A SEQUENTIAL TEST WITH THREE POSSIBLE DECISIONS FOR COMPARING TWO UNKNOWN PROBABILITIES, BASED ON GROUPS OF OBSERVATIONS [1]

by

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1. INTRODUCTION

Sequential tests for comparing two unknown probabilities p and p' with two possible decisions, i.e. tests resulting in one of the decisions

$$p < p' \text{ or } p > p' \tag{1.1}$$

have been developed by A. Wald [8] for pairs of observations and by the Statistical Research Group of the Columbia University [6] for groups of observations.

Such sequential tests with two possible decisions have the drawback that they always result in one of the decisions $p < p'$ or $p > p'$, even if $p = p'$; therefore it is very useful for practical purposes to develop tests with three possible decisions, i.e. tests resulting in one of the decisions

$$p < p', p > p' \text{ or } p \gtrless p'. \tag{1.2}$$

A generalization in this direction of the above-mentioned test of Wald has been given by J. de Boer [2]. Whereas his test applies to pairs of observations - a case which will not be considered in this paper - the next pages generalize the sequential test described in [6], applying to groups of observations, to a test with the three possible decisions (1.2). For this generalization an approximation is used, which makes it unsuitable for small groups.

In order to explain the test clearly and to give a picture of its relation with the other tests mentioned, a short description of these is given in the following section.

2. SHORT DESCRIPTION OF RELATED TESTS

a. *Two-decision-test for probabilities based on groups of observations* [6].

Consider two independent series of independent trials, e.g. two processes A and A', each trial resulting in a success or a failure with probabilities p, $1-p$ and p', $1-p'$ for the two processes respectively. Suppose the group of trials constituting the i-th stage of the test, consists of n_i trials for process A and n_i' trials for process A'. If the numbers of success are $\boldsymbol{a}_i$ and $\boldsymbol{a}_i'$ respectively,[2] $\boldsymbol{a}_i$ and $\boldsymbol{a}_i'$ both possess a binomial probability distribution with parameters n_i, p and n_i', p' respectively.

The following transformation of a binomial probability distribution is then used: if, in general, $\boldsymbol{m}$ possesses a binomial probability distribution with parameters M (numbers of trials) and p (probability of a success), then the random variable

[1] Report SP 34 of the Statistical Department of the Mathematical Centre, Amsterdam.

[2] Random variables will be distinguished from numbers (e.g. from the value they take in an experiment) by printing them in bold type.

$$\begin{cases} y = 2 \arcsin \sqrt{m/M} & \text{for } 0 < m < M \\ y = \sqrt{2/M} & \text{for } m = 0 \\ y = \pi - \sqrt{2/M} & \text{for } m = M \end{cases} \tag{2.1}$$

is, for large M, approximately normally distributed with mean

$$\mu = 2 \arcsin \sqrt{p} \tag{2.2}$$

and variance

$$\sigma^2 = (n+1)/n^2.\ ^3 \tag{2.3}$$

In this way the variable $\boldsymbol{m}$ is transformed in a variable $\boldsymbol{y}$ with a mean which is practically independent of M and a variance which is practically independent of p.

The transformation (2.1) is applied to $\boldsymbol{a}_i$ and to $\boldsymbol{a}_i'$; after this transformation these random variables will be denoted by $\boldsymbol{u}_i$ and $\boldsymbol{u}_i'$ respectively and the sequential test of Wald [8] with two possible decisions for the mean of a normal distribution with known variance is then applied to

$$\boldsymbol{x}_i \stackrel{\text{def}}{=} \boldsymbol{u}_i - \boldsymbol{u}_i'.$$

This test will be generalized by applying to the $\boldsymbol{x}_i$ the sequential test of Sobel and Wald with three possible decisions for the mean of a normal distribution with known variance. [7]

b. *Two-decision-test for the mean of a normal variate with known variance* [8].

For the case that the successive observations $x_1, x_2, \ldots$ are independent observations of one random variable $\boldsymbol{x}$, possessing a normal probability distribution with mean μ and known variance σ^2, Wald's sequential test with two possible decisions for μ has been described in [8, p. 117—124]. This test will be described here for the case that the variance is not constant [6]. This results in a small change in Wald's test; the proof of the validity of this test follows at once from Wald's own proofs.

For the test a value μ_0 of μ must be chosen, the two possible decisions being: $\mu < \mu_0$ and $\mu > \mu_0$, where we may substitute $\leqq$ resp. $\geqq$ for $<$ resp. $>$.

