KONINKL. NEDERL. AKADEMIE VAN WETENSCHAPPEN – AMSTERDAM
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MATHEMATICS

ON ARBITRARY HEREDITARY TIME-DISCRETE STOCHASTIC PROCESSES, CONSIDERED AS STATIONARY MARKOV CHAINS, AND THE CORRESPONDING GENERAL FORM OF WALD'S FUNDAMENTAL IDENTITY

BY

D. VAN DANTZIG AND C. SCHEFFER

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1. The object of the present paper is to prove (cf. theorem 1) by means of three lemmas that an arbitrary time-discrete stochastic process over a sequence of arbitrary sets (in particular: within one arbitrary set) is equivalent to a stationary MARKOV chain in the corresponding set of all finite paths. This result was anticipated by R. P. FEYNMAN [3] who, however, did not prove his result rigorously.

In a previous paper [1], one of us proved, i.a., a generalization of WALD's fundamental identity [7] to stationary MARKOV chains with discrete time parameter in arbitrary sets, and a partial generalization of the same to arbitrary time-discrete stochastic processes. The present paper removes a restriction on the generality, so that our result (theorem 2) includes also KEMPERMAN's theorem [5] as a special case.

2. Let $\{E_i\}$ be a sequence of non-empty sets, on each of which a σ-field [1]) $\sigma(E_i)$ of subsets of E_i, which contains E_i, is defined; i runs through the set N of all natural numbers [2]). E_0 is a set consisting of one point π_0 only, and $\sigma(E_0)$ consists of the empty set, denoted by O, and E_0. We define

$$\forall_n^{\mathrm{N}}\ F_n \overset{\text{def}}{=} E_0 \times E_1 \times \ldots \times E_n \quad ; \quad (F_0 = E_0)\ ^{3)},\ ^{4)},\ F_\omega \overset{\text{def}}{=} \bigcup_0^\infty F_\nu\ ^{5)}.$$

The sets

$$X^{(n)} = X_0 \times X_1 \times \ldots \times X_n \text{ with } X_i \in \sigma(E_i),\ i = 0, 1, \ldots, n,$$

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1) A **σ-field** of sets is a system of subsets of a given set, containing with any one of them its complement, and with any sequence of them its union.

2) Zero is considered to be a natural number.

3) The symbols $\forall_x^S, \exists_x^S$ before a statement $\mathscr{A}(x)$ depending on x, stand for: $\mathscr{A}(x)$ holds for all $x \in S$, and for at least one $x \in S$ respectively. The symbol $\overset{\text{def}}{=}$ denotes an equality defining the left hand member.

4) If $X_1, X_2, \ldots, X_n$ are sets, then

$$X_1 \times X_2 \times \ldots \times X_n \overset{\text{def}}{=} \text{Ens}\ \{(x_1, x_2, \ldots, x_n) \mid x_1 \in X_1,\ x_2 \in X_2, \ldots, x_n \in X_n\},$$

where Ens $\{x \mid \mathscr{A}(x)\}$ (and in the same way Ens $\{x \in S \mid \mathscr{A}(x)\}$ and Ens $\{f(x) \mid \mathscr{A}(x)\}$) denotes the set of all x (or all $x \in S$, or all $f(x)$ respectively) for which the statement $\mathscr{A}(x)$ is true. Thus F_n is the set of all "paths" of "length" n.

5) I.e. F_ω is the set of all paths of finite length.

generate a σ-field on F_n, which will be denoted by σ_n. Then we have

Lemma 1:

$$\sigma_\omega \stackrel{\text{def}}{=} \text{Ens}\,\{\bigcup_0^\infty \Lambda_n | \forall_n^{\mathrm{N}}\ \Lambda_n \in \sigma_n\}$$

is a σ-field.

Proof:

If $\Lambda_k \in \sigma_\omega$ for all natural k, then $\Lambda_k = \bigcup_n \Lambda_{k,n}$ for all k, and $\Lambda_{k,n} \in \sigma_n$, for all k and n, hence

$$\bigcup_k \Lambda_k = \bigcup_k \{\bigcup_n \Lambda_{k,n}\} = \bigcup_n \{\bigcup_k \Lambda_{k,n}\} \in \sigma_\omega,$$

as $\bigcup_k \Lambda_{k,n} \in \sigma_n$ for all n.

Furthermore we have for each $\Gamma \in \sigma_\omega$:

$$\Gamma = \bigcup \Gamma_n \ ; \ \forall_n^{\mathrm{N}}\, \Gamma_n \in \sigma_n.$$

Hence:

$$F_\omega - \Gamma = \bigcup F_\nu - \bigcup \Gamma_n = \bigcup_\nu \{\bigcap_n (F_\nu - \Gamma_n)\} = \bigcup_\nu (F_\nu - \Gamma_\nu) \in \sigma_\omega,$$

as $F_n - \Gamma_n \in \sigma_n$ for all n. Hence σ_ω is a σ-field.

