theta = \sqrt{n}\,(a-\mu)/\sigma$ and $f = n - 1$ degrees of freedom. This is immediately clear, because for a normal distribution the quantities $\sqrt{n}(\boldsymbol{m}-\mu)/\sigma$ and $(n-1)\boldsymbol{s}^2/\sigma^2$ are distributed independently according to a normal distribution with mean 0 and standard deviation 1, and a χ^2-distribution with $n-1$ degrees of freedom respectively (cf. [5]). The distribution function $F(t_1;\ \sqrt{n}\,\eta, n-1)$ of the variable $\boldsymbol{t}_1$ thus follows from (3.3) by substituting $\theta = \sqrt{n}\,\eta$ and $f = n - 1$.

If t_1 is the observed value of $\boldsymbol{t}_1$, a confidence interval $[n_1, n_2]$ for η with confidence coefficient $1 - \alpha$ is obtained by solving n_1 and n_2 from

$$\begin{aligned} P[\boldsymbol{t}_1 \geq t_1^* \mid \theta = \sqrt{n}\,n_1] &= 1 - F(t_1^*;\ \sqrt{n}\,n_1, n-1) = \alpha_1, \\ P[\boldsymbol{t}_1 \leq t_1^* \mid \theta = \sqrt{n}\,n_2] &= F(t_1^*;\ \sqrt{n}\,n_2, n-1) = \alpha - \alpha_1. \end{aligned} \qquad (4.3)$$

For the computation of n_1 and n_2 use can be made of the tables of Johnson and Welch (cf. [2]). By substituting n_1 and n_2 into (4.1) a confidence interval $[p_1, p_2]$ for P is obtained, with confidence coefficient $1 - \alpha$ *).

§ 5. *The computation of a confidence interval for P from the mean* $\boldsymbol{m}$ *and the mean range* $\boldsymbol{w}_{k,l}$.

5.1. The computation of a confidence interval $[\boldsymbol{p}_1, \boldsymbol{p}_2]$ from the mean $\boldsymbol{m}$ and the mean range $\boldsymbol{w}_{k,l}$ can be carried out in an analogous way as in the foregoing section, because for a normal distribution $\boldsymbol{w}_{k,l}$ is distributed independent of $\boldsymbol{m}$ (cf. [6]) and its distribution can be approximated by a Γ-distribution, which can be transformed into a χ^2-distribution.

For this purpose, Patnaik [8]) supposes that the quantity $\boldsymbol{w}_{k,l}/c_{k,l}$, where $c_{k,l}$ is a properly chosen constant, has approximately the same distribution as the standard deviation $\boldsymbol{s}$ of a sample of size $n = kl$, only with a reduced number $\nu_{k,l}$ of degrees of freedom. He thus supposes that $\boldsymbol{v}^2 = \nu_{k,l}\,(\boldsymbol{w}_{k,l}/c_{k,l}\sigma)^2$ is nearly distributed as a quantity $\boldsymbol{v}'^2$, possessing a χ^2-distribution with $\nu_{k,l}$ degrees of freedom. The constants $c_{k,l}$ and $\nu_{k,l}$ can be evaluated by equalizing the

*) For the special case $\alpha_1 = \frac{1}{2}\alpha$ the interval is called a central confidence interval.

first and second moments of $\boldsymbol{w}_{k,l}/\sigma$ and $\boldsymbol{y} = c_{k,l}\, \boldsymbol{v}'/\sqrt{\nu_{k,l}}$. For the moments of $\boldsymbol{w}_{k,l}/\sigma$ we have

$$\mathscr{E}(\boldsymbol{w}_{k,l}/\sigma) = \mathscr{E}(\boldsymbol{w}_l/\sigma) = d_l = M,$$

$$\mathscr{V}ar(\boldsymbol{w}_{k,l}/\sigma) = \frac{1}{k}\,\mathscr{V}ar(\boldsymbol{w}_l/\sigma) = V.$$