Furthermore two values μ_1 and μ_2 must be chosen with

$$\mu_1 < \mu_0 < \mu_2$$

such that the decision $\mu > \mu_0$ is considered an incorrect decision if $\mu \leqq \mu_1$ and the decision $\mu < \mu_0$ is considered incorrect if $\mu \geqq \mu_2$; for values of μ between μ_1 and μ_2 it is not important which decision is taken.

The concepts "correct" and "incorrect decision" are thus defined as follows.

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[3] Tables of y (in radians) and σ^2 are given in [6] for $M = 10\,(1)\,50$ and $0 \leqq \boldsymbol{m} \leqq M$ (cf. also section 3).

TABLE 1. CORRECT AND INCORRECT DECISIONS

value of μ	correct decision	incorrect decision
$\mu \leqq \mu_1$	$\mu < \mu_0$	$\mu > \mu_0$
$\mu \geqq \mu_2$	$\mu > \mu_0$	$\mu < \mu_0$
$\mu_1 < \mu < \mu_2$	$\mu < \mu_0$ and $\mu > \mu_0$	—

The interval (μ_1, μ_2) is called the indifference region. If:
$\alpha =$ the probability of acceptance of $\mu > \mu_0$ if $\mu = \mu_1$,
$\beta =$ the probability of acceptance of $\mu < \mu_0$ if $\mu = \mu_2$,
and if α and β are chosen both $< 1/2$ the probability of an incorrect decision is $\leqq \alpha$ for $\mu \leqq \mu_1$ and $\leqq \beta$ for $\mu \geqq \mu_2$.

The value μ_0 is of no further importance for the performance of the test.

The test is carried out as follows (σ_i^2 is the known variance of $\boldsymbol{x}_i$). Additional observations are taken as long as:

$$\ln B/(\mu_2 - \mu_1) < \sum_{i=1}^{n} (x_i - \frac{\mu_1 + \mu_2}{2}) \Big/ \sigma_i^2 < \ln A/(\mu_2 - \mu_1), \tag{2.4}$$

where

$$A = (1-\beta)/\alpha > 1$$
$$B = \beta/(1-\alpha) < 1.$$

The test is terminated as soon as (2.4) does not hold and the decision $\mu > \mu_0$ is then taken if

$$\sum_{i=1}^{n} (x_i - \frac{\mu_1 + \mu_2}{2}) \Big/ \sigma_i^2 \geqq \ln A/(\mu_2 - \mu_1)$$

and the decision $\mu < \mu_0$ if

$$\sum_{i=1}^{n} (x_i - \frac{\mu_1 + \mu_2}{2}) \Big/ \sigma_i^2 \leqq \ln B/(\mu_2 - \mu_1).$$

If the random variables $\boldsymbol{x}_i$ all have the same variance σ^2 the test may be carried out graphically, as indicated by Wald [8, p. 118—121].

c. *Three-decision-test for the mean of a normal variate with known variance* [7].

The sequential test with three possible decisions for the mean μ of a normal distribution with known variance, developed by Sobel and Wald has been described in [7] for the case that the variance of $\boldsymbol{x}_i$ is a constant. This restriction is again dropped here.

For the test two values μ_0 and μ_0' and four values μ_1, μ_2, μ_3 and μ_4 must be chosen such that

$$\mu_1 < \mu_0 < \mu_2 < \mu_3 < \mu_0' < \mu_4,$$

the three possible decisions being:

$$\begin{aligned}
&(1). \quad \mu < \mu_0 \\
&(2). \quad \mu > \mu_0' \\
&(3). \quad \mu_0 \leqq \mu \leqq \mu_0'.
\end{aligned}$$

The intervals (μ_1, μ_2) and (μ_3, μ_4) are the indifference regions.

The concepts "correct" and "incorrect decision" are defined as follows.

TABLE 2. CORRECT AND INCORRECT DECISIONS

value of μ	correct	incorrect
	decision	
$\mu \leqq \mu_1$	(1)	(3) and (2)
$\mu_1 < \mu < \mu_2$	(1) and (3)	(2)
$\mu_2 \leqq \mu \leqq \mu_3$	(3)	(2) and (1)
$\mu_3 < \mu < \mu_4$	(3) and (2)	(1)
$\mu \geqq \mu_4$	(2)	(1) and (3)

The values μ_0 and μ_0' are of no further importance for the performance of the test.