3. We introduce the following notations:

If $\pi = (x_1, x_2, \ldots, x_n)$, then π is called a "path" of "length" $l(\pi) = n$, and

$$x(\pi) \stackrel{\text{def}}{=} x_{l(\pi)}\ (= x_n)$$

is its "*endpoint*";

$$\mathscr{I}(\pi) \stackrel{\text{def}}{=} (x_1, x_2, \ldots x_{l(\pi)-1}),$$

its first "*derivative*" (initial segment), and

$$\mathscr{I}^k(\pi) \stackrel{\text{def}}{=} \mathscr{I}\{\mathscr{I}^{k-1}(\pi)\} \text{ for all } k \text{ with } 1 \leqq k < n\ (\mathscr{I}^0(\pi) = \pi),$$

so that

$$\mathscr{I}^k(\pi) = (x_1, x_2, \ldots, x_{l(\pi)-k}).$$

Furthermore

$$\mathscr{I}^{l(\pi)}(\pi) \stackrel{\text{def}}{=} \pi_0$$

is called the "*empty path*" ($l(\pi_0) = 0$). Moreover

$$\pi y \stackrel{\text{def}}{=} (x_1, x_2, \ldots, x_{l(\pi)}, y)$$

is the path obtained from π by direct "prolongation" with y, so that

$$x(\pi y) = y \ ; \ \mathscr{I}(\pi y) = \pi.$$

Finally for all $\pi \in F_\omega$ and all $\Lambda \in \sigma_\omega$

$$\Xi(\pi, \Lambda) \stackrel{\text{def}}{=} \text{Ens}\,\{x \in E_{l(\pi)+1} | \pi x \in \Lambda\}$$

is the set of all endpoints of paths belonging to Λ, which are direct prolongations of π. We shall prove some of the properties of these sets:

1) $\forall_\pi^{F_\omega} \Xi(\pi, \bigcirc) = \bigcirc$, which is trivial.

2) $\forall_\pi^{F_\omega} \forall_\Lambda^{\sigma_\omega} \Xi(\pi, \Lambda) = \Xi(\pi, \Lambda \frown F_{l(\pi)+1})$, as $\pi x \in \Lambda$ is equivalent to $\pi x \in \Lambda \frown F_{l(\pi)+1}$.

3) If Λ is an m-dimensional product set: $\Lambda = X_0 \times X_1 \times \ldots \times X_m$, $X_i \in \sigma(E_i)$, then

$$\Xi(\pi, \Lambda) = \bigcirc \qquad \text{if } \pi \notin X_0 \times X_1 \times \ldots X_{m-1} \text{ and}$$
$$\Xi(\pi, \Lambda) = X_m = X_{l(\pi)+1} \text{ if } \pi \in X_0 \times X_1 \times \ldots X_{m-1} \text{ (so that } l(\pi) = m-1\text{)},$$

so that we have for any countably additive set function $V(X)$, defined on $\sigma(E_m)$:

3a) $V\{\Xi(\pi, \Lambda)\} = \mathbf{I}^\pi_{X_0 \times X_1 \ldots X_{m-1}} V(X_m)$ [7]),

if Λ is the product set $X_0 \times X_1 \times \ldots X_m$, as $V(\bigcirc) = 0$.

Furthermore:

4) $\forall_\Lambda^{\sigma_{l(\pi)+1}} \Xi(\pi, F_{l(\pi)+1} - \Lambda) = E_{l(\pi)+1} - \Xi(\pi, \Lambda)$, as

$$\Xi(\pi, F_{l(\pi)+1} - \Lambda) = \text{Ens}\{x \in E_{l(\pi)+1} | \pi x \notin \Lambda\} = E_{l(\pi)+1} - \text{Ens}\{x \in E_{l(\pi)+1} | \pi x \in \Lambda\},$$

and

5) If $\forall_n^{\mathrm{N}} \Gamma_n \in \sigma_n$ then $\Xi(\pi, \cup \Gamma_n) = \cup \Xi(\pi, \Gamma_n)$, as

$$\Xi(\pi, \cup \Gamma_n) = \text{Ens}\{x \in E_{l(\pi)+1} | \exists_n^{\mathrm{N}} \pi x \in \Gamma_n\} = \cup \text{ Ens}\{x \in E_{l(\pi)+1} | \pi x \in \Gamma_n\}.$$

6) From 1), 4) and 5) it follows that the analogue of 5) for intersections instead of unions also holds, and that $\{\Xi(\pi, \Lambda_\nu)\}$ is a dissection [8]) of $\Xi(\pi, \Lambda)$ if $\{\Lambda_\nu\}$ is a dissection of Λ.

7) From 3), 4) and 5) it follows that the class of all $\Lambda \in \sigma_{l(\pi)+1}$ for which $\Xi(\pi, \Lambda) \in \sigma(E_{l(\pi)+1})$, is a σ-field which contains the $(l(\pi)+1)$-dimensional product sets. As $\sigma_{l(\pi)+1}$ is the smallest σ-field which contains these product sets, it follows that this class is identical with $\sigma_{l(\pi)+1}$. Hence

$$\forall_\pi^{F_\omega} \forall_\Lambda^{\sigma_{l(\pi)+1}} \Xi(\pi, \Lambda) \in \sigma(E_{l(\pi)+1}), \text{ whence by 2)}$$

8) $\forall_\pi^{F_\omega} \forall_\Lambda^{\sigma_\omega} \quad \Xi(\pi, \Lambda) \in \sigma(E_{l(\pi)+1})$.

Finally we shall prove

Lemma 2:

If, for all natural n, for every $X \in \sigma(E_{n+1})$, $F(\pi, X)$ is a σ_n-measurable function of $\pi \in F_n$, which, for all $\pi \in F_n$, is countably additive in $X \in \sigma(E_{n+1})$, then $F\{\pi, \Xi(\pi, \Lambda)\}$, for every $\Lambda \in \sigma_\omega$, is σ_ω-measurable in $\pi \in F_\omega$, and, for all $\pi \in F_\omega$, countably additive in $\Lambda \in \sigma_\omega$.

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7) $\mathbf{I}^x_X \overset{\text{def}}{=} \begin{cases} 1 \text{ if } x \in X \\ 0 \text{ if } x \notin X \end{cases}$, so that for $\pi = (x_1, x_2, \ldots, x_n)$

$$\mathbf{I}^\pi_{X_1 \times X_2 \ldots \times X_n} = \mathbf{I}^{x_1}_{X_1} \mathbf{I}^{x_2}_{X_2} \ldots \mathbf{I}^{x_n}_{X_n}.$$

8) A sequence of sets $\{\Lambda_\nu\}$ $(\nu = 1, 2, \ldots)$ is called a dissection of Λ, if $\cup \Lambda_\nu = \Lambda$ and $\Lambda_\nu \frown \Lambda_\mu = \bigcirc$ if $\nu \neq \mu$.