These moments can be derived from H a r t l e y and P e a r s o n's table [3]) of the exact distribution of $\boldsymbol{w}_{1,l}/\sigma$. As the first two moments of $\boldsymbol{y}$ are given by

$$\mathscr{E}(\boldsymbol{y}) = \frac{c}{\sqrt{\nu_{k,l}}}\sqrt{2}\,\frac{\Gamma(\nu_{k,l}+1)/2}{\Gamma(\nu_{k,l}/2)},$$

$$\mathscr{V}ar(\boldsymbol{y}) = \frac{c^2}{\nu_{k,l}}\left[\nu_{k,l} - 2\left\{\frac{\Gamma(\nu_{k,l}+1)/2}{\Gamma(\nu_{k,l}/2)}\right\}^2\right],$$

$\nu_{k,l}$ and $c_{k,l}$ follow from

$$M = \frac{c}{\sqrt{\nu_{k,l}}}\sqrt{2}\,\frac{\Gamma(\nu_{k,l}+1)/2}{\Gamma(\nu_{k,l}/2)}, \qquad V = \frac{c^2}{\nu_{k,l}}\left[\nu_{k,l} - 2\left\{\frac{\Gamma(\nu_{k,l}+1)/2}{\Gamma(\nu_{k,l}/2)}\right\}^2\right].$$

5.2. In [1]) and [4]) it is argued that the range in groups of about seven or eight observations gives better estimates of σ than for greater or smaller numbers of observations. Consequently we may expect that the exact distribution of $\boldsymbol{w}_{1,8}/c_{1,8}\sigma$ will differ little from the approximate distribution. This expectation is confirmed by numerical calculations, from wich the following table is obtained for the exact and approximate distribution of $\boldsymbol{w}_{1,8}/\sigma$.

TABLE I

The exact and the approximate distribution of $\boldsymbol{w}_{1,8}/\sigma$

w/σ	$F(w/\sigma)$	Approx. $F(w/\sigma)$	Difference
0.000	0.0000	0.0000	0.0000
0.498	0.0000	0.0000	0.0000
1.220	0.0109	0.0131	0.0022
1.575	0.0461	0.0500	0.0035
1.876	0.1086	0.1111	0.0025
2.113	0.1892	0.1904	0.0012
2.336	0.2816	0.2800	— 0.0016
2.818	0.5131	0.5065	— 0.0066
3.228	0.6928	0.6912	— 0.0016
3.522	0.8006	0.7977	— 0.0029
3.922	0.8982	0.8988	0.0006
4.285	0.9498	0.9520	0.0022
4.726	0.9811	0.9833	0.0022
5.502	0.9975	0.9980	0.0005
6.757	1.0000	1.0000	0.0000

TABLE II

A comparison of both methods by means of a number of samples taken from W o l d's table of random normal deviates

	m	s	$(a-m)/s$	p_1	p_2	$D_s=p_2-p_1$	$w_{3,8}$	$(a-m)/w_{3,8}$	p_1	p_2	$D_w=p_2-p_1$
						$a = 0.80$					
1	—0.35	0.8778	1.3101	0.0328	0.2224	0.1896	2.43	0.4733	0.0257	0.2230	0.1973
2	—0.32	1.1564	0.9685	0.0727	0.3174	0.2437	2.94	0.3810	0.0524	0.2884	0.2360
3	0.18	1.1290	0.5492	0.1647	0.4546	0.2899	3.44	0.1802	0.1718	0.4685	0.2967
4*	0.48	1.2544	0.2551	0.2547	0.5612	0.3065	3.36	0.0952	0.2483	0.5553	0.3070
5	0.00	0.7226	1.1057	0.0538	0.2783	0.2245	2.20	0.3632	0.0594	0.3025	0.2431
6	0.29	1.0781	0.4731	0.1865	0.4816	0.2951	2.67	0.1910	0.1630	0.4578	0.2948
7	—0.14	0.8894	1.0569	0.0604	0.2919	0.2315	2.65	0.3547	0.0629	0.3093	0.2464
8	0.05	1.0914	0.6872	0.1301	0.4067	0.2766	3.27	0.2294	0.1344	0.4207	0.2863
9	0.15	0.8399	0.7739	0.1103	0.3779	0.2676	2.49	0.2610	0.1129	0.3907	0.2778
10	—0.35	0.9222	1.2470	0.0379	0.2411	0.2032	2.64	0.4356	0.0348	0.2483	0.2135
11	0.16	0.9503	0.6735	0.1333	0.4113	0.2780	2.54	0.2520	0.1188	0.3993	0.2805
12	—0.08	1.0659	0.8256	0.0996	0.3613	0.2617	3.07	0.2866	0.0966	0.3677	0.2711
13	0.17	0.9627	0.6544	0.1380	0.4179	0.2799	2.76	0.2283	0.1353	0.4215	0.2862
14	0.01	0.9899	0.7981	0.1050	0.3701	0.2651	3.29	0.2401	0.1269	0.4102	0.2833
15	0.13	1.0289	0.6512	0.1388	0.4188	0.2800	2.83	0.2367	0.1292	0.4136	0.2844
						$a = 1.40$					
1	—0.35	0.8778	1.9936	0.0037	0.0989	0.0952	2.43	0.7202	0.0022	0.0985	0.0963
3	0.18	1.1290	1.0806	0.0571	0.2849	0.2278	3.44	0.3547	0.0629	0.3092	0.2463
5	0.00	0.7226	1.9361	0.0045	0.1067	0.1022	2.20	0.6359	0.0057	0.1327	0.1270
9	0.15	0.8399	1.4883	0.0194	0.1855	0.1661	2.49	0.5020	0.0202	0.2047	0.1845
12	– 0.08	1.0659	1.3885	0.0259	0.2073	0.1814	3.07	0.4821	0.0239	0.2172	0.1933
15	0.13	1.0289	1.2343	0.0390	0.2442	0.2052	2.83	0.4488	0.0313	0.2392	0.2079