Suppose T is the sequential test of section 2*b* for testing $\mu = \mu_1$ against $\mu = \mu_2$, then this test leads to a decision as soon as

$$\ln B/(\mu_2 - \mu_1) < \sum_{i=1}^{n} \left(x_i - \frac{\mu_1 + \mu_2}{2}\right) \Big/ \sigma_i^2 < \ln A/(\mu_2 - \mu_1) \tag{2.5}$$

does not hold, where

$$A = (1 - \beta)/\alpha \quad , \quad B = \beta/(1 - \alpha),$$

$\alpha =$ the probability of accepting $\mu \geqq \mu_0$ according to T if $\mu = \mu_1$,

$\beta =$ the probability of accepting $\mu < \mu_0$ according to T if $\mu = \mu_2$.

Suppose furthermore that T' is the analogous sequential test for testing $\mu = \mu_3$ against $\mu = \mu_4$ then T' leads to a decision as soon as

$$\ln B'/(\mu_4 - \mu_3) < \sum_{i=1}^{n} \left(x_i - \frac{\mu_3 + \mu_4}{2}\right) \Big/ \sigma_i^2 < \ln A'/(\mu_4 - \mu_3) \tag{2.6}$$

does not hold, where

$$A' = (1 - \beta')/\alpha' \quad , \quad B' = \beta'/(1 - \alpha'),$$

$\alpha' =$ the probability of accepting $\mu > \mu_0'$ according to T' if $\mu = \mu_3$,

$\beta' =$ the probability of accepting $\mu \leqq \mu_0'$ according to T' if $\mu = \mu_4$.

We introduce the following notation

$$\left\{\begin{array}{ll}
a = \ln A/(\mu_2 - \mu_1) & a' = \ln A'/(\mu_4 - \mu_3) \\
b = \ln B/(\mu_2 - \mu_1) & b' = \ln B'/(\mu_4 - \mu_3) \\
\sum_{i=1}^{n} \left(x_i - \frac{\mu_1 + \mu_2}{2}\right) \Big/ \sigma_i^2 = y_n & \sum_{i=1}^{n} \left(x_i - \frac{\mu_3 + \mu_4}{2}\right) \Big/ \sigma_i^2 = y_n',
\end{array}\right. \tag{2.7}$$

then according to T a decision is taken as soon as

$$b < y_n < a \tag{2.5a}$$

does not hold and according to T' as soon as

$$b' < y'_n < a' \tag{2.6a}$$

does not hold.

From (2.7) it follows that a and a' are positive, b and b' negative and

$$y_n > y'_n. \tag{2.8}$$

If now the inequalities

$$\begin{cases} b \leqq b' \\ a \leqq a' \end{cases} \tag{2.9}$$

are fulfilled, it follows from (2.8) and (2.9) that T' cannot lead to the decision $\mu > \mu'_0$ before the decision $\mu \geqq \mu_0$ has been found according to T and that T cannot give the decision $\mu < \mu_0$ before the decision $\mu \leqq \mu'_0$ has been given by T'.

In that case only the following decisions according to T and T' are possible:

1. T' gives the decision $\mu \leqq \mu'_0$ and T gives (at the same or a later step) the decision $\mu < \mu_0$ or the decision $\mu \geqq \mu_0$.

2. T gives the decision $\mu \geqq \mu_0$ and T' gives (at the same or a later step) the decision $\mu \leqq \mu'_0$ or the decision $\mu > \mu'_0$.

The sequential test with three possible decisions is then defined as follows:

Additional observations are taken as long as not both tests T and T' have given a decision. As soon as both tests are terminated a decision is taken according to the following rules:

1. $\mu < \mu_0$ if T' has given the decision $\mu \leqq \mu'_0$ and T the decision $\mu < \mu_0$,
2. $\mu > \mu'_0$ if T has given the decision $\mu \geqq \mu_0$ and T' the decision $\mu > \mu'_0$,
3. $\mu_0 < \mu \leqq \mu'_0$ if T has given the decision $\mu \geqq \mu_0$ and T' the decision $\mu \leqq \mu'_0$.

If (2.9) does not hold there exists the possibility of accepting $\mu > \mu'_0$ according to T' and of afterwards accepting $\mu < \mu_0$ $(< \mu'_0)$ according to T. This kind of contradictory result is excluded by (2.9).

If the random variables $\boldsymbol{x}_i$ all have the same variance σ^2 the test may be carried out graphically [7].