Proof:

Let n and m be two arbitrary natural numbers. We shall prove that $F\{\pi, \Xi(\pi, \Lambda)\}$, for every $\Lambda \in \sigma_m$, is σ_n-measurable in $\pi \in F_n$. If Λ is an m-dimensional product set: $\Lambda = X_0 \times X_1 \times \ldots \times X_m$, $X_i \in \sigma(E_i)$ for $i = 0, 1, \ldots, m$, then we have by 3a)

$$F\{\pi, \Xi(\pi, \Lambda)\} = \mathbf{I}^{\pi}_{X_0 \times X_1 \times \ldots \times X_{m-1}} F(\pi, X_m),$$

which, as a product of σ_n-measurable functions, again is σ_n-measurable. If furthermore $F\{\pi, \Xi(\pi, \Lambda)\}$ is σ_n-measurable for some $\Lambda \in \sigma_m$, then

$$F\{\pi, \Xi(\pi, F_m - \Lambda)\} = \delta_{n,m-1} F\{\pi, E_{n+1} - \Xi(\pi, \Lambda)\} =$$
$$= \delta_{n,m-1} [F(\pi, E_{n+1}) - F\{\pi, \Xi(\pi, \Lambda)\}], \quad ^{9})$$

as a sum of σ_n-measurable functions, is also σ_n-measurable. And if $\{\Lambda_\nu\}$ is a dissection of $\Lambda \in \sigma_m$, with $\Lambda_\nu \in \sigma_m$, for all natural ν, then

$$F\{\pi, \Xi(\pi, \Lambda)\} = F\{\pi, \cup \Xi(\pi, \Lambda_\nu)\} = \sum F\{\pi, \Xi(\pi, \Lambda_\nu)\},$$

by 6) and 8). Therefore, if, for all natural ν, $F\{\pi, \Xi(\pi, \Lambda_\nu)\}$ is σ_n-measurable in $\pi \in F_n$, then $F\{\pi, \Xi(\pi, \Lambda)\}$ is σ_n-measurable too. It follows that the class of all $\Lambda \in \sigma_m$ for which $F\{\pi, \Xi(\pi, \Lambda)\}$ is σ_n-measurable in $\pi \in F_n$, is a σ-field which contains the m-dimensional product sets. Hence this class is σ_m (using the same argument as in 7)). Therefore we have for every pair of natural numbers n and m, and for all $\Lambda \in \sigma_m$: $F\{\pi, \Xi(\pi, \Lambda)\}$ is σ_n-measurable in $\pi \in F_n$. It is now easy to see that the same holds for all $\Lambda \in \sigma_\omega$ by using 2) and our result with $m = n+1$. Hence

$$\forall_a^{\mathrm{R}} \forall_\Lambda^{\sigma_\omega} \text{Ens}\,\{\pi \in F_\omega | F\{\pi, \Xi(\pi, \Lambda)\} \leqq a\} =$$
$$= \bigcup_\nu \text{Ens}\,\{\pi \in F_\nu | F\{\pi, \Xi(\pi, \Lambda)\} \leqq a\} \in \sigma_\omega, \quad ^{10})$$

as we have for all natural ν:

$$\text{Ens}\,\{\pi \in F_\nu | F\{\pi, \Xi(\pi, \Lambda)\} \leqq a\} \in \sigma_\nu.$$

Hence for all $\Lambda \in \sigma_\omega$ $F\{\pi, \Xi(\pi, \Lambda)\}$ is σ_ω-measurable in $\pi \in F_\omega$.

The countable additivity follows immediately from 6) and 8).

4. We assume a stochastic process $\{\mathbf{x}_n\}$ [11]) to be given on the sequence $\{E_i\}$:

$$\mathscr{P}\{\mathbf{x}_{l(\pi)+1} \in X | (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{l(\pi)}) = \pi\} \stackrel{\text{def}}{=} P_X^\pi,$$

where P_X^π is a given function, defined for all $\pi \in F_\omega$ and all $X \in \sigma(E_{l(\pi)+1})$ which, for all $\pi \in F_\omega$, is a probability distribution [12]) over the sets $X \in \sigma(E_{l(\pi)+1})$ and, for all natural n and all $X \in \sigma(E_{n+1})$, is a σ_n-measurable

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9) $\delta_{a,b}$ is the Kronecker symbol.

10) R denotes the set of all real numbers. The argument given here for real valued functions holds also for complex valued functions.

11) Random variables are printed in bold type.

12) A set function $G(\Lambda)$ defined on a σ-field of subsets of a set Ω is a probability distribution if it is countably additive in Λ and non-negative with $G(\Omega) = 1$.

function of $\pi \in F_n$. P_X^π is the probability that a wandering point, having passed through π, arrives at the next step somewhere in X. We shall now extend the definition of this process to F_ω. To this end we define [13] a *generalized* (E, E') *Markov matrix* as a function M_X^π of an element π of a set E and an element X of a σ-field $\sigma(E')$ of subsets of a set E' (which contains E'), such that

1) for each $X \in \sigma(E')$, M_X^π is measurable with respect to a σ-field $\sigma(E)$ of subsets of E $(E \in \sigma(E))$;

2) for each $\pi \in E$, M_X^π is a probability distribution over the sets $X \in \sigma(E')$.

If E and E' coincide we shall assume that $\sigma(E)$ and $\sigma(E')$ also are identical.