5.3. Putting $n = kl$ it follows from the foregoing that the computation of a confidence interval for P can be based on the statistic

$$\boldsymbol{t}_2 = \sqrt{n}\,\frac{a - \boldsymbol{m}}{\boldsymbol{w}_{k,l}/c_{k.l}} = \frac{\sqrt{n}(a - \mu)/\sigma + \sqrt{n}(\boldsymbol{m} - \mu)/\sigma}{\boldsymbol{w}_{k,l}/c_{k,l}\sigma}$$

which is approximately distributed according to a non-central t-distribution with parameter $\theta = \sqrt{n}\,(a - \mu)/\sigma$ and $\nu_{k,l}$ degrees of freedom. If t_2^* is the observed value of $\boldsymbol{t}_2$, a confidence interval $[n_1, n_2]$ for $\eta = (a - \mu)/\sigma$ is obtained from

$$P[\boldsymbol{t}_2 \geq t_2^* \mid \sqrt{n}\,n_1] = 1 - F(t_2^*;\ \sqrt{n}\,n_1, \nu_{k,l}) = \alpha_1,$$
$$P[\boldsymbol{t}_2 \leq t_2^* \mid \sqrt{n}\,n_2] = F(t_2^*;\ \sqrt{n}\,n_2, \nu_{k,l}) = \alpha - \alpha_1.$$

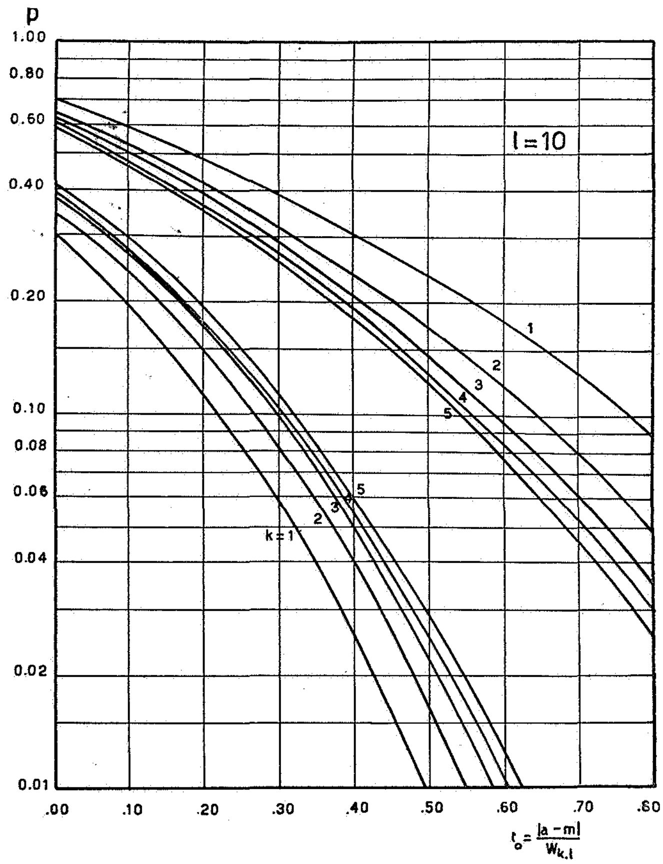

Fig. 1. Central confidence intervals for the percentage defectives in a normal distribution with confidence-coefficient $1 - \alpha = 0.90$; $l = 10$, $k = 1, 2, 3, 4, 5$.