3. SEQUENTAL TEST WITH THREE POSSIBLE DECISIONS FOR THE COMPARISON OF TWO PROBABILITIES

On the basis of the test of section 2 c a sequential test with three possible decisions for comparing two unknown probabilities p and p' may be developed as follows.

To the variables $\boldsymbol{a}_i$ and $\boldsymbol{a}'_i$ (cf. section 1) one of the following transformations is applied

$$y = 2 \arcsin \sqrt{m/M}\,,^{4} \tag{3.1}$$

$$\begin{cases} y' = 2 \arcsin \sqrt{m/M} & \text{for } 0 < m < M,^{5} \\ y' = 2 \arcsin \sqrt{\tfrac{1}{4}M} & \text{for } m = 0, \\ y' = \pi - 2 \arcsin \sqrt{\tfrac{1}{4}M} & \text{for } m = M, \end{cases} \tag{3.2}$$

$$\begin{cases} y'' = 2 \arcsin \sqrt{m/M} & \text{for } 0 < m < M, \\ y'' = \sqrt{2/M} & \text{for } m = 0, \\ y'' = \pi - \sqrt{2/M} & \text{for } m = M, \end{cases} \tag{3.3}$$

where m possesses a binomial probability distribution with parameters M and p.

The transformation (3.1) is introduced by Fisher [3], the transformation (3.2) by Bartlett [1] and (3.3) is given in [6]. For further information about the transformations we refer to [5, p. 395—416].

Denoting the variables a_i and a'_i, after their transformation, by u_i and u'_i the sequential test of section 2c is applied to the random variables

$$x_i \overset{\text{def}}{=} u_i - u'_i \qquad i = 1, 2, \ldots\ ,$$

which possess, for large n_i and n'_i, approximately a normal probability distribution with mean

$$\mu = 2 \arcsin\sqrt{p} - 2 \arcsin\sqrt{p'} = 2 \arcsin(\sqrt{pq'} - \sqrt{p'q}) \quad (q = 1 - p; q' = 1 - p') \tag{3.4}$$

and variance

$$\sigma_i^2 = (n_i + 1)/n_i^2 + (n'_i + 1)/n_i'^2. \tag{3.5}$$

Two values μ_0 and μ'_0 and four values μ_1, μ_2, μ_3 and μ_4 of μ must be chosen, with:

$$\mu_1 < \mu_0 < \mu_2 < \mu_3 < \mu'_0 < \mu_4 \tag{3.6}$$

and four values α, α', β and β' (all $< \tfrac{1}{2}$) with (cf. (2.9))

$$\begin{cases} \ln B/(\mu_2 - \mu_1) \leqq \ln B'/(\mu_4 - \mu_3), \\ \ln A/(\mu_2 - \mu_1) \leqq \ln A'/(\mu_4 - \mu_3), \end{cases} \tag{3.7}$$

where

$$\begin{cases} A = (1 - \beta)/\alpha\ , & B = \beta/(1 - \alpha), \\ A' = (1 - \beta')/\alpha'\ , & B' = \beta'/(1 - \alpha'). \end{cases}$$

Having chosen these values the abovementioned test may be applied, leading to one of the decisions:

$$\begin{cases} 1. & \mu < \mu_0, \\ 2. & \mu > \mu'_0, \\ 3. & \mu_0 \leqq \mu \leqq \mu'_0. \end{cases} \tag{3.8}$$

We shall translate these decisions in terms of p and p'. Let

$$\sqrt{pq'} - \sqrt{p'q} = \delta, \tag{3.9}$$

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[4] Tables of $y = 2 \arcsin\sqrt{x}$ are given in [4, p. 70—71] for $x = 0{,}000\ (0{,}001)\ 1{,}000$, with y in radians.

[5] Tables of $y = 2 \arcsin\sqrt{\tfrac{1}{4}n}$ and $y = \pi - 2 \arcsin\sqrt{\tfrac{1}{4}n}$ are given in [5, p. 406] for $n = 10(1)50$.

then

$$\mu = 2 \arcsin \delta \qquad |\delta| \leqq 1. \tag{3.10}$$

The functional relationship (3.9) between p and p' for given δ^2 consists (fig. 1) of the arcs PQ and RS of the ellipse:

$$p^2 + p'^2 - 2pp'(1 - 2\delta^2) - 2\delta^2(p + p') + \delta^4 = 0. \tag{3.11}$$