Furthermore we define:

$$\forall_\pi^{F_\omega} \forall_\Lambda^{\sigma_\omega} \quad Q_\Lambda^\pi \overset{\text{def}}{=} P^\pi_{\Xi(\pi,\Lambda)}\,. \tag{4.1}$$

It is easy to prove now that Q_Λ^π is a generalized (F_ω, F_ω) Markov matrix by using lemma 2 and the properties of $\Xi(\pi, \Lambda)$. We shall prove more generally that this result holds also for the n-fold convolution of Q_Λ^π:

Lemma 3:

The relations

$$\begin{cases} \forall_\pi^{F_\omega} \forall_\Lambda^{\sigma_\omega} & Q_{(0)}{}_\Lambda^\pi \overset{\text{def}}{=} \mathbf{I}_\Lambda^\pi \\ \forall_n^{\mathrm{N}} \forall_\pi^{F_\omega} \forall_\Lambda^{\sigma_\omega} & Q_{(m)}{}_\Lambda^\pi \overset{\text{def}}{=} \int Q_{(m-1)}{}_{d\tau}^\pi Q_\Lambda^\tau \quad {}^{14)} \end{cases} \tag{4.2}$$

define for all natural m a generalized (F_ω, F_ω) Markov *matrix, with* $Q_{(1)}{}_\Lambda^\pi = Q_\Lambda^\pi$.

Proof:

If $m=0$, the statement of the lemma is trivial. Let us assume that for some integer $m \geqq 1$ $Q_{(m-1)}{}_\Lambda^\pi$ is a generalized (F_ω, F_ω) Markov matrix. The countable additivity of $Q_{(m-1)}{}_\Lambda^\pi$ ensures for every $\Lambda \in \sigma_\omega$ the existence of $Q_{(m)}{}_\Lambda^\pi$, as Q_Λ^π, by its definition (4.1) and by lemma 2, is σ_ω-measurable and bounded. Furthermore it is trivial that $Q_{(m)}{}_\Lambda^\pi$ is (finitely) additive, and from known theorems [15] it follows immediately that $Q_{(m)}{}_\Lambda^\pi$ is countably additive, since Q_Λ^π is. Moreover it is trivial that $Q_{(m)}{}_\Lambda^\pi \geqq 0$ and that $Q_{(m)}{}_{F_\omega}^\pi = 1$.

Finally the integral $\int Q_{(m-1)}{}_{d\tau}^\pi Q_\Lambda^\tau$, being a limit of σ_ω-measurable functions (the approximating sums are finite sums of σ_ω-measurable functions, hence σ_ω-measurable), again is σ_ω-measurable. This completes the proof of the lemma.

5. We shall now prove that the newly defined quantities $Q_{(n)}{}_\Lambda^\pi$ satisfy the Chapman–Kolmogorov equation:

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[13] Cf [1] § 8.

[14] When the domain of integration is not mentioned, the integrations are to be extended over the whole space F_ω.

[15] Cf e.g. [1] § 8.

Theorem 1:

The relation

$$Q_{(n+m)A}^{\ \pi} = \int Q_{(n)d\tau}^{\ \pi}\, Q_{(m)A}^{\ \tau} \tag{5.1}$$

holds for all $\pi \in F_\omega$, all $A \in \sigma_\omega$, and for every pair of natural numbers n and m.

Proof:

For $m=0$ (5.1) holds for all n, which follows immediately from the definition of $Q_{(0)A}^{\ \pi}$. We assume now that, for some integer $m_0 \geqq 1$ and for all n, (5.1) holds with $m=m_0-1$. Then we have, by using (4.2) and interchanging integrations:

$$\int Q_{(n)d\tau}^{\ \pi}\, Q_{(m_0)A}^{\ \tau} = \int Q_{(n)d\tau}^{\ \pi} \{\int Q_{(m_0-1)d\sigma}^{\ \tau}\, Q_A^\sigma\} =$$
$$= \int \{\int Q_{(n)d\tau}^{\ \pi}\, Q_{(m_0-1)d\sigma}^{\ \tau}\}\, Q_A^\sigma = \int Q_{(n+m_0-1)d\sigma}^{\ \pi}\, Q_A^\sigma = Q_{(n+m_0)A}^{\ \pi}.$$

The interchange of integrations is now justified by the existence of $\int Q_{(n)d\tau}^{\ \pi}\, Q_{(m_0-1)M}^{\ \tau}$ and of $\int Q_{(m_0-1)d\sigma}^{\ \tau}\, Q_A^\sigma$, and by the *boundedness* of the result, according to [1], lemma 4 [16]). Hence (5.1) holds for all n if $m=m_0$ which completes the proof of the theorem.

From this theorem it follows that on F_ω a stationary Markov chain $\{\boldsymbol{\pi}_n\}$, with transition probabilities $Q_{(n)A}^{\ \pi}$, can be defined:

$$\forall_n^{N}\, \forall_m^{N}\, \forall_\pi^{F_\omega}\, \forall_A^{\sigma_\omega}\, \mathscr{P}\{\boldsymbol{\pi}_{n+m} \in A \mid \boldsymbol{\pi}_m = \pi\} \overset{\text{def}}{=} Q_{(n)A}^{\ \pi}, \tag{5.2}$$

and

$$\mathscr{P}\{\boldsymbol{\pi}_{n_k+n} \in A \mid \boldsymbol{\pi}_{n_1} = \pi_1, \ldots, \boldsymbol{\pi}_{n_k} = \pi_k\} \overset{\text{def}}{=} Q_{(n)A}^{\ \pi_k}, \tag{5.3}$$

whenever $0 \leqq n_1 \leqq \ldots \leqq n_k$; $n \geqq 0$, and if $(\pi_1, \pi_2, \ldots, \pi_k)$ is a set of mutual consistent points of F_ω, i.e. if

$$\mathscr{I}^m(\pi_j) = \pi_i,$$

for all i and j with $i \leqq j$, and $n_j - n_i = m$.