A confidence interval $[p_1, p_2]$ for P with confidence coefficient $1 - \alpha$ is then given by (4.1).

§ 6. *A comparison of both methods.* The two methods for computing a confidence interval for P, as described in the foregoing sections, may be compared by means of a number of samples taken from Wold's table [7]) of random normal deviates. This comparison is carried out for the particular case $k = 3$ and $l = 8$, because the approximating distribution of the range in groups of eight observations does not differ much from the exact distribution and a sample of about the size $n = 24$ is usual in quality control.

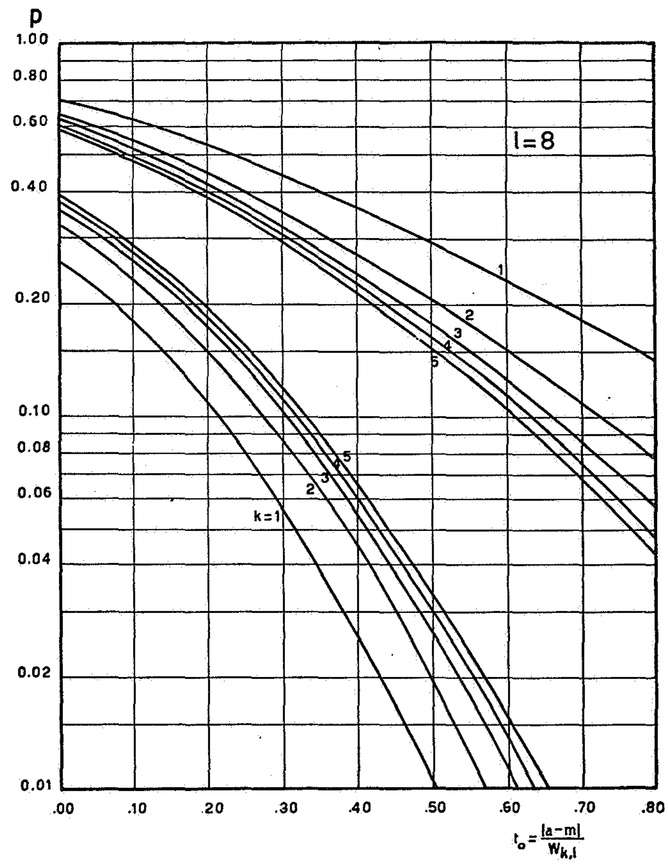

Fig. 2. Central confidence intervals for the percentage defectives in a normal distribution with confidence-coefficient $1 - \alpha = 0.90$; $l = 8$, $k = 1, 2, 3, 4, 5$.

Central confidence-intervals with confidence coefficient $1 - \alpha = 0.95$ are computed for the probabilities P_1 and P_2 that the normal deviate $\boldsymbol{x}$ exceeds the levels $a = 0.80$ and $a = 1.40$, the true values of P_1 and P_2 being 0.2119 and 0.0808 respectively.

The numerical results obtained are tabulated in table II. It appears that the intervals, obtained in both ways, are practically the same. Only the intervals, obtained from the mean and the mean range, are slightly larger than those obtained from the mean and the standard deviation. This is evident, because the variables $(a - \boldsymbol{m})/\boldsymbol{s}$ and $(a - \boldsymbol{m})/\boldsymbol{w}_{3,8}$ are highly correlated and the range is a less efficient