Choosing two values δ_0 and δ_0' and four values δ_1, δ_2, δ_3 and δ_4 of δ with

$$\delta_1 < \delta_0 < \delta_2 < \delta_3 < \delta_0' < \delta_4, \tag{3.12}$$

the decisions (3.8) are equivalent with the following decisions for δ (3.10):

$$\begin{cases} 1. \quad \delta < \delta_0, \\ 2. \quad \delta > \delta_0', \\ 3. \quad \delta_0 \leqq \delta \leqq \delta_0' \end{cases} \tag{3.13}$$

and hence with the following decisions for p and p':

1. the point (p, p') lies above the arc RS of figure 1 with $\delta = \delta_0$,
2. the point (p, p') lies below the arc PQ with $\delta = \delta_0'$, (3.14)
3. the point (p, p') lies on or between the arcs PQ and RS with $\delta = \delta_0'$ and $\delta = \delta_0$ respectively.

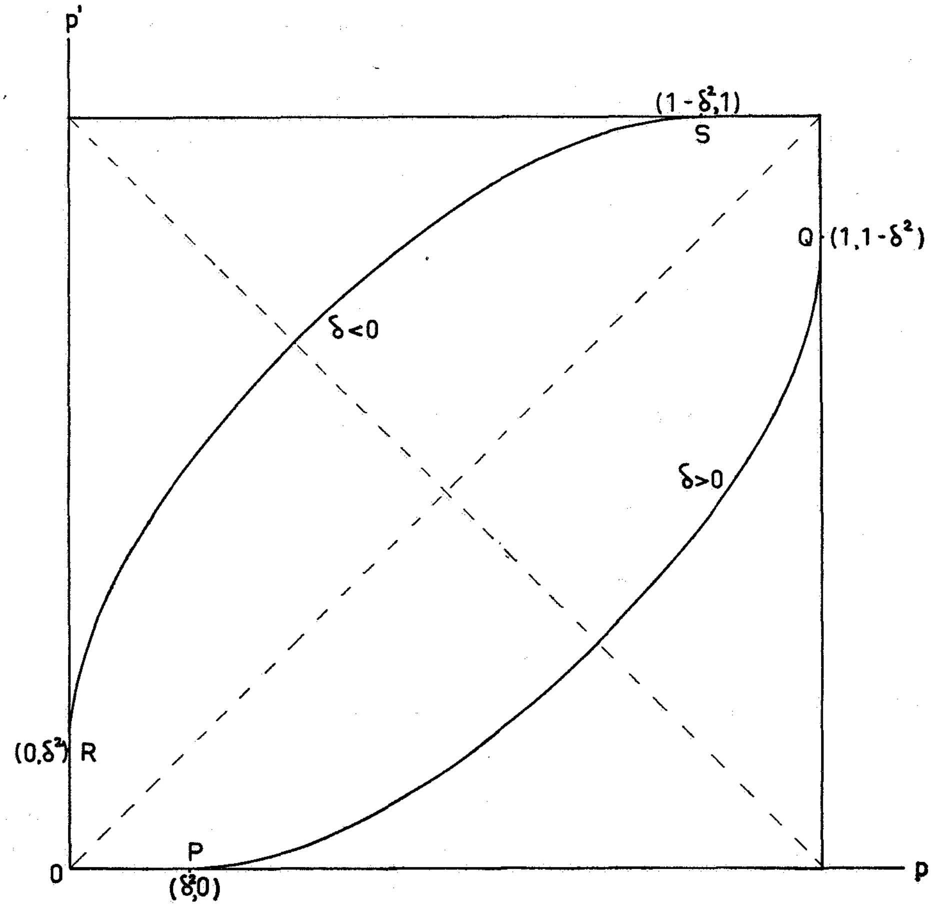

FIG. 1. Functional relationship between p and p' for given value of δ^2.

The values δ_1, δ_2, δ_3 and δ_4 may be chosen by means of fig. 2, where the arcs PQ and RS are given for several values of δ.

One may also choose these values as follows:

1. Wald [8] uses the ratio

$$u = pq'/p'q.$$

On the line $p + p' = 1$, δ can be expressed in terms of u:

$$u = (1+\delta)^2/(1-\delta)^2, \tag{3.15}$$

which is equivalent to

$$\begin{cases} \delta = 0 & \text{if} \quad u = 1 \\ \delta = (u + 1 - 2\sqrt{u})/(u-1) & \text{if} \quad u \neq 1. \end{cases} \tag{3.16}$$