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[16]) This lemma states that in the repeated integral $\int_{E_1} F_{dx} \int_{E_2} R_{dy}^x f^y$, the order of the integration may be reversed:

$$\int_{E_1} F_{dx} \int_{E_2} R_{dy}^x f^y = \int_{E_2} \{\int_{E_1} F_{dx} R_{dy}^x\} f^y, \tag{I}$$

if both the integrals $H_A = \int_{E_1} |F|_{dx} |R^x|_A$ and $h^x = \int_{E_2} |R^x|_{dy} |f^y|$, exist for all $A \varepsilon \sigma(E_2)$ and for all $x \varepsilon E_1$ respectively, and if one of the members of (I) is absolutely convergent. I.e. if either

$$\int_{E_2} H_{dy} |f^y| < \infty \text{ or } \int_{E_1} |F|_{dx} h^x < \infty.$$

(E_1 and E_2 are sets on each of which a σ-field $\sigma(E_i)$ ($i = 1, 2$) is defined, F_A is a countably additive set function defined on $\sigma(E_1)$, R_M^x is a generalized (E_1, E_2) Markov matrix and f^y is a $\sigma(E_2)$-measurable function of $y \varepsilon E_2$. The absolute value function $|F|_A$ of a countably additive set function F, defined on a σ-field, is defined by $|F|_A \overset{\text{def}}{=} \sup \{\sum_{\nu=1}^{\infty} |F_{A_\nu}|\}$, where the supremum is taken over all possible dissections of A.

Theorem 1 ensures now the consistency of the definitions (5.2) and (5.3) [17]).

6. In the sequel we shall need repeatedly the following

Lemma 4:

If g^π and h^π are σ_ω-measurable functions defined for all $\pi \in F_\omega$, and if for all $\pi \in F_\omega$:

$$\int Q^\pi_{d\tau} \; |h^\tau| < \infty$$

then

$$\int Q^\pi_{d\tau}\, g^{\mathscr{I}(\tau)}\, h^\tau = g^\pi \int Q^\pi_{d\tau}\, h^\tau \tag{6.1}$$

holds for all $\pi \in F_\omega$.

Proof:

By (4.1) we have for all $\pi \in F_\omega$, and all $\Gamma \in \sigma_\omega$

$$Q^\pi_\Gamma = P^\pi_{\Xi(\pi,\Gamma)} = \int\limits_{E_{l(\pi)+1}} P^\pi_{dx}\, \mathbf{I}^x_{\Xi(\pi,\Gamma)} = \int\limits_{E_{l(\pi)+1}} P^\pi_{dx}\, \mathbf{I}^{\pi x}_\Gamma ,$$

where the identity $\mathbf{I}^x_{\Xi(\pi,\Gamma)} = \mathbf{I}^{\pi x}_\Gamma$ follows immediately from the definition of $\Xi(\pi, \Gamma)$. Hence:

$$\int Q^\pi_{d\tau}\, g^{\mathscr{I}(\tau)}\, h^\tau = \int \{\int\limits_{E_{l(\pi+1)}} P^\pi_{dx}\, \mathbf{I}^{\pi x}_{d\tau}\}\, g^{\mathscr{I}(\tau)}\, h^\tau ,$$

Now both the integrals

$$\int\limits_{E_{l(\pi)+1}} P^\pi_{dx}\, \mathbf{I}^{\pi x}_\Lambda = Q^\pi_\Lambda , \text{ and } \int \mathbf{I}^{\pi x}_{d\tau}\, |g^{\mathscr{I}(\tau)}\, h^\tau| = |g^\pi\, h^{\pi x}|$$

exist, so that we may, according to lemma 4 in [1], interchange integrations unless the resulting integral is divergent. Hence

$$\int Q^\pi_{d\tau}\, g^{\mathscr{I}(\tau)}\, h^\tau = \int\limits_{E_{l(\pi)+1}} P^\pi_{dx} \int \mathbf{I}^{\pi x}_{d\tau}\, g^{\mathscr{I}(\tau)}\, h^\tau = \int\limits_{E_{l(\pi)+1}} P^\pi_{dx}\, g^\pi\, h^{\pi x} = g^\pi \int\limits_{E_{l(\pi)+1}} P^\pi_{dx}\, h^{\pi x} .$$

Applying this derivation again, now with $g^\pi = 1$ we get:

$$\int\limits_{E_{l(\pi)+1}} P^\pi_{dx}\, h^{\pi x} = \int Q^\pi_{d\tau}\, h^\tau .$$

Hence:

$$\int Q^\pi_{d\tau}\, g^{\mathscr{I}(\tau)}\, h^\tau = g^\pi \int Q^\pi_{d\tau}\, h^\tau .$$

The integral in the right hand member is absolutely convergent, and therefore the derivation of our result is justified.

Corollary:

Under the same conditions we have

$$\int Q_{(n)}{}^\pi_{d\tau}\, g^{\mathscr{I}(\tau)}\, h^\tau = \int Q_{(n-1)}{}^\pi_{d\sigma}\, g^\sigma \int Q^\sigma_{d\tau}\, h^\tau , \tag{6.2}$$

if the integral in the right hand member is absolutely convergent. In

[17]) Cf [6].

particular, taking $h^\pi = 1$ for all $\pi \in F_\omega$ (for this choice of h^π, the condition of lemma 4 is certainly satisfied):

(6.3) $$\int Q_{(n)d\tau}^{\pi} \, g^{\mathscr{I}(\tau)} = \int Q_{(n-1)d\sigma}^{\pi} \, g^{\sigma},$$

if $\int Q_{(n-1)d\sigma}^{\pi} |g^{\sigma}| < \infty$.