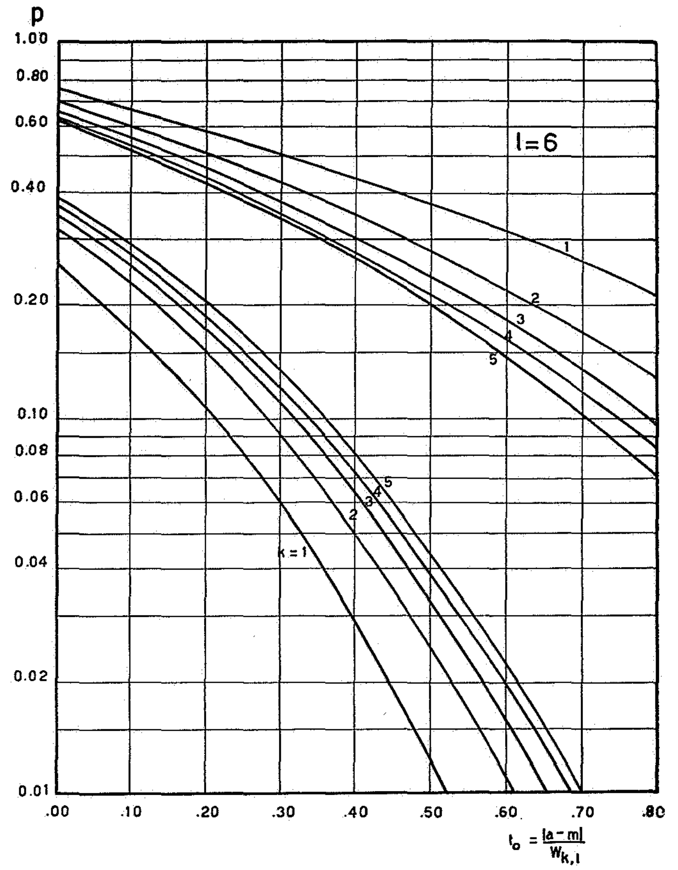

Fig. 3. Central confidence intervals for the percentage defectives in a normal distribution with confidence-coefficient $1 - \alpha = 0.90$; $l = 6$, $k = 1, 2, 3, 4, 5$.

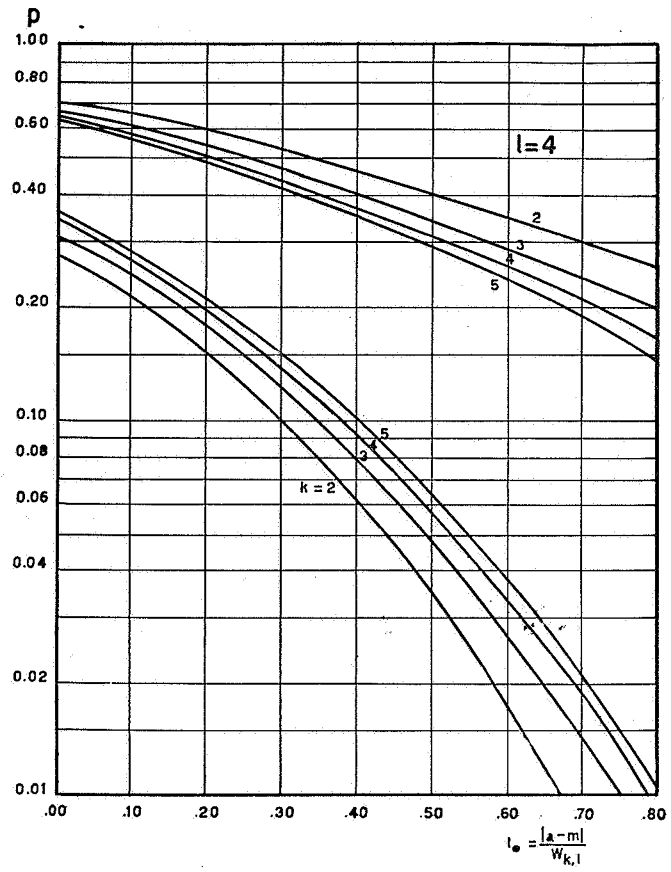

Fig. 4. Central confidence intervals for the percentage defectives in a normal distribution with confidence-coefficient $1 - \alpha = 0.90$; $l = 4$, $k = 2, 3, 4, 5$.

estimate of σ than the standard deviation. For only one of the samples (denoted by *) the interval does not contain the true value P_1.

In figs. 1, 2, 3 and 4 nomograms for special values of k and l are given for computation of $[p_1, p_2]$ from $(a - m)/w_{k,l}$. These diagrams were constructed by van der Heem Ltd., The Hague, and kindly delivered for publication.

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