Choosing four values for u with

$$u_1 < u_2 < 1 < u_3 < u_4 \tag{3.17}$$

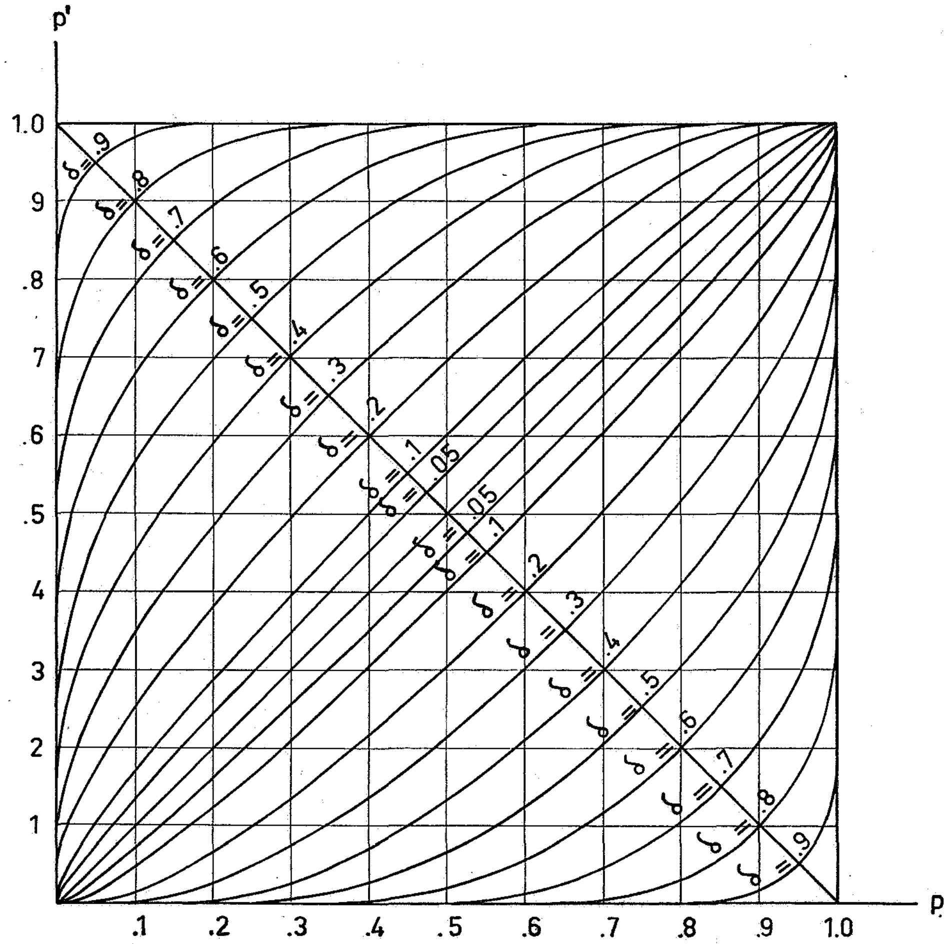

FIG. 2. Functional relationship between p and p' for several values of δ.

one finds four values for δ such that

$$\delta_1 < \delta_2 < 0 < \delta_3 < \delta_4. \tag{3.18}$$

2. On the line $p + p' = 1$ the equality

$$\delta = p - p' \tag{3.19}$$

holds.

Choosing four values for $p - p'$ one finds four values for δ.

The four values of δ (or u respectively) must furthermore be chosen such that (3.7) holds.

Usually one will choose these values symmetrically, i.e. such that

$$\begin{cases} \mu_4 = -\mu_1 \\ \mu_3 = -\mu_2 \end{cases} \tag{3.20}$$

which is equivalent to

$$\begin{cases} \delta_4 = -\delta_1 \\ \delta_3 = -\delta_2 \end{cases} \tag{3.21}$$

and to

$$u_1 u_4 = u_2 u_3 = 1. \tag{3.22}$$

If (3.20) holds, (3.7) is equivalent to

$$\begin{cases} B \leq B' \\ A \leq A'. \end{cases} \tag{3.23}$$

Remark

It is not necessary that p and p' are constants. We only need a constant δ.

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Résumé: Le "Statistical Research Group" de l'Université de Columbia a développé un test séquentiel avec deux décisions possibles pour examiner l'égalité de deux probabilités inconnues, basé sur la comparaison successive de groupes d'observations. Dans le présent article l'auteur décrit un test analogue avec trois décisions possibles – ce qui est plus utile en pratique – en employant la méthode de MM. M. Sobel et A. Wald. Elle donne aussi un exposé des méthodes mentionnées par d'autres auteurs.