7. We shall now define a collective matrix [18]) for this stochastic process. If the wandering point is in $x(\pi)$, having passed through $\mathscr{I}(\pi)$, we assume that there is a probability A^π that it will be "absorbed", and that in this case there is a probability $1-U^\pi$ that some definite event, called a "catastrophe" $\mathscr{C}$, will occur. If no absorption takes place in $x(\pi)$ (probability $B^\pi = 1 - A^\pi$) the probability of $\mathscr{C}$ is $1 - T^\pi$. We now shall determine the conditional probability D_A^π, that the wandering point, having passed through π, without being absorbed on the way, will be absorbed somewhere in A, without being absorbed previously and without a catastrophe having happened previously. For this purpose we define

$$\beta^\pi \overset{\text{def}}{=} \prod_{\nu=0}^{l(\pi)} B^{\mathscr{I}^\nu(\pi)} \quad ; \quad \alpha^\pi \overset{\text{def}}{=} A^\pi \, \beta^{\mathscr{I}(\pi)},$$

$$V^\pi \overset{\text{def}}{=} \prod_{\nu=0}^{l(\pi)} T^{\mathscr{I}^\nu(\pi)} \quad ; \quad W^\pi \overset{\text{def}}{=} U^\pi \, V^{\mathscr{I}(\pi)}.$$

I.e. β^π is the probability that the wandering point will not be absorbed in $x(\pi)$ (nor previously), α^π is the probability that the wandering point will be absorbed in $x(\pi)$, but not previously. In the first case there is a probability V^π that $\mathscr{C}$ will occur in $x(\pi)$ and has not occurred before. In the latter case this probability equals W^π. It is easy to see now that D_A^π must satisfy the integral equation

(7.1) $$D_A^\pi = \alpha^\pi \, W^\pi \, \mathbf{I}_A^\pi + \int Q_{d\tau}^\pi \, D_A^\tau,$$

as the two terms in the right hand member represent the probabilities of the following two events:

I) The event, of which D_A^π is the probability, takes place in π, and
II) This event takes place in some point of F_ω, which can be reached from π.

A second expression for D_A^π is obtained by remarking that the event, of which D_A^π is the probability, can only take place in π, or in a point of F_ω which can be reached from π in a finite number of steps:

(7.2) $$D_A^\pi = \sum_{n=0}^{\infty} \int Q_{(n)d\tau}^{\pi} \, \alpha^\tau \, W^\tau \, \mathbf{I}_A^\tau = \sum_{n=0}^{\infty} \int_A Q_{(n)d\tau}^{\pi} \, \alpha^\tau \, W^\tau.$$

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[18]) In the sequel we shall not use the interpretation of the quantities A^π, B^π, T^π and U^π. We mention these interpretations in connection with [1], where they are used repeatedly. For the method of collective marks and its applications to time discrete stochastic processes cf also [2] and [5].

8. We now take $U^\pi = 1$ for all $\pi \in F_\omega$ (hence $W^\pi = V^{\mathscr{I}(\pi)}$; that means that $\mathscr{C}$ can only occur if the wandering point is absorbed) and allow T^π to take arbitrary complex values, subject only to the following conditions: For all $\pi \in F_\omega$ a real positive number T_0^π exists, such that for some ϑ, independent of π, with $0 \leqq \vartheta < 1$, the relations

$$\forall_\pi^{F_\omega} \quad \beta^\pi |T^\pi| \leqq \vartheta \, \beta^\pi \, T_0^\pi \tag{8.1}$$ [19]

and

$$\forall_\pi^{F_\omega} \quad \beta^\pi \int Q_{\mathrm{d}\tau}^\pi \, B^\tau T_0^\tau \leqq \beta^\pi \tag{8.2}$$

are satisfied. We have now:

Lemma 5:

The conditions (8.1) *and* (8.2) *entail the absolute convergence of* (7.2), *and in this case* (7.1) *is satisfied by* (7.2).

Proof:

If $n \geqq 1$, we have, applying lemma 4, its corollary and theorem 1,

$$\left| \int Q_{(n)\mathrm{d}\tau}^{\pi} \alpha^\tau \, V^{\mathscr{I}(\tau)} \, \mathbf{I}_\Lambda^\tau \right| \leqq \int Q_{(n)\mathrm{d}\tau}^{\pi} \beta^{\mathscr{I}(\tau)} \left| V^{\mathscr{I}(\tau)} \right| = \int Q_{(n-1)\mathrm{d}\sigma}^{\pi} \beta^\sigma \left| V^\sigma \right|,$$

if this last integral is convergent. Proceeding in the same way, now using (6.2) and (8.2), we have moreover, if $n \geqq 2$

$$\int Q_{(n-1)\mathrm{d}\sigma}^{\pi} \beta^\sigma \left| V^\sigma \right| = \int Q_{(n-1)\mathrm{d}\sigma}^{\pi} \beta^{\mathscr{I}(\sigma)} \left| V^{\mathscr{I}(\sigma)} \right| B^\sigma |T^\sigma| \leqq$$
$$\leqq \int Q_{(n-2)\mathrm{d}\varrho}^{\pi} \beta^\varrho \left| V^\varrho \right| \vartheta \int Q_{\mathrm{d}\sigma}^{\varrho} \, B^\sigma T_0^\sigma \leqq \vartheta \int Q_{(n-2)\mathrm{d}\varrho}^{\pi} \beta^\varrho \left| V^\varrho \right|.$$

Hence by induction we have for all $n \geqq 1$, all $\pi \in F_\omega$ and all $\Lambda \in \sigma_\omega$

$$\left| \int Q_{(n)\mathrm{d}\tau}^{\pi} \alpha^\tau \, V^{\mathscr{I}(\tau)} \, \mathbf{I}_\Lambda^\tau \right| \leqq \vartheta^{n-1} \int Q_{(0)\mathrm{d}\tau}^{\pi} \beta^\tau \left| V^\tau \right| = \vartheta^{n-1} \left| V^\pi \right| \beta^\pi \tag{8.3}$$

so that the result is bounded indeed. Hence we have for the series in (7.2)

$$\forall_\pi^{F_\omega} \forall_\Lambda^{\sigma_\omega} \quad |D_\Lambda^\pi| \leqq |D^\pi|_\Lambda \leqq \alpha^\pi \left| V^{\mathscr{I}(\pi)} \right| + \frac{\beta^\pi |V^\pi|}{1-\vartheta}. \tag{8.4}$$ [20]

Substituting this series into the right hand member of (7.1) we get

$$\alpha^\pi V^{\mathscr{I}(\pi)} \mathbf{I}_\Lambda^\pi + \int Q_{\mathrm{d}\tau}^\pi \sum_{n=0}^{\infty} \int Q_{(n)\mathrm{d}\sigma}^{\tau} \alpha^\sigma V^{\mathscr{I}(\sigma)} \mathbf{I}_\Lambda^\sigma = \alpha^\pi \, V^{\mathscr{I}(\pi)} \, \mathbf{I}_\Lambda^\pi + \sum_{n=0}^{\infty} \int Q_{\mathrm{d}\tau}^\pi \int Q_{(n)\mathrm{d}\sigma}^{\tau} \alpha^\sigma V^{\mathscr{I}(\sigma)} \mathbf{I}_\Lambda^\sigma =$$

$$= \alpha^\pi \, V^{\mathscr{I}(\pi)} \, \mathbf{I}_\Lambda^\pi + \sum_{n=0}^{\infty} \int Q_{(n+1)\mathrm{d}\sigma}^{\pi} \alpha^\sigma \, V^{\mathscr{I}(\sigma)} \, \mathbf{I}_\Lambda^\sigma = \sum_{n=0}^{\infty} \int Q_{(n)\mathrm{d}\sigma}^{\pi} \alpha^\sigma \, V^{\mathscr{I}(\sigma)} \, \mathbf{I}_\Lambda^\sigma = D_\Lambda^\pi .$$

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[19] I.e. $|T^\pi| \leqq \vartheta \, T_0^\pi$, unless $\beta^\pi = 0$.

[20] As $|D_\Lambda^\pi| \leqq F_\Lambda$ for some positive valued countably additive set function F_Λ implies $|D^\pi|_\Lambda \leqq F_\Lambda$. $|D^\pi|_\Lambda$ denotes for every $\pi \in F_\omega$ the absolute value function of the countably additive set function D_Λ^π.

The interchange of integration and summation is allowed because [21])
1) each term of the series in the second member exists absolutely by (8.3),
2) the series in the first member converges for every $\pi \in F_\omega$ by (8.4),
3) the upper bound (8.4) is an integrable function, which may easily be verified.

As to the interchange of integrations in the second member: by using (8.3) it can be shown easily that the conditions of lemma 4 in [1] are satisfied.

9. We now pass to the generalization of WALD's fundamental identity:

Theorem 2:

If T^π satisfies the conditions of lemma 5, if $U^\pi=1$ and if

$$\exists_{f_0}^{\Omega_P} \forall_\pi^{F_\omega} \quad \beta^\pi\, T_0^\pi \int Q_{\mathrm{d}\tau}^\pi f_0^\tau \leqq \beta^\pi f_0^\pi \tag{9.1}$$ [22])

then for all complex valued σ_ω-measurable functions f^π, satisfying

$$\exists_c^{\mathrm{P}} \forall_\pi^{F_\omega - F_0} \quad \beta^{\mathscr{I}(\pi)} |f^\pi| \leqq c\, \beta^{\mathscr{I}(\pi)} f_0^\pi \tag{9.2}$$ [23]),

the relations

$$\int D_{\mathrm{d}\tau}^\pi f^\tau = \beta^{\mathscr{I}(\pi)} V^{\mathscr{I}(\pi)} f^\pi \tag{9.3}$$

and

$$\beta^\pi V^{\mathscr{I}(\pi)} f^\pi = \beta^\pi V^\pi \int Q_{\mathrm{d}\tau}^\pi f^\tau, \tag{9.4}$$

are equivalent for all $\pi \in F_\omega - F_0$.

Proof:

We shall need the following two propositions

I) $\int |D^\pi|_{\mathrm{d}\tau} |f^\tau| < \infty$.

Proof:

$$\int |D^\pi|_{\mathrm{d}\tau} |f^\tau| \leqq \int \{\sum_{n=0}^{\infty} \int Q_{(n)\mathrm{d}\sigma}^{\pi}\, \alpha^\sigma\, |V_{\mathrm{u}}^{\mathscr{I}(\sigma)}|\, \mathrm{I}_{\mathrm{d}\tau}^\sigma\}\, |f^\tau|,$$

and (8.4) shows that the series between brackets is bounded uniformly in Λ. By a classical argument it may be shown that we may interchange summation and integration, provided that the resulting series converges absolutely and that each of its terms exists absolutely. Now, by an argument analogous to the one we used for the derivation of (8.3), we get

$$\int \{\int Q_{(n)\mathrm{d}\sigma}^{\pi}\, \alpha^\sigma\, |V^{\mathscr{I}(\sigma)}|\, \mathrm{I}_{\mathrm{d}\tau}^\sigma\} \cdot |f^\tau| = \int Q_{(n)\mathrm{d}\sigma}^{\pi}\, \alpha^\sigma\, |V^{\mathscr{I}(\tau)}| \cdot |f^\sigma| \leqq c\, \vartheta^n \beta^{\mathscr{I}(\pi)}\, |V^{\mathscr{I}(\pi)}|\, f_0^\pi,$$

[21]) Cf e.g. [4], 15.2.31 and 15.2.311.
[22]) Ω_{P} is the set of all positive valued σ_ω-measurable functions.
[23]) P is the set of all positive numbers.

so that summation and integration may be interchanged. Hence:

$$\int |D^\pi|_{d\tau} |f^\tau| \leqq \beta c^{\mathscr{I}(\pi)} \, | V^{\mathscr{I}(\pi)} | \, f_0^\pi \frac{1}{1-\vartheta}. \tag{9.5}$$

Our second proposition is a direct consequence of the first:

II) $\int D^\pi_{d\tau} f^\tau = \sum_{n=0}^{\infty} \int Q^{\ \pi}_{(n)d\tau} \, \alpha^\tau \, V^{\mathscr{I}(\tau)} f^\tau .$

We now proceed to the proof of the theorem. If (9.3) holds, then we have by (7.1):

$$\beta^{\mathscr{I}(\pi)} V^{\mathscr{I}(\pi)} f^\pi = \alpha^\pi V^{\mathscr{I}(\pi)} f^\pi + \int \{\int Q^\pi_{d\tau} D^\tau_{d\sigma}\} f^\sigma .$$

Using now (8.4), (8.2) and I) one sees that the conditions of [1], lemma 4 are satisfied, therefore, by (9.3) and lemma 4:

$$\beta^{\mathscr{I}(\pi)} V^{\mathscr{I}(\pi)} f^\pi = \alpha^\pi V^{\mathscr{I}(\pi)} f^\pi + \int Q^\pi_{d\tau} \int D^\tau_{d\sigma} f^\sigma = \alpha^\pi V^{\mathscr{I}(\pi)} f^\pi + \beta^\pi V^\pi \int Q^\pi_{d\tau} f^\tau ,$$

whence (9.4), as $\beta^{\mathscr{I}(\pi)} - \alpha^\pi = \beta^\pi$.

If (9.4) is true we have, as $\alpha^\tau = \beta^{\mathscr{I}(\tau)} - \beta^\tau$:

$$\int Q^{\ \pi}_{(n)d\tau} \alpha^\tau V^{\mathscr{I}(\tau)} f^\tau = \int Q^{\ \pi}_{(n)d\tau} \beta^{\mathscr{I}(\tau)} V^{\mathscr{I}(\tau)} f^\tau - \int Q^{\ \pi}_{(n)d\tau} \beta^\tau V^{\mathscr{I}(\tau)} f^\tau ,$$

for all $n \geqq 1$, i.e., applying (6.2) in the first term and (9.4):

$$= \int Q^{\ \pi}_{(n-1)d\sigma} \beta^\sigma V^\sigma \int Q^\sigma_{d\tau} f^\tau - \int Q^{\ \pi}_{(n)d\tau} \beta^\tau V^{\mathscr{I}(\tau)} f^\tau =$$

$$= \int Q^{\ \pi}_{(n-1)d\sigma} \beta^\sigma V^{\mathscr{I}(\sigma)} f^\sigma - \int Q^{\ \pi}_{(n)d\tau} \beta^\tau V^{\mathscr{I}(\tau)} f^\tau = K^\pi_{(n-1)} - K^\pi_{(n)} ,$$

where $K^\pi_{(n)} \overset{\text{def}}{=} \int Q^{\ \pi}_{(n)d\tau} \beta^\tau V^{\mathscr{I}(\tau)} f^\tau$. Hence by II)

$$\int D^\pi_{d\tau} f^\tau = \int Q^{\ \pi}_{(0)d\tau} \alpha^\tau V^{\mathscr{I}(\tau)} f^\tau + \sum_{n=1}^{\infty} (K^\pi_{(n-1)} - K^\pi_{(n)}) =$$

$$= \alpha^\pi V^{\mathscr{I}(\pi)} f^\pi + K^\pi_{(0)} - \lim_{n\to\infty} K^\pi_{(n)} .$$

Repeating once more the argument, used in the derivation of (8.3) shows:

$$| K^\pi_{(n)} | \leqq c \vartheta^n \beta^\pi f_0^\pi | V^{\mathscr{I}(\pi)} | .$$

Hence

$$\int D^\pi_{d\tau} f^\tau = \alpha^\pi V^{\mathscr{I}(\pi)} f^\pi + \int Q^{\ \pi}_{(0)d\tau} \beta^\tau V^{\mathscr{I}(\tau)} f^\tau = \beta^{\mathscr{I}(\pi)} V^{\mathscr{I}(\pi)} f^\pi .$$

Hence (9.3) holds for all $\pi \in F_\omega$. This completes the proof of the theorem.

[24]) This theorem passes into Kemperman's theorem [5, p. 14], if we make the following substitutions:

Let E_i be the real axis for all $i > 0$; let $\{\mathrm{x}_n\}$ be a process by independent increases (i.e. a process with $\mathrm{x}_n = \sum_{k=0}^{n} \mathrm{y}_k$, where $\{\mathrm{y}_k\}$ is a set of independent isodistributed random variables). If we take furthermore f^π, and f_0^π to be of the form e^{tx} and e^{tx_0} respectively our $\int D^\pi_{d\tau} f^\tau$ passes into Kemperman's $r_n(t)$ and we get

$$\sum_{n=0}^{\infty} r_n(t) \, e^{-tx_0} \varphi(t)^{-n} = 1,$$

where $\varphi(t)$ is the characteristic function of y_n.

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